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CATEGORIES OF SEMIGROUPS IN QUANTUM COMPUTATIONAL STRUCTURES

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ABSTRACT. We investigate a categorial duality between quasi MV-algebras (a variety of algebras arising from quantum computation and tightly connected with fuzzy logic) and a reflective subcategory of l-groups with strong units.

Introduction

Recently, increasing attention has been paid to algebraic structures arising from quantum computation [3], [4], [8]. More precisely these structures stem from mathematical description of circuits obtained by combinations of quantum gates and operations acting in the bidimensional complex Hilbert space $\mathbf{C}^2$ ([9]). In this case the information processed by means of quantum gates is represented by *qubits* (unit vectors in the bidimensional complex Hilbert space $\mathbf{C}^2$) or by *qumixes* (density operators[1] in $\mathbf{C}^2$) according as they correspond to maximal or to possibly incomplete pieces of information. As is well known, each density operator σ in $\mathbf{C}^2$ has a matrix representation via the Pauli matrices

$$\sigma = \frac{1}{2}(I + r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3)$$

where:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and r_1, r_2, r_3 are real numbers s.t. $r_1^2+r_2^2+r_3^2 \le 1$. We will denote by $\mathscr{D}(\mathbf{C}^2)$ the set of all density operators of $\mathbf{C}^2$. It can be noticed that density operators are

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[1]Hermitian, positive operators with trace 1.

in one-one correspondence with the points of the Poincaré sphere $\mathbf{D}^3$ of radius 1. An interesting feature of density operators is the fact that any real number $0 \leq \lambda \leq 1$ uniquely determines a density operator $\rho_\lambda = \frac{1}{2}(I + (1-2\lambda)\sigma_3)$. For each $\sigma = \frac{1}{2}(I + r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3)$ in $\mathscr{D}(\mathbf{C}^2)$ we can associate, as dictated by the Born rule, a probability value $\mathtt{p}(\sigma)$ in the following manner: $\mathtt{p}(\sigma) = \mathtt{Tr}(\rho_1\sigma) = \frac{1-r_3}{2}$. In this perspective, quantum gates can be represented (in a probabilistic way) as operations in $\mathbf{D}^3$. The analysis of the structural properties of these transformations, fully described in [3], [7], [6], has suggested the introduction of appropriate algebraic structures in order to provide an abstract overview of them. In previous works [8], [10], the most basic of such class of structures is motivated by the following operations in $\mathscr{D}(\mathbf{C}^2)$:

- $\sigma \oplus \tau = \rho_{\mathtt{p}(\sigma)\oplus\mathtt{p}(\tau)}$, where $\mathtt{p}(\sigma) \oplus \mathtt{p}(\tau) = \min\{\mathtt{p}(\sigma) + \mathtt{p}(\tau), 1\}$,
- $\mathtt{NOT}(\tau) = \sigma_1\tau\sigma_1$,
- ρ_0 and ρ_1 as constant operations.

For sake of notational simplicity, in what follows, $\mathscr{D}(\mathbf{C}^2)$ will denote the structure $\langle \mathscr{D}(\mathbf{C}^2), \oplus, \mathtt{NOT}, \rho_0 \rangle$. It has been shown in [8] that the equational theory of $\mathscr{D}(\mathbf{C}^2)$ is algebraically represented by the class of *quasi MV-algebras*.

In this paper we provide a categorial duality between quasi MV-algebras and a category of semigroups, which turns out to be a reflective subcategory of the category of l-groups with strong unit.

The paper is organized as follows: in §1 we recall some basic definitions and properties about MV-algebras and l-groups. In §2 we introduce the notion of *quasi l-group* and we outline its relations with quasi MV-algebras. In Section §3 we introduce an analogous for quasi MV-algebras of the Γ functor for MV-algebras [5]. Finally, section §4 is dedicated to the study of categorical duality between these quasi l-groups and quasi MV-algebras.

1. Basic notions

We recall from [2] and [5] some basic notions about l-groups and MV-algebras, respectively.

A *lattice ordered abelian group* or l-group for short, [2] is an algebra $\langle G, +, -, \vee, \wedge, 0\rangle$, of type $\langle 2, 1, 2, 2, 0\rangle$, which satisfies the following conditions:

(1) $\langle G, +, -, 0\rangle$ is an abelian group,

(2) $\langle G, \wedge, \vee\rangle$ is a lattice,

(3) $w + (x \vee y) = (w + x) \vee (w + y)$,

(4) $w + (x \wedge y) = (w + x) \wedge (w + y)$.

If x is an element of a l-group G we define the *absolute value* of x as $|x| = (x \vee 0) + (-x \vee 0)$. Moreover an element $u \geq 0$ in G is a *strong unit* of G iff for each $x \in G$ there exists a natural number n such that $|x| \leq nu$. We denote by $\mathscr{LG}_u$ the category of l-groups whose objects are l-groups with strong unit and whose arrows are l-groups homomorphisms preserving strong units.

An *MV-algebra* [5] is an algebra $\langle A, \oplus, \neg, 0\rangle$ of type $\langle 2, 1, 0\rangle$ satisfying the following axioms:

(1) $\langle A, \oplus, 0\rangle$ is an abelian monoid,

(2) $\neg\neg x = x$,

(3) $x \oplus \neg 0 = \neg 0$,

(4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. (*Łukasiewicz axiom*)

By means of the primitive MV-algebraic functions, we can define:

$$1 = \neg 0, \quad x \vee y = \neg(\neg x \oplus y) \oplus y,$$
$$x \odot y = \neg(\neg x \oplus \neg y), \quad x \wedge y = \neg(\neg x \vee \neg y).$$

An important example of an MV-algebra is given by the algebra $\text{Ł}_{[0,1]} = \langle [0, 1], \oplus, \neg, 0, 1\rangle$, where $x \oplus y = \min\{x + y, 1\}$ and $\neg x = 1 - x$. It is well known that if G is an l-group and $u \in G$ is a strong unit then the interval algebra $\langle [0, u], \oplus, \neg, 0, u\rangle$ where $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$ is an MV-algebra. In this way, $\text{Ł}_{[0,1]}$ can be obtained from the l-group $\mathbf{R}$ of the real numbers with strong unit 1.

On the other hand, it is possible to construct out of an MV-algebra an l-group as we will see in what follows. Let A be an MV-algebra. A *good sequence* in A is a sequence $\mathbf{a} = (a_1, a_2, \dots)$ of elements of A such that $a_i \oplus a_{i+1} = a_i$ for each $i = 1, 2 \dots$ and there exists an integer n such that $a_i = 0$ if $i > n$. We denote by (a) the sequence $(a, 0 \dots)$. The set of good sequences of A is noted by M_A.

Let $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$ be arbitrary good sequences. If we consider the following operations in M_A:

M1: $\mathbf{a} + \mathbf{b} = (c_1, c_2, \dots)$ where $c_i = a_i \oplus (a_{i-1} \odot b_1) \oplus (a_{i-2} \odot b_2) \oplus \cdots \oplus (a_2 \odot b_{i-2}) \oplus (a_1 \odot b_{i-1}) \oplus b_i$ for each $i = 1, 2, \dots$,

M2: $\mathbf{a} \vee \mathbf{b} = (a_1 \vee b_1, a_2 \vee b_2, \dots)$,

M3: $\mathbf{a} \wedge \mathbf{b} = (a_1 \wedge b_1, a_2 \wedge b_2, \dots)$,

then $\langle M_A, +, \vee, \wedge, (0)\rangle$ is an abelian cancellative lattice monoid whose order is given as $\mathbf{a} \leq \mathbf{b}$ iff $a_i \leq b_i$ for each $i = 1, 2 \dots$. This order is *translation invariant*, in the sense that $\mathbf{a} \leq \mathbf{b}$ implies that $\mathbf{a} + \mathbf{d} \leq \mathbf{b} + \mathbf{d}$ for each good sequence $\mathbf{d}$.

From the abelian lattice monoid M_A we can obtain an l-group as follows: we consider the equivalence relation $\equiv$ in $M_A \times M_A$ given by $(\mathbf{a}, \mathbf{b}) \equiv (\mathbf{a}', \mathbf{b}')$ iff

$\mathbf{a}+\mathbf{b}' = \mathbf{a}'+\mathbf{b}$. Denoting by $[\mathbf{a},\mathbf{b}]$ the equivalence class of $(\mathbf{a},\mathbf{b})$, and by G_A the set of equivalence classes we can consider the following operations in G_A:

$$[\mathbf{a},\mathbf{b}]+[\mathbf{c},\mathbf{d}] = [\mathbf{a}+\mathbf{c},\mathbf{b}+\mathbf{d}],$$
$$[\mathbf{a},\mathbf{b}]\vee[\mathbf{c},\mathbf{d}] = [(\mathbf{a}+\mathbf{d})\vee(\mathbf{c}+\mathbf{b}),\mathbf{b}+\mathbf{d}],$$
$$[\mathbf{a},\mathbf{b}]\wedge[\mathbf{c},\mathbf{d}] = [(\mathbf{a}+\mathbf{d})\wedge(\mathbf{c}+\mathbf{b}),\mathbf{b}+\mathbf{d}],$$
$$-[\mathbf{a},\mathbf{b}] = [\mathbf{b},\mathbf{a}],$$
$$0 = [(0),(0)].$$

In this case $\langle G_A,+,\vee,\wedge,-,0\rangle$ is an l-group called *Chang's l-group* of the MV-algebra A and the order in G_A is given by $[\mathbf{a},\mathbf{b}]\leq_G[\mathbf{c},\mathbf{d}]$ iff $\mathbf{a}+\mathbf{d}\leq\mathbf{b}+\mathbf{c}$.

2. Quantum computational structures

In this section we introduce quasi MV-algebras ([8]). From an intuitive point of view, a quasi MV-algebra can be seen as an MV-algebra which fails to satisfy the equation $x\oplus 0 = x$.

Definition 2.1. A *quasi MV-algebra* ([8]) is an algebra $\langle A,\oplus,\neg,0,1\rangle$ of type $\langle 2,1,0,0\rangle$ satisfying the following equations:

Q1: $x\oplus(y\oplus z) = (x\oplus z)\oplus y$,

Q2: $\neg\neg x = x$,

Q3: $x\oplus 1 = 1$,

Q4: $\neg(\neg x\oplus y)\oplus y = \neg(\neg y\oplus x)\oplus x$,

Q5: $\neg(x\oplus 0) = \neg x\oplus 0$,

Q6: $(x\oplus y)\oplus 0 = x\oplus y$,

Q7: $\neg 0 = 1$.

Axioms Q5 and Q6 are introduced for the sake of mathematical smoothness, due to the general failure of $x\oplus 0 = x$. We denote by $q\mathscr{MV}$ the variety of qMV-algebras. We define $\odot,\vee,\wedge$ as we did for MV-algebras. The following lemma can be easily proved:

Lemma 2.2. *Let A be a quasi MV-algebra. Then we have:*

(1) $x\vee y = x\vee(y\oplus 0) = (x\vee y)\oplus 0$,

(2) $x\wedge y = x\wedge(y\oplus 0) = (x\wedge y)\oplus 0$,

(3) $x\odot y = x\odot(y\oplus 0) = (x\odot y)\oplus 0$.

Let A be a quasi MV-algebra. Then we define a binary relation $\leq$ on A as follows:

$$a \leq b \iff 1 = \neg a \oplus b.$$

It is clear that $\langle A, \leq \rangle$ is a preorder. One can also easily prove that $a \leq b$ iff $a \wedge b = a \oplus 0$ iff $a \vee b = b \oplus 0$. Moreover $a \leq a \oplus 0$ and $a \oplus 0 \leq a$. If we define $A \oplus 0 = \{x \oplus 0 : x \in A\}$ it is not very hard to see that $\langle A \oplus 0, \oplus, \neg, 0, 1 \rangle$ is an MV-algebra. An element $a \in A$ is *regular* iff $a \oplus 0 = 0$. Clearly, $A \oplus 0$ is the set of regular elements.

Lemma 2.3. *Let A be quasi MV-algebra and $a \in A$. Then, $a \in A \oplus 0$ iff $\neg a \in A \oplus 0$.*

Proof. It follows from the fact that, $a = a \oplus 0$ iff $\neg a = \neg(a \oplus 0) = \neg a \oplus 0$. □

Example 2.4. Given the standard Łukasiewicz algebra $Ł_{[0,1]}$, the standard qMV-algebra is built from $[0, 1] \times [0, 1]$ with the following operations:

$$(a, b) \oplus (c, d) = (a \oplus c, 1/2),$$
$$0 = (0, 1/2),$$
$$1 = (1, 1/2),$$
$$\neg(a, b) = (\neg a, \neg b).$$

The standard qMV-algebra is particularly important since an equation holds in the whole variety $q\mathscr{MV}$ if ad only if it holds in the standard qMV-algebra, see [8].

Definition 2.5. A *quasi l-group* (shortly, ql-group) is an algebra $\langle G, +, \vee, \wedge, -, 0 \rangle$ of type $\langle 2, 2, 2, 1, 0 \rangle$ such that, upon defining $G + 0 = \{x + 0 : x \in G\}$, the following conditions are satisfied:

QL1: $\langle G + 0, +, \vee, \wedge, -, 0 \rangle$ is an l-group,

QL2: $x + (-x) = 0$,

QL3: $-(-x) = x$,

QL4: $-(x + 0) = -x + 0$,

QL5: $x + y = (x + 0) + (y + 0)$,

QL6: $x \vee y = (x + 0) \vee (y + 0)$,

QL7: $x + (y \vee z) = (x + y) \vee (x + z)$.

We denote by $q\mathscr{LG}$ the variety of ql-groups. For sake of notational clarity in what follows we will write $x - y$ instead of $x + (-y)$. We inductively define nx as follows $1x = x$ and $(n + 1)x = nx + x$. It can be easily seen that a ql-group is an l-group iff it satisfies the equation $x + 0 = x$.

Example 2.6. Let $\mathbb{R}$ be the set of real number. We consider the ql-group $S_{\mathbb{R}}$ given by $(\mathbb{R} \times \{1/2\}) \cup ([-1,1] \times [0,1])$ equipped with the following operations:

(1) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 1/2)$,
(2) $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, 1/2)$,
(3) $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, 1/2)$,
(4) $-(x, y) = (-x, 1 - y)$,
(5) $0 = (0, 1/2)$.

We note that $S_{\mathbb{R}} + 0 = \mathbb{R} \times \{1/2\}$. As we shall see in what follows the ql-group $S_{\mathbb{R}}$ is the natural dual of the standard qMV-algebra introduced in Example 2.4.

Proposition 2.7. *Let G be a ql-group and let $a, b, c \in G$ then we have:*

(1) $-(x \vee y) = -x \wedge -y$,
(2) $-(x \wedge y) = -x \vee -y$,
(3) $x \wedge y = (x + 0) \wedge (y + 0)$,
(4) $\langle G, + \rangle$, $\langle G, \vee \rangle$ *and* $\langle G, \wedge \rangle$ *are abelian semigroups,*
(5) $(x + y) + 0 = x + (y + 0) = x + y$,
(6) $-(x + y) = -x - y$,
(7) $x \vee x = x \wedge x = x + 0$,
(8) $x \vee (x \wedge y) = x \wedge (x \vee y) = x + 0$,
(9) $x + (y \wedge z) = (x + y) \wedge (x + z)$,
(10) $x \vee y = x \vee (y + 0) = (x \vee y) + 0$,
(11) $x \wedge y = x \wedge (y + 0) = (x \wedge y) + 0$.

Proof.

(1) By axioms QL3 and QL4, $-(x \vee y) = -((x + 0) \vee (y + 0)) = -(x + 0) \wedge -(y + 0) = (-x + 0) \wedge (-y + 0) = -x \wedge -y$ since $G + 0$ is an l-group. The same argument allows to prove (2).

(3) Using item (2) we have that $x \wedge y = -(-x \vee -y) = -((-x+0) \vee (-y+0)) = -(-(x + 0) \vee -(y + 0)) = (x + 0) \wedge (y + 0)$.

(4) We prove that $\langle G, + \rangle$ is an abelian semigroup. Since $x+0, y+0 \in G+0$ then using axiom QL4 we have that $x+y = (x+0)+(y+0) = (y+0)+(x+0) = y+x$ and $(x+y)+z = ((x+0)+(y+0))+(z+0) = (x+0)+((y+0)+(z+0)) = x+(y+z)$. In similar manner we can prove that $\langle G, \vee \rangle$ and $\langle G, \wedge \rangle$ are abelian monoids.

(5) By axiom QL5, $x + y \in G + 0$. Therefore $x + y = (x + y) + 0 = (x + 0) + (y + 0) = (x + 0) + ((y + 0) + 0) = x + (y + 0)$.

(6) $-(x+y) = -((x+0)+(y+0)) = -(x+0)-(y+0) = (-x+0)+(-y+0) = -x-y$.

The remaining items are left to the reader. □

Definition 2.8. Let G be a ql-group. Then we define the binary relation $\leq$ on G as

$$a \leq b \iff a+0 = a \wedge b.$$

Proposition 2.9. *Let G be a ql-group $a, b, c \in G$. Then we have:*

(1) *$\langle G, \leq \rangle$ is a preorder,*
(2) *$a \leq b$ iff $b+0 = a \vee b$,*
(3) *$a \leq a+0$ and $a+0 \leq a$,*
(4) *if $a \leq b$ then, for any $c \in G$, $a+c \leq b+c$, $a \wedge c \leq b \wedge c$, $a \vee c \leq b \vee c$,*
(5) *$a \wedge b \leq a \leq a \vee b$,*
(6) *if $a \leq b$ then $-b \leq -a$,*
(7) *if $0 \leq a \leq b$ then $0 \leq b-a \leq b$, $a+0 = a \wedge (a+b)$.*

Proof.

(1) It is clear that $a \leq a$. Now we prove that if $a \leq b$ and $b \leq c$ then $a \leq c$. In fact if $a \leq b$ then $a+0 = a \wedge b$ and $b \leq c$ then $b+0 = b \wedge c$. By Proposition 2.7(11), $a+0 = a \wedge b = a \wedge (b+0) = a \wedge (b \wedge c) = (a \wedge b) \wedge c = (a+0) \wedge c = a \wedge c$. Thus $a \leq c$.

(2) If $a \wedge b$ then $a+0 = a \wedge b$ and $b \vee (a+0) = (a \wedge b)$. By Proposition 2.7(8) $b \vee a = b+0$. The other direction is analogous.

(3) and (4) are immediate.

(5) Suppose that $a \leq b$, then $a+0 = a \wedge b$. $(a+c)+0 = (a+0)+c = (a \wedge b)+c = (a+c) \wedge (b+c)$ resulting $a+c \leq b+c$. Moreover $(a \wedge c)+0 = (a+0) \wedge c = (a \wedge b) \wedge c = (a \wedge (b+0)) \wedge (c \wedge (b+0)) = ((a \wedge b)+0) \wedge ((a \wedge c)+0) = (a \wedge b) \wedge ((a \wedge c)$. In the case of $\vee$ we use item (2).

(6) $(a \wedge b) \wedge a = (a \wedge a) \wedge b = (a+0) \wedge b = (a \wedge b)+0$. In the case of $\vee$ we use item (2).

(7) If $a \leq b$ then $a+0 = a \wedge b$, whence $-b+0 = (a \wedge b)-a-b = ((a-a) \wedge (b-a))-b = (0-b) \wedge (0-a) = -b \wedge -a$ and $-b \leq -a$.

(8) Using item (5), if $a \leq b$ then $0 \leq b-a$. Using item (7), since $-a \leq -0 = 0$. Therefore $b-a \leq b+0 \leq b$. In the second case, using item (5), we see that $a \leq a+b$. □

Definition 2.10. Let G be a ql-group. A function $u\colon G \to G$ is a *quasi unit* (q-unit for short) iff it satisfies:

(1) $0 \leq u(0)$,

(2) $u(x + 0) = u(0) - x$,

(3) if $0 \leq x \leq u(0)$ then $u(0) - u(x) = 0 + x$,

(4) $uu(x) = x$.

It is not hard to verify that every ql-group admits a q-unit. For instance, it suffices to consider

$$u(x) = \begin{cases} -x, & \text{if } x \neq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, it can also be seen that if G is an l-group, then, for any $a \geq 0$, the function $u_a(x) = a - x$ is a quasi unit in G. It can be noticed that if $a \in G$ is a strong unit for an l-group G, then the function $u_a(x)$ is the unique function which allows us to define the negation of the MV-algebra associated to the interval $[0, a]$ of G.

Proposition 2.11. *Let G be a ql-group and u be a q-unit. Then we have:*

1. $u(0) \in G + 0$,
2. *if v is a q-unit such that $v(0) = u(0)$, then for each $x \in G$, $u(x + 0) = v(x + 0)$,*
3. *if $x \leq y$ then $u(y + 0) \leq u(x + 0)$.*

Moreover if $0 \leq x, y \leq u(0)$ then:

4. *if $x \leq y$ then $u(y) \leq u(x)$,*
5. $0 \leq u(x) \leq u(0)$,
6. $u(0) - x = u(x) + 0$.

Proof.

1) $u(0) = u(0 + 0) = u(0) - 0 = u(0) + 0$, resulting $u(0) \in G + 0$.

2) $v(x + 0) = v(0) - x = u(0) - x = u(x + 0)$.

3) Suppose that $x \leq y$. Using Proposition 2.9-6 and 4, $-y \leq -x$. Thus $u(y + 0) = u(0) + (-y) \leq u(0) + (-x) = u(x + 0)$.

Suppose that $0 \leq x, y \leq u(0)$.

4) If $x \leq y$ then $x + 0 \leq y + 0$. By definition of q-unit, $u(0) - (u(x) + 0) = u(0) - u(x) = x + 0 \leq y + 0 = u(0) - u(y) = u(0) - (u(y) + 0)$ and this is an inequality in the l-group $G + 0$, whence $-(u(x) + 0) \leq -(u(y) + 0)$ and then $u(y) + 0 \leq u(x) + 0$, that is $u(y) \leq u(x)$.

5) From item 4, if $0 \leq x$ then $u(x) \leq u(0)$ and $x \leq u(0)$, thus $0 = uu(0) \leq u(x)$.

6) $u(x) + 0 = u(0) - uu(x) = u(0) - x$. □

For simplicity we use u_0 as abbreviation of $u(0)$.

Definition 2.12. Let $(G, u) \in q\mathscr{L}\mathscr{G}_u$. Let $[0, u_0] = \{x \in G : \ 0 \leq x \leq u_0\}$, for each $x, y \in [0, u_0]$ we define:

(1) $x \oplus y = u_0 \wedge (x + y)$,

(2) $\neg x = u(x)$.

The structure $\langle [0, u_0], \oplus, \neg, 0, u_0 \rangle$ will be denoted by $\Gamma_q(G, u)$. By basic l-group properties and Proposition 2.11-5, it is clear that $[0, u_0]$ is closed w.r.t. the operations defined above.

Proposition 2.13. *$\Gamma_q(G, u)$ is a quasi MV-algebra.*

Proof. Let $0 \leq x, y \leq u(0) = u_0$.

Q1: $(x \oplus y) \oplus z = u_0 \wedge (z + (u_0 \wedge (x + y))) = (u_0 \wedge (z + u_0)) \wedge (z + y + x) = u_0 \wedge (z + y + x)$. By the same argument we prove that $x \oplus (y \oplus z) = u_0 \wedge (z + y + x)$.

Q2 and Q3: Straightforward.

Q4: $\neg(\neg x \oplus y) \oplus y = u(u(x) \oplus y) \oplus y = u_0 \wedge (y + u(u(x) \oplus y)) = u_0 \wedge (y + u(u_0 \wedge (u(x) + y))) = u_0 \wedge (y + u((u_0 \wedge (u(x) + y)) + 0)) = u_0 \wedge (y + u_0 - (u_0 \wedge (u(x) + y))) = u_0 \wedge (y + u_0 + (-u_0 \vee (-u(x) - y))) = u_0 \wedge ((y + u_0 - u_0) \vee (y + u_0 - u(x) - y)) = u_0 \wedge ((y + 0) \vee (u_0 - u(x) + 0)) = u_0 \wedge ((y + 0) \vee (x + 0)) = u_0 \wedge (y \vee x)$. This shows that x, y are interchangeable, thus this qMV-axiom is verified.

Q5: Since u_0 and $x + 0$ lies in $G + 0$ being $x + 0 \leq u_0$ then $\neg(x \oplus 0) = u(u_0 \wedge (x + 0)) = u(x + 0) = u_0 - x$. On the other hand, since $u(x) \leq u_0$, $\neg x \oplus 0 = u_0 \wedge (u(x) + 0) = u(x) + 0$. Using Proposition 2.11-6 we have that $\neg(x \oplus 0) = \neg x \oplus 0$.

Q6: $(x \oplus y) \oplus 0 = u_0 \wedge ((u_0 \wedge (x + y)) + 0) = u_0 \wedge ((u_0 + 0) \wedge (x + y + 0)) = u_0 \wedge (u_0 \wedge (x + y + 0) = u_0 \wedge (x + y) = x \oplus y$ since $u_0 \in G + 0$.

Q7: By definition. □

Let us stress the fact that the image of the operation $\neg$ is not, in general, contained in the set of regular element of a qMV-algebra. Therefore, it has been necessary to define the notion of the function $\neg$ in $\Gamma_q(G, u)$ independently from any binary function of the ql-group G. This observation justify the choice of a function, instead of an element, as a q-unit in a ql-group.

Proposition 2.14. *Let (G, u) be a ql-group with q-unit, then we have:*

$$\Gamma_q(G, u) \oplus 0 = \Gamma_q(G + 0, u(0))$$

Proof. Let $x \in \Gamma_q(G,u) \oplus 0$ then $0 \leq x = x \oplus 0 = x + 0 \leq u(0)$ whence $x \in \Gamma_q(G+0, u(0))$. On the other hand, if $x \in \Gamma_q(G+0, u(0))$ then $0 \leq x = x + 0 \leq u(0)$; thus $x = u(0) \wedge (x+0) = x \oplus 0$ and then $x \in \Gamma_q(G,u) \oplus 0$. □

Example 2.15. If we consider the ql-group $S_{\mathbb{R}}$ of Example 3.3, and for each $x = (x_1, x_2) \in S_{\mathbb{R}}$ we define

$$u(x_1,x_2) = \begin{cases} (1-x_1, 1-x_2), & \text{if } (x_1,x_2) \in S_R + 0 \text{ or } 0 \leq x_1 \leq 1 \\ (x_1,x_2), & \text{otherwise} \end{cases}$$

then u is a q-unit in $S_{\mathbb{R}}$. In fact:

(1) $0 = (0,1/2) \leq (1,1/2) = (1-0, 1-1/2) = u(0,1/2) = u(0)$.

(2) $u(x+0) = u((x_1,x_2)+(0,1/2)) = u(x_1,1/2) = (1-x_1,1/2) = (1,1/2) - (x_1,x_2) = u(0,1/2) - (x_1,x_2) = u(0) - x$.

(3) $(0,1/2) \leq x = (x_1,x_2) \leq (1,1/2)$ then, $u(0) - u(x) = u(0,1/2) - u(x_1,x_2) = (1,1/2) - (1-x_1, 1-x_2) = (x_1,1/2) = (x_1,x_2) + (0,1/2) = x + 0$.

(4) $uu(x) = u(u(x_1,x_2)) = u(1-x_1,1-x_2) = (x_1,x_2) = x$, if $x = (x_1,x_2) \in S_{\mathbb{R}} + 0$ or $0 \leq x_1 \leq 1$. The other case is direct.

Moreover it is clear that $[0,u_0] = [0,1] \times [0,1]$, whereby $\Gamma_q(S_{\mathbb{R}}, u)$ is the standard quasi MV-algebra of Example 2.4.

3. The functor Γ_q

Let G be a ql-group. For each $a \in G$ the *absolute value* of a is defined as $|a| = a \vee -a$.

Lemma 3.1. *If G is a ql-group and $a \in G$ then $|a| = |a+0|$.*

Proof. $|a+0| = (a+0) \vee (-a+0) = a \vee -a = |a|$. □

Definition 3.2. Let G be a ql-group. A q-unit u on G is said to be *strong* iff for each $x \in G$ there is an integer $n \geq 0$ such that $|x| \leq nu_0$.

Example 3.3. Let us consider the ql-group $S_{\mathbb{R}}$ with the q-unit u given in Example 2.15. Let $x = (x_1,x_2)$ and suppose that $x_1 > 0$. Since 1 is a strong unit in the real l-group $\mathbb{R}$, there exists a natural number n such that $x_1 \leq n = n1$. Thus $|x| = (x_1,1/2) \leq (n,1/2) = n(1-0,1/2) = nu(0,1/2) = nu(0)$. Thus u is a strong q-unit in $S_{\mathbb{R}}$.

From Proposition 2.11-2, the following lemma establishes the relation between l-groups with strong unit and ql-group with strong q-unit.

Lemma 3.4. *Let G be an l-group. If u is a strong unit in G, then $u(x) = u_0 - x$ is the unique quasi strong unit such that $u = u_0$. On the other hand, if $u(x)$ is a quasi strong unit, then u_0 is the unique strong unit on G such that $u(x) = u_0 - x$.*

Definition 3.5. A ql-group with strong q-unit (G, u) is called *bounded* iff for each $x \notin G + 0$, $-u(0) \leq x \leq 0$ or $0 \leq x \leq u(0)$.

We denote by $q\mathscr{L}\mathscr{G}_u$ the category whose objects are bounded ql-groups (G, u), and whose arrows are $f\colon (G_1, u_1) \to (G_2, u_2)$ such that f is a homomorphism of ql-group satisfying $f(u_1(0)) = u_2(0)$. These homomorphisms are referred to as *unital homomorphisms.*

Theorem 3.6. *$q\mathscr{L}\mathscr{G}_u$ is a reflective subcategory* [1] *of $\mathscr{L}\mathscr{G}_u$.*

Proof. If $A \in q\mathscr{L}\mathscr{G}_u$, we define $\mathscr{S}(A, u) = (A + 0, u(0))$, and for each unital qc-homomorphism $f\colon A \to A'$, we let $\mathscr{S}(f)\colon A + 0 \to A' + 0$ be defined by $\mathscr{S}(f)(x+0) = f(x)+0$. Upon noticing that u_0 is a strong unit in $A+0$ it is clear that $\mathscr{S}(f)$ is an $\mathscr{L}\mathscr{G}_u$-homomorphism. It is easy to check that $\mathscr{S}\colon q\mathscr{L}\mathscr{G}_u \to \mathscr{L}\mathscr{G}_u$ is a functor. In view of Lemma 3.4, first consider the unital qc-homorphism $p_A\colon (A, u) \to (A + 0, u_0 - x)$ such that $p_A(x) = x + 0$. A routinary check will assure that the following diagram is commutative:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\equiv"] & A' \arrow[d, "p_{A'}"] \\
A + 0 \arrow[r, "\mathscr{S}(f)"'] & A' + 0
\end{tikzcd}$$

Suppose that $B \in \mathscr{L}\mathscr{G}_u$ and $f\colon A \to B$ be a unital qc-homomorphism. The mapping $g = f \restriction A + 0$ defines[2] a $\mathscr{L}\mathscr{G}_u$-homomorphism $g : A + 0 \to B$ that makes the following diagram commutative:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p_A"'] \arrow[r, phantom, shift right=4ex, "\equiv"] & B \\
A + 0 \arrow[ur, "g"'] &
\end{tikzcd}$$

and it is obvious that g is the only $\mathscr{L}\mathscr{G}_u$-homomorphism making the triangle commutative. Therefore we have proved that $\mathscr{S}$ is a reflector. $\square$

[2]We will denote by $f \restriction A$ the restriction of a function f to the set A.

Proposition 3.7. *Let (G, u) be a ql-group with strong q-unit, and consider the following sets:*

$$M_1 = \{x \notin G + 0 :\ 0 \leq x \leq u(0)\},$$
$$M_2 = \{x \notin G + 0 : -u(0) \leq x \leq 0\}.$$

Then we have:

(1) $B(G) = \langle (G+0) \cup M_1 \cup M_2, +, \vee, \wedge, -, 0 \rangle$ *is a bounded sub ql-group of* G*, being* $u \upharpoonright (G + 0) \cup M_1 \cup M_2$ *the strong q-unit,*

(2) $\Gamma_q(G, u) = \Gamma_q(B(G), u)$.

Proof.

1) We first prove that $(G+0) \cup M_1 \cup M_2$ is closed w.r.t. u. If $x \in G+0$ then $u(x) = u(x+0) = u(0) - x \in G + 0$. In view of Proposition 2.11-5, $M_1 \cup M_2$ is closed w.r.t. $u(x)$. $B(G)$ is a sub ql-group of G since $M_1 \cup M_2$ is closed w.r.t. $-$, and binary operations in G have the form $G \times G \to G + 0$.

2) It follows from the previous item. □

The following proposition can be easily proved.

Proposition 3.8. *If we define* $\Gamma_q \colon q\mathscr{LG}_u \to q\mathscr{MV}$ *such that* $(G, u) \to \Gamma_q(G, u)$ *for each* $(G, u) \in q\mathscr{LG}_u$*, and for each unital homomorphism* $f \colon (G_1, u_1) \to (G_2, u_2)$*,* $\Gamma_q(f) = f \upharpoonright [0, u_0]$*, then* Γ_q *is a functor between the category of bounded ql-groups and the category of quasi MV-algebras.*

We are now ready to introduce the notion of quasi good sequence, that will play an analogous role as the one played by the notion of good sequence with respect to MV-algebras.

Definition 3.9. Let A be a quasi-MV algebra. A sequence $\mathbf{a} = (a_1, a_2, \dots)$ of elements on A is said to be *quasi good* iff

(1) $a_1 \oplus a_2 = a_1 \oplus 0$,

(2) $(a_2, a_3, \dots)$ is a good sequence in $A \oplus 0$.

We denote by M_A^q the set of quasi good sequences in A.

Lemma 3.10. *Let A be quasi MV-algebra. Then the following assertions are equivalent:*

(1) $(a_1, \dots, a_n, \dots)$ *is a quasi good sequence in* A*,*

(2) $(a_1 \oplus 0, \dots, a_n, \dots)$ *is good sequence in* $A \oplus 0$.

Proof. Suppose that $(a_1, \dots, a_n, \dots)$ is a quasi good sequence in A. Then $(a_1 \oplus 0) \oplus a_2 = a_1 \oplus a_2 = a_1 \oplus 0$. On the other hand if $(a_1 \oplus 0, \dots, a_n, \dots)$ is good sequence in $A \oplus 0$, then $a_1 \oplus 0 = (a_1 \oplus 0) \oplus a_2 = a_1 \oplus a_2$, so that $(a_1, \dots, a_n, \dots)$ a good sequence. □

In view of the above Lemma, if $\mathbf{a} = (a_1, \dots, a_n, \dots)$ is a quasi good sequence, we denote by $\mathbf{a} \oplus 0 = (a_1, \dots, a_n \dots) \oplus 0$ the good sequence $(a_1 \oplus 0, \dots, a_n, \dots)$ in the MV-algebra $A \oplus 0$. It is clear that $M_{A\oplus 0} = \{\mathbf{a} \oplus 0 : \ \mathbf{a} \in M_A^q\} \subseteq M_A^q$.

Proposition 3.11. *Let A be a quasi MV-algebra and consider the structure $\langle M_A^q, +, \vee, \wedge\rangle$ defined as* M1, M2, M3 *in the monoid of good sequences. Then for each $\mathbf{a}, \mathbf{b} \in M_A^q$ we have that:*

(1) $\mathbf{a} + \mathbf{b} = (\mathbf{a} + \mathbf{b}) \oplus 0 = \mathbf{a} + (\mathbf{b} \oplus 0)$,

(2) $\mathbf{a} \vee \mathbf{b} = (\mathbf{a} \vee \mathbf{b}) \oplus 0 = \mathbf{a} \vee (\mathbf{b} \oplus 0)$,

(3) $\mathbf{a} \wedge \mathbf{b} = (\mathbf{a} \wedge \mathbf{b}) \oplus 0 = \mathbf{a} \wedge (\mathbf{b} \oplus 0)$.

Thus these operations have the form $M_A^q \times M_A^q \to M_{A\oplus 0}$.

Proof. It is a consequence of Lemma 2.2. □

Lemma 3.12. *Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in M_A^q$ then we have:*

(1) *if $\mathbf{a} + \mathbf{b} = \mathbf{a} + \mathbf{c}$ then $\mathbf{b} \oplus 0 = \mathbf{c} \oplus 0$,*

(2) *if $\mathbf{a} + \mathbf{b} = (0)$ then $\mathbf{a} \oplus 0 = \mathbf{b} \oplus 0 = (0)$.*

Proof. Follows from Proposition 3.11 and [5, Proposition 2.3.1]. □

Definition 3.13. Let A be a quasi MV-algebra and $\mathbf{a} = (a_1, \dots, a_n, \dots), \mathbf{b} = (b_1, \dots, b_n, \dots) \in M_A^q$. Let us define the binary relation $\leq$ on M_A^q as:

$$\mathbf{b} \leq \mathbf{a} \iff b_i \leq a_i \qquad \text{for each} \quad i = 1, \dots, n.$$

Proposition 3.14. *Let A be a quasi MV-algebra then we have:*

(1) *$\langle M_A^q, \leq\rangle$ is a preorder,*

(2) *For each $\mathbf{a}$, $\mathbf{b}$ in M_A^q, $\mathbf{b} \leq \mathbf{a}$ iff there exists a good sequence $\mathbf{c} \in A \oplus 0$ such that $\mathbf{b} + \mathbf{c} = \mathbf{a} \oplus 0$. The element $\mathbf{c}$ is unique and $\mathbf{c}$ is noted by $\mathbf{a} - \mathbf{b}$.*

Proof.

1) Follows from the fact that A is preordered.

2) Suppose that $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_m)$ and assume that there exists a good sequence $\mathbf{c}$ such that $\mathbf{b} + \mathbf{c} = \mathbf{a} \oplus 0$. Then $(\mathbf{b} \oplus 0) + (\mathbf{c} \oplus 0) = \mathbf{a} \oplus 0$. Since $\mathbf{b} \oplus 0$, $\mathbf{c} \oplus 0$ and $\mathbf{a} \oplus 0$ are good sequences in $A \oplus 0$, by [5, Proposition 2.3.2] we have that $b_i \oplus 0 \leq a_i \oplus 0$ whence $b_i \leq a_i$, for each $i = 1, \dots, n$. The converse uses the same argument. The unicity follows from [5, Proposition 2.3.4 i]. □

We consider the following sets:

$$M_{A_1} = \{((x), \mathbf{0}) : \ x \notin A \oplus 0\}$$
$$M_{A_2} = \{(\mathbf{0}, (y)) : \ y \notin A \oplus 0\}$$
$$M(A) = (M_{A\oplus 0})^2 \cup M_{A_1} \cup M_{A_2}$$

It is clear that $M(A) \subseteq M_A^q \times M_A^q$ and by Lemma 2.3 if $((a), \mathbf{0}) \in M_{A_1}$ then $((\neg a), \mathbf{0}) \in M_{A_1}$ and the same obtains for M_{A_2}. Also we consider the binary relation $\equiv$ in $M(A)$ defined as follows:

$$(\mathbf{a}, \mathbf{b}) \equiv (\mathbf{a}', \mathbf{b}') \iff \begin{cases} \mathbf{a} + \mathbf{b}' = \mathbf{a}' + \mathbf{b}, & \text{if } \mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}' \in M_{A\oplus 0} \\ (\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}'), & \text{otherwise.} \end{cases}$$

It is clear that $\equiv$ is reflexive and symmetric. Transitivity follows either from the fact that $(M_{A\oplus 0})^2$ is the monoid of good sequences, or by the transitivity of the equality. Thus $\equiv$ is transitive, and therefore it is an equivalence relation.

We denote by $[\mathbf{a}, \mathbf{b}]$ the equivalence class determined by the pair $(\mathbf{a}, \mathbf{b})$ and by G_A^q the set of equivalence classes.

Proposition 3.15. *Let us define the following operations on G_A^q:*

$$[\mathbf{a}, \mathbf{b}] + [\mathbf{c}, \mathbf{d}] = [\mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{d}],$$
$$[\mathbf{a}, \mathbf{b}] \vee [\mathbf{c}, \mathbf{d}] = [(\mathbf{a} + \mathbf{d}) \vee (\mathbf{c} + \mathbf{b}), \mathbf{b} + \mathbf{d}],$$
$$[\mathbf{a}, \mathbf{b}] \wedge [\mathbf{c}, \mathbf{d}] = [(\mathbf{a} + \mathbf{d}) \wedge (\mathbf{c} + \mathbf{b}), \mathbf{b} + \mathbf{d}],$$
$$-[\mathbf{a}, \mathbf{b}] = [\mathbf{b}, \mathbf{a}],$$
$$0 = [\mathbf{0}, \mathbf{0}].$$

Then we have:

(1) $[\mathbf{a}, \mathbf{b}] + [\mathbf{c}, \mathbf{d}] = [\mathbf{a} \oplus 0, \mathbf{b}] + [\mathbf{c}, \mathbf{d}] = [\mathbf{a}, \mathbf{b} \oplus 0] + [\mathbf{c}, \mathbf{d}]$.

(2) $[\mathbf{a}, \mathbf{b}] \vee [\mathbf{c}, \mathbf{d}] = [\mathbf{a} \oplus 0, \mathbf{b}] \vee [\mathbf{c}, \mathbf{d}] = [\mathbf{a}, \mathbf{b} \oplus 0] \vee [\mathbf{c}, \mathbf{d}]$.

(3) $[\mathbf{a}, \mathbf{b}] \wedge [\mathbf{c}, \mathbf{d}] = [\mathbf{a} \oplus 0, \mathbf{b}] \wedge [\mathbf{c}, \mathbf{d}] = [\mathbf{a}, \mathbf{b} \oplus 0] \wedge [\mathbf{c}, \mathbf{d}]$.

(4) *The binary operations have the form $G_A^q \times G_A^q \to G_{A\oplus 0}$.*

(5) *$\langle G_A^q, +, \vee, \wedge, -, 0\rangle$ is a ql-group.*

P r o o f. We first note that operations are well defined.

(1), (2), (3) Follow from Propositions 3.11.

(4) Follows from the previous items.

(5) Now we see that $\langle G_A^q, +, \vee, \wedge, -, 0\rangle$ is a ql-group:

QL1: It is clear that $\langle G_Q(A) + 0, +, \vee, \wedge, -, 0\rangle$ is an l-group, more precisely the Chang's l-group of the good sequences of $A \oplus 0$.

QL2, QL3, QL4, QL5, QL6: Straightforward calculation.

QL7: Using items (2), (3) and the fact that G_{A+0} is an l-group we have that, $[\mathbf{a}, \mathbf{b}] + ([\mathbf{c}, \mathbf{d}] \vee [\mathbf{e}, \mathbf{f}]) = [\mathbf{a} \oplus 0, \mathbf{b} \oplus 0] + ([\mathbf{c} \oplus 0, \mathbf{d} \oplus 0] \vee [\mathbf{e} \oplus 0, \mathbf{f} \oplus 0]) = ([\mathbf{a} \oplus 0, \mathbf{b} \oplus 0] + [\mathbf{c} \oplus 0, \mathbf{d} \oplus 0]) \vee ([\mathbf{a} \oplus 0, \mathbf{b} \oplus 0] + [\mathbf{e} \oplus 0, \mathbf{f} \oplus 0]) = ([\mathbf{a}, \mathbf{b}] + [\mathbf{c}, \mathbf{d}]) \vee ([\mathbf{a}, \mathbf{b}] + [\mathbf{e}, \mathbf{f}])$.

Thus $\langle G_A^q, +, \vee, \wedge, -, 0\rangle$ is ql-group. □

We will refer to $\langle G_A^q, +, \vee, \wedge, -, 0\rangle$ as *Chang's ql-group*.

Remark 3.16. We can see G^q_A as the Chang's l-group $G_{A\oplus 0}$ with new *non regular elements* given by the equivalence classes $[(a), \mathbf{0}] = \{((a), \mathbf{0})\}$ and $[\mathbf{0}, (b)] = \{(\mathbf{0}, (b))\}$ with $a, b \notin A \oplus 0$. Thus G^q_A has a structure

$$G^q_A = G_{A\oplus 0} \cup M_{A_1} \cup M_{A_2}.$$

Proposition 3.17. *Let A be a quasi MV-algebra and $[\mathbf{a}, \mathbf{b}] \in G^q_A$. Then the following assertions are equivalent:*

1. $0 \leq [\mathbf{a}, \mathbf{b}]$.
2. $\mathbf{b} + (0) \leq \mathbf{a} + (0)$.
3. *There exists a unique good sequence $\mathbf{e} = (e_1, \dots, e_n, 0, 0, \dots)$ in $A \oplus 0$ such that $[\mathbf{a}, \mathbf{b}] + [(0), (0)] = [\mathbf{e}, (0)]$.*

Moreover in the case $[\mathbf{a}, \mathbf{b}] \leq [(1), (0)]$ there exists unique $e_1 \in A \oplus 0$ such that:

4. *If $\mathbf{a}, \mathbf{b} \in G_{A+0}$ then $[\mathbf{a}, \mathbf{b}] = [(e_1), 0]$.*
5. *Otherwise, $[\mathbf{a}, \mathbf{b}] = [(a), \mathbf{0}]$ and $a \oplus 0 = e_1$.*

Proof.

1) $\iff$ 2) Since G^q_A is a ql-group $0 = [(0), (0)] \leq [\mathbf{a}, \mathbf{b}]$ iff $[(0), (0)] = [(0), (0)] \wedge [\mathbf{a}, \mathbf{b}] = [(\mathbf{b} + (0)) \wedge (\mathbf{a} + (0)), \mathbf{b} + (0)]$ iff $\mathbf{b} + (0) + (0) = ((\mathbf{b} + (0)) \wedge (\mathbf{a} + (0))) + (0)$ iff $\mathbf{b} + (0) = (\mathbf{b} + (0)) \wedge (\mathbf{a} + (0))$ iff $\mathbf{b} + (0) \leq \mathbf{a} + (0)$ since $\mathbf{b} + (0), \mathbf{a} + (0) \in M_{A\oplus 0}$ and $M_{A\oplus 0}$ is a lattice monoid.

2) $\Longrightarrow$ 3) Since $\mathbf{b} + (0), \mathbf{a} + (0) \in M_{A\oplus 0}$ and $\mathbf{b} + (0) \leq \mathbf{a} + (0)$, by [5, Proposition 2.3.4] there exists a unique good sequence $\mathbf{e} = (e_1, \dots, e_n, 0, 0, \dots)$ in $A \oplus 0$ such that $(\mathbf{b} + (0)) + \mathbf{e} = \mathbf{a} + (0)$. By Proposition 3.15, $[\mathbf{a}, \mathbf{b}] + [(0), (0)] = [\mathbf{a} \oplus 0, \mathbf{b} \oplus 0] + [(0), (0)] = [\mathbf{a} \oplus 0, \mathbf{b} \oplus 0]$. Using Proposition 3.11 $(\mathbf{b} \oplus 0) + \mathbf{e} = ((\mathbf{b} \oplus 0) + (0)) + \mathbf{e} = (\mathbf{b} + (0)) + \mathbf{e} = \mathbf{a} + (0) = (\mathbf{a} \oplus 0) + (0)$, resulting $[\mathbf{a}, \mathbf{b}] + [(0), (0)] = [\mathbf{a} \oplus 0, \mathbf{b} \oplus 0] = [\mathbf{e}, (0)]$.

3) $\Longrightarrow$ 2) If $[\mathbf{a}, \mathbf{b}] + [(0), (0)] = [\mathbf{e}, (0)]$, then $[\mathbf{a} \oplus 0, \mathbf{b} \oplus 0] = [\mathbf{a} \oplus 0, \mathbf{b} \oplus 0] + [(0), (0)] = [\mathbf{a}, \mathbf{b}] + [(0), (0)] = [\mathbf{e}, (0)]$, resulting $(\mathbf{b} \oplus 0) + \mathbf{e} = (\mathbf{a} \oplus 0) + (0) = \mathbf{a} \oplus 0$. Using [5, Proposition 2.3.4] we have that $\mathbf{b} \oplus 0 \leq \mathbf{a} \oplus 0$. By Proposition 3.15, $\mathbf{b} + (0) = (\mathbf{b} \oplus 0) + (0) = \mathbf{b} \oplus 0 \leq \mathbf{a} \oplus 0 = (\mathbf{a} \oplus 0) + (0) = \mathbf{a} + (0)$.

Suppose also that $[\mathbf{a}, \mathbf{b}] \leq [(1), (0)]$. Then $[\mathbf{a}, \mathbf{b}] + [(0), (0)] = [\mathbf{e}, (0)] \leq [(1), (0)]$. Since $[\mathbf{e}, (0)] \in G_{A\oplus 0}$ by [5, Proposition 2.4.5], $[\mathbf{e}, (0)] = [(e_1), (0)]$. If $\mathbf{a}, \mathbf{b} \in G_{A+0}$, then from the fact that $[\mathbf{a}, \mathbf{b}] + [\mathbf{0}, \mathbf{0}] = [\mathbf{a}, \mathbf{b}]$, it follows that $[\mathbf{a}, \mathbf{b}] = [(e_1), 0]$.

Otherwise, by definition of $M(A)$, $[\mathbf{a}, \mathbf{b}] = [\mathbf{a}, \mathbf{b}] \wedge [(1), (0)] = [(\mathbf{a} + (0)) \wedge (\mathbf{b}+(1)), \mathbf{b}+(0)] = [\mathbf{a}+(0), \mathbf{b}+(0)] = [\mathbf{a}, \mathbf{b}]+0$. By item 3, $[\mathbf{a}, \mathbf{b}] = [\mathbf{a}, (0)]$. Therefore $[\mathbf{a}, \mathbf{0}] + [\mathbf{0}, \mathbf{0}] = [\mathbf{a} \oplus 0, \mathbf{0}] = [(a_1 \oplus 0), \mathbf{0}] = [(e_1), \mathbf{0}]$, resulting $(e_1, 0, 0, \dots) = (a_1 \oplus 0, 0, \dots)$. Finally $a_1 \oplus 0 = e_1$ and $[\mathbf{a}, \mathbf{b}] = [(a_1, 0, \dots), \mathbf{0}]$. ☐

Proposition 3.18. *Let A be quasi MV-algebra. If we consider the ql-group G_A^q then:*

$$u([\mathbf{x},\mathbf{y}]) = \begin{cases} [(1),(0)] - [\mathbf{x},\mathbf{y}], & \textit{if } \mathbf{x},\mathbf{y} \in M_{A\oplus 0}, \\ [(\neg a),\mathbf{0}], & \textit{if } (\mathbf{x},\mathbf{y}) = ((a),\mathbf{0}) \in M_{A_1}, \\ [\mathbf{0},(\neg b)], & \textit{if } (\mathbf{x},\mathbf{y}) = (\mathbf{0},(b)) \in M_{A_2} \end{cases}$$

is a strong q-unit in G_A^q. Thus (G_A^q, u) is an object in $q\mathscr{L}\mathscr{G}_u$.

Proof.

1) $[(0),(0)] \le [(1),(0)] = u([(0),(0)])$.

2) $u([\mathbf{x},\mathbf{y}] + [(0),(0)]) = u([\mathbf{x}\oplus 0, \mathbf{y}\oplus 0] + [(0),(0)]) = u([\mathbf{x}\oplus 0,\mathbf{y}\oplus 0]) = [(1),(0)] - [\mathbf{x}\oplus 0,\mathbf{y}\oplus 0] = u([(0),(0)]) - [\mathbf{x},\mathbf{y}]$.

3) Suppose that $0 \le [\mathbf{x},\mathbf{y}] \le [(1),(0)]$. Using Proposition 3.17, $[\mathbf{x},\mathbf{y}] = [(a),\mathbf{0}]$. In this case, $u([(0),(0)]) - u([(a),\mathbf{0}]) = [(1),(0)] - [(\neg a),\mathbf{0}] = [(1),(0)] + [\mathbf{0},(\neg a)] = [(1),(0)] + [\mathbf{0},(\neg a \oplus 0)] = [(1),(\neg a\oplus 0)] = [(1),(\neg(a\oplus 0))] = [(1),(1) - (a\oplus 0)]$, in view of [5, Proposition 2.3.4] and the fact that $a\oplus 0 \in A\oplus 0$. On the other hand $[(0),(0)] + [(a),\mathbf{0}] = [(0),(0)] + [(a\oplus 0),\mathbf{0}] = [(a\oplus 0),\mathbf{0}]$. We need to see that $[(1),(1) - (a\oplus 0)] = [(a\oplus 0),\mathbf{0}]$, but this follows from the fact that $(1) = (1) + \mathbf{0}$ and $(1) = ((1) - (a\oplus 0)) + (a\oplus 0)$, resulting $u([(0),(0)]) - u([(a),\mathbf{0}]) = [(0),(0)] + [(a),\mathbf{0}]$.

4) By definition of u, it is clear that $uu([\mathbf{x},\mathbf{y}]) = [\mathbf{x},\mathbf{y}]$.

5) Since $u([(0),(0)]) = [(1),(0)]$ is a strong unit of the Chang's l-group $G_{A\oplus 0}$, it is clear that for each $[\mathbf{x},\mathbf{y}] \in G_A^q$, $|[\mathbf{x},\mathbf{y}]| \le nu([(0),(0)])$ for some integer $n \ge 0$.

Thus u is a strong q-unit in G_A^q. Moreover, by Remark 3.16 and Proposition 3.17, G_A^q is a bounded ql-group. □

Theorem 3.19. *Let A be a quasi MV-algebra and consider the ql-group $\langle G_A^q, u\rangle$. Then $\varphi\colon A \to \Gamma_q(G_A^q, u)$ such that $a \to \varphi(a) = [(a),\mathbf{0}]$, is a quasi MV-isomorphism.*

Proof. By Proposition 3.17, φ is a bijection. It is not very hard to see that the restriction $\varphi\restriction_{A\oplus 0}$ is the well known MV-isomorphism from the MV-algebra $A\oplus 0$ onto the MV-algebra $\Gamma(G_{A\oplus 0}, [(1),(0)])$ (see [5, Proposition 2.4.5]). Let $x, y \in A$. Then $\varphi(x\oplus y) = \varphi((x\oplus 0)\oplus(y\oplus 0)) = \varphi\restriction_{A\oplus 0}((x\oplus 0)\oplus(y\oplus 0)) = \varphi\restriction_{A\oplus 0}(x\oplus 0) \oplus \varphi\restriction_{A\oplus 0}(y\oplus 0) = [(x\oplus 0),\mathbf{0}] \oplus [(y\oplus 0),\mathbf{0}] = [(1),(0)] \wedge ([(x\oplus 0),\mathbf{0}] + [(y\oplus 0),\mathbf{0}]) = [(1),(0)] \wedge ([(x),\mathbf{0}] + [(y),\mathbf{0}]) = [(1),(0)] \wedge (\varphi(x) + \varphi(y)) = \varphi(x)\oplus\varphi(y)$. With the same argument we can see that $\varphi(x\vee y) = \varphi(x)\vee\varphi(y)$.

If $x \in A\oplus 0$, then $\varphi(\neg x) = \varphi\restriction_{A\oplus 0}(\neg x) = [(1),(0)] - [(x),(0)] = u([(x),(0)]) = \neg[(x),(0)] = \neg\varphi\restriction_{A\oplus 0}(x) = \neg\varphi(x)$. If $x \notin A\oplus 0$, $\varphi(\neg x) = [(\neg x),\mathbf{0}] = u([(x),\mathbf{0}]) = \neg\varphi(x)$. Thus φ is a quasi MV-isomorphism. □

4. Inverting the functor Γ_q

Lemma 4.1. *Let A, B be a qMV-algebras and $\varphi\colon A \to B$ be a quasi MV-homomorphism. If $\mathbf{a} = (a_1, a_2 \dots a_n, \dots)$ is a quasi good sequence in M_A^q, then $\varphi^*(\mathbf{a}) = (\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n), \dots)$ is a quasi good sequence in M_B^q. Moreover:*

(1) *φ^* define a $\langle +, \vee, \wedge, \mathbf{0}\rangle$-homomorphism $\varphi^*\colon M_A^q \to M_B^q$,*

(2) *if we consider the Chang's ql-group $\langle G_A^q, u_A\rangle$ and $\langle G_B^q, u_B\rangle$ then $\varphi^\# : \langle G_A^q, u_A\rangle \to \langle G_B^q, u_B\rangle$ such that $\varphi^\#([\mathbf{a}, \mathbf{b}]) = [\varphi^*(\mathbf{a}), \varphi^*(\mathbf{b})]$ is a unital ql-group homomorphism.*

Proof. Since φ is qMV-homomorphism, we have that $\varphi(a_1) \oplus \varphi(a_2) = \varphi(a_1 \oplus a_2) = \varphi(a_1 \oplus 0) = \varphi(a_1) \oplus \varphi(0) = \varphi(a_1) \oplus 0$. Since $(a_2 \dots a_n, \dots)$ is a good sequence in $A \oplus 0$, $(\varphi(a_2), \dots, \varphi(a_n), \dots)$ is a good sequence in $B \oplus 0$ (see [5, §7.1]). Thus $\varphi^*(\mathbf{a}) \in M_B^q$. From this it is clear that $\varphi^*(\mathbf{0}) = \mathbf{0}$.

1) We first note that $\varphi^*(\mathbf{a} \oplus \mathbf{0}) = \varphi^*(\mathbf{a}) \oplus \mathbf{0}$. Let $\triangle$ be a binary operation in M_A^q. Using Proposition 3.11 and [5, §7.1] again, $\varphi^*(\mathbf{a} \triangle \mathbf{b}) = \varphi^*((\mathbf{a} \oplus 0)\triangle (\mathbf{b} \oplus 0)) = \varphi^*(\mathbf{a} \oplus 0) \triangle \varphi^*(\mathbf{b} \oplus 0) = (\varphi^*(\mathbf{a}) \oplus 0) \triangle (\varphi^*(\mathbf{b}) \oplus 0) = (\varphi^*(\mathbf{a}) \triangle \varphi^*(\mathbf{b})) \oplus 0 = \varphi^*(\mathbf{a}) \triangle \varphi^*(\mathbf{b})$. Finally φ^* is a $\langle +, \vee, \wedge, \mathbf{0}\rangle$-homomorphism.

2) Straightforward calculation. □

Proposition 4.2. *$\Xi\colon q\mathscr{MV} \to q\mathscr{LG}_u$ such that for each $A \in q\mathscr{MV}$, $\Xi(A) = \langle G_A^q, u_A\rangle$ and for each qMV-homomorphism $\varphi\colon A \to B$, $\Xi(\varphi) = \varphi^\#$ is a functor.*

Theorem 4.3. *The composite functor $\Gamma_q \Xi\colon q\mathscr{MV} \to q\mathscr{MV}$ is naturally equivalent to the identity functor $1_{q\mathscr{MV}}$.*

Proof. Let $\psi\colon A \to B$ be a qMV-homomorphism and $\varphi_A\colon A \to \Gamma_q\Xi(A)$, $\varphi_B\colon B \to \Gamma_q\Xi(B)$ be the qMV-isomorphisms given in Theorem 3.19. We will prove that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\varphi_A"'] & B \arrow[d, "\varphi_B"] \\ \Gamma_q\Xi(A) \arrow[r, "\Gamma_q\Xi(\psi)"'] & \Gamma_q\Xi(B) \end{tikzcd}$$

If $a \in A$ then $\varphi_B\psi(a) = [(\psi(a)), \mathbf{0}]$. On the other hand, by Lemma 4.1 $(\Gamma_q\Xi(\psi)\varphi_A)(a) = \Gamma_q\Xi(\psi)([(a), \mathbf{0}]) = [(\psi(a)), \mathbf{0}] \in \Gamma_q\Xi(B)$. Since $\Gamma_q(\Xi(\psi))$ is the restriction of $\Xi(\psi)$ to $\Gamma_q\Xi(B)$, we can write $\Gamma_q(\Xi(\psi))(\varphi_A(a)) = [(\psi(a)), \mathbf{0}] = \varphi_B\psi(a)$. So the diagram is commutative. □

Proposition 4.4. *If (G,u) is an l-group with strong unit then we have:*

(1) *For each $0 \leq a \in G$ there is a unique good sequence $g(a) = (a_1, \dots, a_n)$ in $\Gamma(G,u)$ such that $a = a_1 + \dots + a_n$,*

(2) *$g\colon G^+ \to M_{\Gamma(G,u)}$, such that $g(u) = (u)$, is a $(+, \vee, \wedge)$-isomorphism,*

(3) *$\psi_0\colon G \to G_{\Gamma(G,u)}$ defined by $\psi_0(x) = [g(x^+), g(x^-)]$ is an l-group isomorphism such that $\psi(u) = [(u), \mathbf{0}]$.*

Proof. See [5, Lemma 7.1.3, Lemma 7.1.5, Corollary 7.16]. □

Taking into account Proposition 2.14 and Remark 3.16 if $(G,u) \in q\mathscr{L}\mathscr{G}_u$, we can consider $G^q_{\Gamma_q(G,u)}$ as

$$G^q_{\Gamma_q(G,u)} = G_{\Gamma_q(G+0,u(0))} \cup G_{\Gamma_q(G,u)_1} \cup G_{\Gamma_q(G,u)_2}.$$

Lemma 4.5. *Let $(G,u) \in q\mathscr{L}\mathscr{G}_u$ and consider the map $\psi\colon G \to G^q_{\Gamma_q(G,u)}$ defined as follows*

$$\psi(x) = \begin{cases} \psi_0(x), & \text{if } x \in G+0, \\ [(x), \mathbf{0}], & \text{if } x \in G_1, \\ [\mathbf{0}, (x)], & \text{if } x \in G_2. \end{cases}$$

Then ψ is a $q\mathscr{L}\mathscr{G}_u$-isomorphism.

Proof. By definition of bounded ql-group and Chang's ql-group it is clear that ψ is a bijection.

If $x \in G+0$ then $\psi(-x) = \psi_0(-x) = -\psi_0(x) = -\psi(x)$. If $x \in G_1$ then $-x \in G_2$, thus $\psi(-x) = [\mathbf{0}, (x)] = -[(x), \mathbf{0}] = -\psi(x)$. If $x \in G_2$ we use the same argument.

We first prove that $\psi(x+0) = \psi(x) + \psi(0)$. In fact: if $x \in G_1$ then $0 \leq x+0 \leq u(0)$. By Proposition 4.4, $g(x+0) = g((x+0)^+) = (x+0)$ and $g((x+0)^-) = \mathbf{0}$ and then $\psi(x+0) = [(x+0), \mathbf{0}]$. On the other hand $\psi(x)+0 = [(x), \mathbf{0}] + [\mathbf{0}, \mathbf{0}] = [(x+0), \mathbf{0}]$. Resulting $\psi(x+0) = \psi(x) + \psi(0)$. If $x \in G_2$, $-x \in G_1$ and then $\psi(x+0) = \psi(-(x+0)) = -\psi(-x+0) = -(-\psi(x) + \psi(0)) = \psi(x) + \psi(0)$.

If $*$ is a binary operation then $\psi(x) * \psi(y) = (\psi(x)+0) * (\psi(y)+0) = (\psi(x)+\psi(0)) * (\psi(y)+\psi(0)) = \psi_0(x+0) * \psi_0(y+0) = \psi_0((x+0)*(y+0)) = \psi_0(x*y) = \psi(x*y)$.

Since $u(0) \in G+0$, $\psi(u(0)) = [(u(0)), \mathbf{0}]$ whence ψ is unital. Finally ψ is a $q\mathscr{L}\mathscr{G}_u$-isomorphism. □

Theorem 4.6. *The composite functor $\Xi\Gamma_q\colon q\mathscr{L}\mathscr{G}_u \to q\mathscr{L}\mathscr{G}_u$ is naturally equivalent to the identity functor $1_{q\mathscr{L}\mathscr{G}_u}$.*

Proof. Let $(G,u), (H,v) \in q\mathscr{L}\mathscr{G}_u$ and $f\colon G \to H$ be a $q\mathscr{L}\mathscr{G}_u$-homomorphism. We will see that the following diagram is commutative:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\psi_G"'] & H \arrow[d, "\psi_H"] \\ \Xi\Gamma_q(G) \arrow[r, "\Xi\Gamma_q(f)"'] & \Xi\Gamma_q(H) \end{tikzcd}$$

If $x \in G + 0$ then $f(x) \in H + 0$, thus $((\Xi\Gamma_q(f))\psi_G)(x) = (\psi_H f)(x)$ in view of [5, Theorem 7.1.7]. If $x \in G_1$ then $\psi_G(x) = [(x), \mathbf{0}]$ and $((\Xi\Gamma_q(f))\psi_G)(x) = [(f(x)), (f(0))] = [(f(x)), \mathbf{0}]$. On the other hand, since $0 \leq f(x) \leq f(u(0)) = v(0)$, $\psi_h(f(x)) = [(f(x)), \mathbf{0}]$. If $x \in G_2$ then $-x \in G_1$, since $-x = x$. Finally the diagram is commutative. □

Theorem 4.7. *$q\mathscr{MV}$ is categorically equivalent to $q\mathscr{LG}_u$.*

Proof. Follows from the Theorems 4.5, 3.19, 4.3 and 4.6. □

Corollary 4.8. *$\mathscr{MV}$ is a reflective subcategory of $q\mathscr{MV}$.*

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