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ON A RELATIVE UNIFORM COMPLETION OF AN ARCHIMEDEAN LATTICE ORDERED GROUP

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ABSTRACT. The notion of a relatively uniform convergence (ru-convergence) has been used first in vector lattices and then in Archimedean lattice ordered groups.

Let G be an Archimedean lattice ordered group. In the present paper, a relative uniform completion (ru-completion) G_{ω_1} of G is dealt with. It is known that G_{ω_1} exists and it is uniquely determined up to isomorphisms over G. The ru-completion of a finite direct product and of a completely subdirect product are established. We examine also whether certain properties of G remain valid in G_{ω_1}. Finally, we are interested in the existence of a greatest convex l-subgroup of G, which is complete with respect to ru-convergence.

The notion of a relatively uniform convergence (ru-convergence) has been studied first in vector lattices (cf. monographs by B. Z. Vulikh [25], W. A. J. Luxemburg and A. C. Zaanen [21]). A. I. Veksler [24] modified the definition of ru-convergence in vector lattices for applying in lattice ordered groups. The notion of a relative uniform completion (ru-completion) of vector lattices has been introduced by A. I. Veksler [24]. A. W. Hager and J. Martinez [15], R. N. Ball and A. W. Hager [3], and J. Martinez [22] considered ru-completion in Archimedean lattice ordered groups. A related notion of a uniform convergence (with a fixed regulator) in various structures

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was studied, e.g., in [19], [5], [10], [8], [9]. Convergences in MV-algebras are discussed in Section 5.

An ru-completion of an Archimedean lattice ordered group G can be constructed by a successive use of the Cantor's method, which consists in adding limits of relatively uniformly Cauchy sequences. To obtain an ru-completion, there are needed ω_1 steps, in general. This completion is denoted by G_{ω_1}. A lattice ordered group obtained in the first step of this process is called a Cantor extension of G. It was investigated in [11].

This paper can be considered as a continuation of the paper [11]. Let G be an Archimedean lattice ordered group. First we prove that if A is a direct factor of G, then A_{ω_1} is a direct factor of G_{ω_1} and that if G is a completely subdirect product of Archimedean lattice ordered groups G_i $(i \in I)$, then G_{ω_1} is a completely subdirect product of $(G_i)_{\omega_1}$ $(i \in I)$. Then we prove that if every disjoint upper bounded subset of G is finite (Conrad's condition), then the same holds in G_{ω_1}. We give also an example to show that the ru-completion of an epiarchimedean lattice ordered group need not be epiarchimedean. Finally, we are interested in the existence of the greatest convex l-subgroup of G which is G- ru-complete (for the definition see Paragraph 4).

1. Preliminaries

The standard terminology and notation will be used for lattice ordered groups (cf. [1], [12], [14]). The group operation will be written additively.

Suppose that G is a lattice ordered group. Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{Q}$ and $\mathbb{R}$ the additive groups of all rationals and reals with the natural linear order, respectively. If for all $0 \leq x, y \in G$, $nx \leq y$ for each $n \in \mathbb{N}$ implies $x = 0$, then G is called *Archimedean*. Archimedean lattice ordered groups are Abelian ([14, Lemma 4.1.2]). An l-subgroup H of G will be called *dense* if for each $0 < g \in G$ there exists $h \in H$ with $0 < h \leq g$.

G is said to be *complete* if every non-void subset of G bounded from above has a least upper bound in G. An equivalent condition is that every nonempty subset of G bounded from below has a greatest lower bound in G. Complete lattice ordered groups are Archimedean ([12, Proposition 54.2]).

Definition 1.1. (cf. [1, p. 71]) Let $G^\wedge$ be a lattice ordered group satisfying the following conditions:

(α) G is an l-subgroup of $G^\wedge$.

(β) $G^\wedge$ is complete.

(γ) Each element of $G^\wedge$ is the least upper bound of a subset of G (or equivalently the dual statement).

Then $G^\wedge$ is called the *Dedekind completion* of G.

Theorem 1.2. ([1, Theorem 8.2.2]) *Every Archimedean lattice ordered group admits a unique Dedekind completion.*

W. A. J. Luxemburg and A. C. Zaanen [21] studied the notions of a uniform convergence and of a relatively uniform convergence of sequences in vector lattices. We recall a modification of these concepts and related results for lattice ordered groups.

In what follows G is assumed to be an Archimedean lattice ordered group.

Definition 1.3. ([19], [10]) Let (x_n) be a sequence in G, $x \in G$ and $0 < u \in G$. We say that the sequence (x_n) *u-uniformly converges* to x, written $x_n \stackrel{u}{\to} x$, if for each $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that

$$p\,|x_n - x| \le u \qquad \text{for each} \quad n \in \mathbb{N},\ \ n \ge n_0\,.$$

We will refer to u as a *convergence regulator.*

Definition 1.4. ([24]) A sequence (x_n) in G is said to be *relatively uniformly convergent* (ru-*convergent*) to an element $x \in G$ (or x is a *limit* of (x_n)), written $x_n \to x$, if there exists $0 < u \in G$ such that $x_n \stackrel{u}{\to} x$.

Lemma 1.5. ([11, Lemma 2.5]) *Limits in G are uniquely determined.*

Lemma 1.6. ([11, Lemma 2.8]) *Let $a, b \in G$, $x_n \in [a, b]$ for all $n \in \mathbb{N}$ and let $x_n \to x$. Then $x \in [a, b]$.*

Definition 1.7. Let $0 < u \in G$. A sequence (x_n) in G is called *u-uniformly Cauchy* if for each $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that

$$p\,|x_n - x_m| \le u \qquad \text{for each} \quad m, n \in \mathbb{N},\ \ m \ge n \ge n_0\,.$$

Definition 1.8. A sequence (x_n) in G is called *relatively uniformly Cauchy* (ru-*Cauchy*) if (x_n) is u-uniformly Cauchy for some $0 < u \in G$.

Lemma 1.9. ([11, Lemma 2.14]) *Every sequence in G which is* ru-*Cauchy is bounded in G.*

Let F denote the set of all sequences in G which are ru-Cauchy.

It is easy to prove that if a sequence (x_n) in G is ru-convergent then $(x_n) \in F$ (see [11, Corollary 2.12]). If also the converse holds then G is called *relatively uniformly complete* (ru-*complete*).

Example 1.10. Let G be the set of all sequences in $\mathbb{R}$ with a finite support. If the operation $+$ and the partial order are defined componentwise, then G is an Archimedean lattice ordered group. We intend to show that G is ru-complete.

Let (X^n) be an ru-Cauchy sequence in G, $(X^n) = (x_1^n, x_2^n, \dots, 0, 0, \dots)$. We have to prove that the sequence (X^n) is ru-convergent in G.

There exists $0 < U \in G$, $U = (u_1, u_2, \dots, 0, 0, \dots)$ such that for each $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ with the property

$$p\,|X^n - X^m| \le U \qquad \text{for each} \quad m, n \in \mathbb{N},\ \ m \ge n \ge n_0\,,$$

so

$$p\,|x_i^n - x_i^m| \le u_i \qquad \text{for each} \quad i \in \mathbb{N},\ \ m, n \in \mathbb{N},\ \ m \ge n \ge n_0\,.$$

Hence $(x_i^n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ for each $i \in \mathbb{N}$, thus (x_i^n) is convergent for each $i \in \mathbb{N}$. Let $\lim\limits_{n\to\infty} x_i^n = x_i$.

There exists $i_0 \in \mathbb{N}$ such that $x_i^{n_0} = 0$ and $u_i = 0$ for each $i \in \mathbb{N}$, $i > i_0$. Hence $x_i^n = 0$ for each $n \ge n_0$, $i \in \mathbb{N}$, $i > i_0$.

Denoting $u = \max\{u_1, \dots, u_{i_0}\}$ we get

$$p\,|x_i^n - x_i^m| \le u \qquad \text{for each} \quad m, n \in \mathbb{N},\ \ m \ge n \ge n_0,\ \ i \in \mathbb{N},\ \ i \le i_0\,.$$

Assuming that $i \in \mathbb{N}$, $i \le i_0$ and $n \in \mathbb{N}$, $n \ge n_0$ are fixed, we get

$$x_i^n - \frac{u}{p} \le x_i^m \le x_i^n + \frac{u}{p} \qquad \text{for all} \quad m \ge n\,.$$

Consequently

$$x_i^n - \frac{u}{p} \le x_i \le x_i^n + \frac{u}{p}\,.$$

Then

$$p\,|x_i^n - x_i| \le u \qquad \text{for each} \quad i \in \mathbb{N},\ \ i \le i_0,\ \ n \in \mathbb{N},\ \ n \ge n_0\,.$$

Under the notations $X = (x_1, \dots, x_{i_0}, 0, 0, \dots)$, $V = (u, \dots, u, 0, 0, \dots)$ with i_0 copies of u, the elements X and V belong to G and

$$p\,|X^n - X| \le V \qquad \text{for each} \quad n \in \mathbb{N},\ \ n \ge n_0\,,$$

i.e., $X^n \to X$.

Let K' be an l-subgroup of a lattice ordered group K, (x_n) a sequence in K and $x \in K$. If $x_n \xrightarrow{u} x$ for some $0 < u \in K$ (K') we will often write $x_n \to x(K)$ $(x_n \to x(K'))$.

Definition 1.11. Let H be an Archimedean lattice ordered group with the following properties:

(i) G is an l-subgroup of H.

(ii) For every sequence $(x_n) \in F$ there exists $x \in H$ such that $x_n \to x(H)$.

(iii) For every $x \in H$ there exists a sequence $(x_n) \in F$ such that $x_n \to x(H)$.

Then H will be called a *Cantor extension* of G.

Lemma 1.12. *Let (x_n) be a sequence in G and $0 < u, v \in G$. If (x_n) is u-uniformly Cauchy and $x_n \xrightarrow{v} x$ then $x_n \xrightarrow{u} x$.*

Proof. Assume that (x_n) is a u-uniformly Cauchy sequence and $x_n \xrightarrow{v} x$. Let $p \in \mathbb{N}$. There exists $n_p \in \mathbb{N}$ with

$$p\,|x_n - x_m| \le u \qquad \text{for each} \quad m, n \in \mathbb{N},\ \ m \ge n \ge n_p\,.$$

Then

$$p\,x_n - u \le p\,x_m \le p\,x_n + u \qquad \text{for each} \quad m, n \in \mathbb{N},\ \ m \ge n \ge n_p\,.$$

Assume that $n \ge n_p$ is fixed. From $x_m \xrightarrow{v} x$ it follows $p\,x_m \xrightarrow{v} p\,x$. By 1.6, for each $n \ge n_p$ we get

$$p\,x_n - u \le p\,x \le p\,x_n + u\,,$$
$$p\,|x_n - x| \le u\,,$$

i.e, $x_n \xrightarrow{u} x$. □

Remark. By using 1.12, the conditions (ii) and (iii) are equivalent to the conditions (ii_1) and (iii_1), respectively.

(ii_1) For every sequence (x_n) in G which is u-uniformly Cauchy for some $0 < u \in G$, there exists $x \in H$ such that $x_n \xrightarrow{u} x$.

(iii_1) For every $x \in H$ there exist a sequence (x_n) in G and $0 < u \in G$ such that $x_n \xrightarrow{u} x$.

In view of Remark, Definition 1.11 is equivalent with Definition 3.2 of a Cantor extension given in [11].

Theorem 1.13. ([11, Theorems 4.8, 4.10]) *Let G be an Archimedean lattice ordered group. Then a Cantor extension H of G exists and it is uniquely determined up to isomorphisms over G.*

Cantor extension of G is not ru-complete, in general. A. I. Veksler [24] has proved that Cantor extension of a vector lattice with projections is ru-complete and remarked that the same result is valid for a lattice ordered group with projections (i.e., for a strongly projectable lattice ordered group [1]).

If $x_n \to 0$, then (x_n) is called a *zero* sequence. Denote by E the set of all zero sequences in G.

We will apply the Cantor sequence completion method to obtain a Cantor extension of G (cf. [11]).

Let $(x_n), (y_n) \in F$. If the operation $+$ and the partial order are defined componentwise then F turns to an Archimedean lattice ordered group and E becomes an l-ideal of F. We can form the quotient group $G^* = F/E$. Given $(x_n) \in F$ we denote by $(x_n)^*$ the corresponding element of G^*. We have $(x_n)^* + (y_n)^* = (x_n + y_n)^*$. If we put $(x_n)^* \leq (y_n)^*$ if and only if there exist $(x'_n) \in (x_n)^*$ and $(y'_n) \in (y_n)^*$ with $(x'_n) \leq (y'_n)$, then G^* is an Archimedean lattice ordered group in which $(x_n)^* \vee (y_n)^* = (x_n \vee y_n)^*$ and dually, $|(x_n)^*| = (|x_n|)^*$.

If we define $f\colon G \to G^*$ by $f(x) = (x, x, \dots)^*$ for each $x \in G$, then f is an embedding of the lattice ordered group G and G can be considered as an l-subgroup of G^*. Moreover, we have:

Theorem 1.14. ([11, Theorem 4.8]) *G^* is a Cantor extension of G.*

Theorem 1.15. ([11, Theorem 4.6]) *G^* is an l-subgroup of $G^\wedge$.*

Corollary 1.16. ([11, Corollary 4.9]) *G is a dense l-subgroup of G^*.*

Definition 1.17. (cf. [3], [22]) Let $\mathrm{ru}(G)$ be an Archimedean lattice ordered group with the following properties:

(a) G is an l-subgroup of $\mathrm{ru}(G)$.

(b) $\mathrm{ru}(G)$ is ru-complete.

(c) If G is an l-subgroup of H, H is an l-subgroup of $\mathrm{ru}(G)$ and H is ru-complete, then $H = \mathrm{ru}(G)$.

Then $\mathrm{ru}(G)$ is said to be a *relative uniform completion* (ru-*completion*) of G.

Example 1.18. Let G be the set of all eventually constant sequences and H the set of all convergent sequences in $\mathbb{R}$. If the operation $+$ and the partial order are performed componentwise, then H is an Archimedean lattice ordered group and G is an l-subgroup of H. There is established in [11] that G fails to be ru-complete, H is ru-complete and $H = G^*$. Therefore H is an ru-completion of G. Hence $H = G_{\omega_1}$.

Having shown that G^* is a Cantor extension of G (see 1.14), we can find an ru-completion of G. It suffices only to define lattice ordered groups G_λ for each ordinal $\lambda \leq \omega_1$ (ω_1 is the first uncountable ordinal) as follows (cf. [3]):

$G_0 = G$,

$G_\lambda = (G_{\lambda-1})^*$ if λ is an ordinal less than ω_1 having a predecessor $\lambda - 1$,

$G_\lambda = \Big(\bigcup\limits_{\tau<\lambda} G_\tau\Big)^*$ if λ is a limit ordinal $\lambda < \omega_1$,

$$G_{\omega_1} = \bigcup_{\tau<\omega_1} G_\tau.$$

Apparently, if $0 \le \lambda_1 < \lambda_2 \le \omega_1$ then G_{λ_1} is an l-subgroup of G_{λ_2}.

By using the transfinite induction, 1.14 and 1.15, the following results are easy to derive.

Theorem 1.19. (cf. [3]) *G_{ω_1} is an* ru-*completion of G.*

Lemma 1.20. *G_{ω_1} is an l-subgroup of $G^\wedge$.*

From 1.1 it follows that G is a dense l-subgroup of $G^\wedge$. Then as a consequence of 1.20 we get:

Corollary 1.21. *G is a dense l-subgroup of G_{ω_1}.*

Lemma 1.22. *Let H be an* ru-*complete Archimedean lattice ordered group and G be an l-subgroup of H. Then there exists an l-isomorphism φ of G_{ω_1} into H leaving all elements of G fixed.*

Proof. By the transfinite induction we will prove the assertion

(δ) For any ordinal $0 \le \lambda < \omega_1$ there exists an l-isomorphism $\varphi_\lambda\colon G_\lambda \to H$ such that φ_λ is an extension of φ_α for any $\alpha < \lambda$.

Let $\lambda = 0$. We have $G_0 = G$. The identity mapping is a desired l-isomorphism $\varphi_0\colon G \to H$.

Let λ be a non-limit ordinal, $\lambda > 0$. Suppose that there exists an l-isomorphism $\varphi_{\lambda-1}\colon G_{\lambda-1} \to H$ such that $\varphi_{\lambda-1}$ is an extension of φ_β for each $\beta < \lambda - 1$. Let $x \in G_\lambda$. With respect to (iii$_1$) there exist a sequence (x_n) in $G_{\lambda-1}$ and $0 < u \in G_{\lambda-1}$ with $x_n \xrightarrow{u} x$. Consequently $(\varphi_{\lambda-1}(x_n))$ is $\varphi_{\lambda-1}(u)$-uniformly Cauchy in $\varphi_{\lambda-1}(G_{\lambda-1}) \subseteq H$. Hence there exist $y \in H$ and $0 < v \in H$ such that $\varphi_{\lambda-1}(x_n) \xrightarrow{v} y$. Let us put $\varphi_\lambda(x) = y$. It is easy to see that φ_λ is correctly defined and that φ_λ fulfills (δ).

Let λ be a limit ordinal, $\lambda > 0$. Assume that for any $\alpha < \lambda$ there is an l-isomorphism $\varphi_\alpha\colon G_\alpha \to H$ such that φ_α is an extension of φ_β for each $\beta < \alpha$. Define the mapping $\psi\colon \bigcup\limits_{\alpha<\lambda} G_\alpha \to H$ by putting $\psi(x) = \varphi_\alpha(x)$ if $x \in G_\alpha$. Clearly, ψ is an l-isomorphism of $\bigcup\limits_{\alpha<\lambda} G_\alpha$ into H such that it is an extension of φ_α for any $\alpha < \lambda$. Let $x \in G_\lambda$. Again by (iii$_1$), there exist a sequence (x_n) in $\bigcup\limits_{\alpha<\lambda} G_\alpha$ and $0 < u \in \bigcup\limits_{\alpha<\lambda} G_\alpha$ such that $x_n \xrightarrow{u} x$. Further we define the mapping $\varphi_\lambda\colon G_\lambda \to H$ and prove that φ_λ fulfills (δ) in an analogous way to the previous case.

Finally, define the mapping $\varphi\colon G_{\omega_1} \to H$ by the rule $\varphi(x) = \varphi_\lambda(x)$, if $x \in G_\lambda$. With respect to (δ), φ is a required l-isomorphism. □

Lemma 1.23. *Let H be an* ru-*completion of G. Then there exists an l-isomorphism of G_{ω_1} onto H leaving all elements of G fixed.*

P r o o f. According to 1.22 there exists an l-isomorphism φ of $G\omega_1$ into H fixing all elements of G. It remains to show that φ is surjective. Apparently, $H_1 = \varphi(G\omega_1)$ is an ru-complete l-subgroup of H and $G = \varphi(G)$ is an l-subgroup of H_1. Then by 1.17, $H_1 = H$ which completes the proof. □

From 1.19 and 1.23 we immediately obtain:

Theorem 1.24. *Let G be an Archimedean lattice ordered group. Then there exists an* ru-*completion* $\mathrm{ru}(G)$ *of G and it is uniquely determined up to isomorphisms over G.*

The result presented in the last theorem is not new, but we have not been able to find its proof in the papers.

2. Direct factors of G_{ω_1}

Let G be a lattice ordered group and X a subset of G. The set

$$X^\delta = \bigl\{g \in G:\ |g| \wedge |x| = 0 \ \text{for all}\ x \in X\bigr\}$$

is a convex l-subgroup of G.

Let A be a convex l-subgroup of G. If there exists a convex l-subgroup B of G such that $A \cap B = \{0\}$, $G = A + B$ then A (and also B) is called a *direct factor* of G and G is said to be the *direct product* of A and B. This is expressed by writing $G = A \times B$. If $G = A \times B$ then $B = A^\delta$.

It is well-known that a convex l-subgroup A of G is a direct factor of G if and only if the following condition is fulfilled:

For each $0 \le g \in G$ the set $S = \{a \in A:\ 0 \le a \le g\}$ has a greatest element.

If A is a direct factor of G and $0 \le g \in G$ then the greatest element of S is denoted by $g(A)$ and called the component of g in A.

In this section we continue with the assumption that G is an Archimedean lattice ordered group. It will be shown how to construct an ru-completion of a convex l-subgroup of G that is a direct factor of G. This result will be applied to find an ru-completion of the direct product of lattice ordered groups with a finite number of direct factors.

Let A be a convex l-subgroup of G. Then $c(A)$ stands for the convex l-subgroup of G_{ω_1} generated by A, i.e.,

$$c(A) = \left\{x \in G_{\omega_1} :\ a_1 \le x \le a_2 \ \text{for some}\ a_1, a_2 \in A\right\}.$$

Lemma 2.1. *Let A be a direct factor of G. Then $c(A)$ is a direct factor of G_{ω_1}.*

Proof. Evidently, $G \cap c(A) = A$. Assume that $0 \le y \in G_{\omega_1}$. The proof will be done if we show that the set $S = \left\{0 \le x \in c(A) :\ x \le y\right\}$ possesses a greatest element. By 1.20, $G_{\omega_1} \subseteq G^\wedge$. Then 1.1 yields that $y \le g$ for some $g \in G$. The element $h = y \wedge g(A)$ belongs to $c(A)$ and $0 \le h \le y$, so $h \in S$. We claim that h is the greatest element of S. Suppose that there exists $s \in S$, $h < s$. From $s \le y$ and $s \le a$ for some $a \in A$ we obtain $s \le y \wedge a$. Convexity of A in G yields that $a \wedge g \in A$. Taking into account that A is a direct factor of G, we get $s > h = y \wedge g(A) \ge y \wedge g \wedge a = y \wedge a$, a contradiction. □

The proof of 2.1 is similar to that of [18, Proposition 2.6] where the relation between the direct factors in G and $G^\wedge$ has been stated.

Theorem 2.2. *Let A be a direct factor of G. Then $c(A) \cong A_{\omega_1}$.*

Proof. We have to verify that the conditions (a), (b) and (c) are satisfied with $\mathrm{ru}(G)$ and G replaced by $c(A)$ and A, respectively.

(a) It is easy to check that A is an l-subgroup of $c(A)$.

(b) To prove that $c(A)$ is ru-complete, assume that (x_n) is a sequence in $c(A)$ and that it is ru-Cauchy. Hence there exists $0 < u \in c(A)$ such that (x_n) is u-uniformly Cauchy. By 1.9, there are $v, w \in c(A)$ with $v \le x_n \le w$ for any $n \in \mathbb{N}$. Further, there are $a_1, a_2 \in A$ such that $a_1 \le v$ and $w \le a_2$. Whence $a_1 \le x_n \le a_2$ for all $n \in \mathbb{N}$. G_{ω_1} being ru-complete, there is $x \in G_{\omega_1}$ with $x_n \to x(G_{\omega_1})$. As $c(A) \subseteq G_{\omega_1}$, 1.12 entails $x_n \xrightarrow{u} x$. In view of 1.6, $a_1 \le x \le a_2$, so $x \in c(A)$.

(c) Let H be an ru-complete lattice ordered group such that A is an l-subgroup of H and H is an l-subgroup of $c(A)$.

Observe that $B_\tau = G_\tau \cap c(A) \supseteq A$ for any $\tau < \omega_1$. We want to prove that $c(A) \subseteq H$. It suffices to show that $B_\tau \subseteq H$ for each $\tau < \omega_1$, because $\bigcup\limits_{\tau < \omega_1} B_\tau = c(A)$. Assume that $x \in B_\tau$ for some $\tau < \omega_1$. We are going to prove that $x \in H$.

Three cases (α), (β) and (γ) can occur.

(α) $\tau = 0$.

Then $x \in B_0 = G_0 \cap c(A) = G \cap c(A) = A \subseteq H$.

(β) τ is a limit ordinal.
Assume that for any $\lambda < \tau$, $B_\lambda \subseteq H$ is valid. We can assume that $x \geq 0$. From $G_\tau = \Big(\bigcup\limits_{\lambda<\tau} G_\lambda\Big)^*$ and $x \in G_\tau$ we infer that there exists a sequence (x_n) in $\bigcup\limits_{\lambda<\tau} G_\lambda$ with $(x_n) \geq 0$, $x_n \to x\Big(\bigcup\limits_{\lambda<\tau} G_\lambda\Big)$. Then there exist $\lambda_1 < \tau$ and $0 < u \in G_{\lambda_1}$ such that $x_n \xrightarrow{u} x$. As $x \in c(A)$, there is $a \in A$, $x \leq a$. Consequently, $x'_n = x_n \wedge a \xrightarrow{u} x \wedge a = x$. We have $0 \leq x'_n \leq a$ for all $n \in \mathbb{N}$. Since $a \in B_\lambda$ for any $\lambda < \tau$, for each $n \in \mathbb{N}$ there exists $\lambda_n < \tau$ such that $x'_n \in B_{\lambda_n}$. The assumption implies that $B_{\lambda_n} \subseteq H$ for each $n \in \mathbb{N}$, so (x'_n) is a sequence in H.

We prove now that (x'_n) is an ru-Cauchy sequence in H. We are looking for a regulator lying in H. Remark that u need not have such a property.

Let $p \in \mathbb{N}$. There exists $n_0 \in \mathbb{N}$ with

$$p\,|x'_n - x| \leq u \qquad \text{for all} \quad n \in \mathbb{N},\ \ n \geq n_0\,;$$

(x'_n) is a sequence in $H \subseteq c(A)$ and $x \in c(A)$. By 2.1, $c(A)$ is a direct factor of G_{ω_1}. Then

$$p\,|x'_n - x| = p\,|x'_n - x|(c(A)) \leq u\big(c(A)\big) \leq a'$$

for each $n \in \mathbb{N}$, $n \geq n_0$ and some $a' \in A$, i.e., $x'_n \xrightarrow{a'} x$. This yields that (x'_n) is an ru-Cauchy sequence in H because of $a' \in A \subseteq H$.

Taking into account the assumption that H is ru-complete and 1.5 we get $x \in H$.

(γ) τ is a non-limit ordinal.
Assume that $B_{\tau-1} \subseteq H$. Again we suppose that $x \geq 0$. From $G_\tau = (G_{\tau-1})^*$ and $x \in G_\tau$ we deduce that there exist a sequence (x_n) in $G_{\tau-1}$ and $0 < u \in G_{\tau-1}$ with $x_n \xrightarrow{u} x$. In the same way as in the previous case we construct the sequence (x'_n) and prove that $x'_n \in B_{\tau-1}$ for each $n \in \mathbb{N}$. Applying the assumption we obtain that (x'_n) is a sequence in H. Further, repeating the procedure from (β) we conclude the proof. $\square$

Lemma 2.3. *Let $G = A \times B$. Then $G_{\omega_1} = c(A) \times c(B)$.*

Proof. By 2.1, $c(A)$ is a direct factor of G_{ω_1}. We wish to prove that $(c(A))^\delta = c(B)$ holds.

The relation $A \subseteq c(A)$ yields $(c(A))^\delta \subseteq A^\delta = B \subseteq c(B)$.

To prove the inclusion $c(B) \subseteq (c(A))^\delta$, assume that $0 \leq x \in c(B)$ and $y \in c(A)$. There are elements $b \in B$ and $a \in A$ with $0 \leq x \leq b$ and $0 \leq |y| \leq a$. Then $a \wedge b = 0$ implies $x \wedge |y| = 0$. Hence $x \in (c(A))^\delta$. $\square$

As a consequence of 2.2 and 2.3 we get:

Theorem 2.4. *Let $G = A \times B$. Then $G_{\omega_1} \cong A_{\omega_1} \times B_{\omega_1}$.*

What has been shown now for two direct factors can be generalized to an arbitrary finite number of direct factors. It is an open question, whether it is possible to generalize this result to an infinite number of direct factors.

3. ru-completion of a completely subdirect product of lattice ordered groups

Let G_1 be an l-subgroup of G and $F_1(E_1)$ the set of all sequences in G_1 which are ru-Cauchy (zero) (regulators are taken from G_1). Given $(x_n) \in F_1$, symbol $(x_n)^1$ denotes the element of the quotient lattice ordered group $G_1^* = F_1/E_1$. The mapping defined by $f((x_n)^1) = (x_n)^*$ for each $(x_n)^1 \in G_1^*$ is an embedding of the lattice ordered group G_1^* into G^*. Hence G_1^* can be viewed as an l-subgroup of G^*.

Let I be a nonempty set and let G_i be an Archimedean lattice ordered group for any $i \in I$. Assume that G is the direct product of G_i, $G = \prod\limits_{i \in I} G_i$. Then G is an Archimedean lattice ordered group. The component of an element $x \in G$ in the direct factor G_i will be denoted also by $x(i)$. Let $F_i(E_i)$ be the set of all ru-Cauchy (zero) sequences in G_i for each $i \in I$. The following lemmas are easy to verify (cf. [11, Lemmas 6.5, 6.6]). In both lemmas we suppose that (x_n) is a sequence in G.

Lemma 3.1. *If $(x_n) \in F$ then $(x_n(i)) \in F_i$ for any $i \in I$.*

Lemma 3.2. *If $(x_n) \in E$ then $(x_n(i)) \in E_i$ for any $i \in I$.*

Assume that $(x_n)^* \in G^*$. By 3.1, from $(x_n) \in F$ it follows that $(x_n(i)) \in F_i$ for each $i \in I$. Thus $(x_n(i))^* \in G_i^*$ for each $i \in I$. Denote by X the element of $K = \prod\limits_{i \in I} G_i^*$ such that $X(i) = (x_n(i))^*$ for any $i \in I$. Define the mapping $\phi\colon G^* \to K$ by putting $\phi((x_n)^*) = X$ for any $(x_n)^* \in G^*$.

Lemma 3.3. ([11, Theorem 6.8]) *Let $G = \prod\limits_{i \in I} G_i$. Then ϕ is an l-isomorphism of G^* onto K.*

Let us put $G_i^0 = \{g \in G : \ g(j) = 0 \ \text{for each} \ j \in I, \ j \neq i\}$. Let H be an l-subgroup of G such that $G_i^0 \subseteq H$ for each $i \in I$. Then H is said to be a *completely subdirect product* of G_i $(i \in I)$. This notion is due to F (Šik [23]).

Theorem 3.4. *Let φ be an l-isomorphism of a lattice ordered group H onto a completely subdirect product of G_i $(i \in I)$. Then there exists an l-isomorphism φ^* of H^* onto a completely subdirect product of G_i^* $(i \in I)$ such that φ^* is an extension of φ.*

P r o o f. H and H^* can be considered as l-subgroups of $G = \prod_{i \in I} G_i$ and G^*, respectively. Applying 3.3 and setting $\varphi^* = \phi \restriction H^*$, φ^* is an l-isomorphism of H^* into $K = \prod_{i \in I} G_i^*$.

Let $i \in I$, $X \in K$ with $X(i) = (x_n^i)^*$ and $X(j) = 0$ for each $j \in I$, $j \neq i$. It remains to show that X has an origin in H^* under the mapping φ^*. From $(x_n^i) \in F_i$ we infer that (x_n^i) is u^i-uniformly Cauchy sequence for some $0 < u^i \in G_i$. Consider the sequence (x_n) in G with $x_n(i) = x_n^i$, $x_n(j) = 0$ for any $j \in I$, $j \neq i$, and the element $u \in G$ with $u(i) = u^i$, $u(j) = 0$ for any $j \in I$, $j \neq i$. We have $0 < u \in H$, (x_n) is a sequence in H and (x_n) is u-uniformly Cauchy. Therefore $(x_n)^* \in H^*$ is valid. We conclude $\varphi^*((x_n)^*) = X$.

Apparently, φ^* is an extension of φ. □

Theorem 3.5. *Let φ be an l-isomorphism of a lattice ordered group H onto a completely subdirect product of G_i $(i \in I)$. Then there exists an l-isomorphism $\bar{\varphi}$ of H_{ω_1} onto a completely subdirect product of $(G_i)_{\omega_1}$ $(i \in I)$.*

P r o o f. We first prove the assertion:

(∗) For any ordinal $\tau < \omega_1$ there exists an l-isomorphism φ_τ of G_τ onto a completely subdirect product of $(G_i)_\tau$ $(i \in I)$ such that φ_τ is an extension of φ_λ for any $\lambda < \tau$.

Let $\tau = 0$. Since $\varphi_0 = \varphi$, the assertion (∗) follows from the assumption.

Let τ be a non-limit ordinal. Suppose that $\varphi_{\tau-1}$ is an l-isomorphism of $H_{\tau-1}$ onto a completely subdirect product of $(G_i)_{\tau-1}$ $(i \in I)$ and that $\varphi_{\tau-1}$ is an extension of φ_λ for each $\lambda < \tau - 1$. Then according to definition of H_τ and 3.4, there is an l-isomorphism φ_τ of H_τ onto $(G_i)_\tau$ $(i \in I)$ extending $\varphi_{\tau-1}$. Therefore φ_τ fulfills (∗).

Let τ be a limit ordinal. Assume that for each $\lambda < \tau$ there exists an l-isomorphism φ_λ of H_λ onto a completely subdirect product of $(G_i)_\lambda$ $(i \in I)$ and that φ_λ is an extension of φ_β for each $\beta < \lambda$. Define the mapping $\psi\colon \bigcup_{\lambda<\tau} H_\lambda \to \prod_{i \in I} \bigcup_{\lambda<\tau} (G_i)_\lambda$ by the rule $\psi(x) = \varphi_\lambda(x)$, if $x \in H_\lambda$. Then ψ is correctly defined and it is an l-isomorphism of $\bigcup_{\lambda<\tau} H_\lambda$ into $\prod_{i \in I} \bigcup_{\lambda<\tau} (G_i)_\lambda$.

Let $i \in I$ and $y \in \prod_{i \in I} \bigcup_{\lambda<\tau} (G_i)_\lambda$ such that $y(j) = 0$ for each $j \in I$, $j \neq i$. Suppose that $y(i) \in (G_i)_\beta$, $\beta < \tau$. By the assumption, φ_β is an l-isomorphism of H_β onto a completely subdirect product of $(G_i)_\beta$ $(i \in I)$. Hence there exists $x \in H_\beta$ with $\varphi_\beta(x) = y$ and so $\psi(x) = y$.

Therefore ψ is an l-isomorphism of $\bigcup_{\lambda<\tau} H_\lambda$ onto a completely subdirect product of $\bigcup_{\lambda<\tau} (G_i)_\lambda$ $(i \in I)$. Definition of ψ implies that ψ is an extension of φ_λ for each $\lambda < \tau$. Again, by definition of H_τ and 3.4, there exists an l-isomorphism φ_τ of H_τ onto a completely subdirect product of $(G_i)_\tau$ $(i \in I)$ such that φ_τ is an extension of ψ. Consequently, φ_τ satisfies $(*)$.

Applying $(*)$ we show that the mapping $\bar{\varphi}\colon H_{\omega_1} \to \prod_{i \in I} (G_i)_{\omega_1}$ defined by $\bar{\varphi}(x) = \varphi_\tau(x)$ whenever $x \in G_\tau$, is a desired l-isomorphism. $\square$

4. Further results on G and G_{ω_1}

In the present section we investigate which properties of G remain valid in $G\omega_1$. Further, we are interested in the question whether there exists a greatest G- ru-complete convex l-subgroup of a lattice ordered group G. As in earlier sections, the lattice ordered group G is supposed to be Archimedean.

Let D be a subset of G consisting of strictly positive elements of G. If $x \wedge y = 0$ whenever x and y are distinct elements of D, then D will be called a *disjoint subset* of G.

An element $0 < x \in G$ is referred to as *basic* if the set $\{g \in G : \ 0 \leq g \leq x\}$ is a chain. We will say that a subset B of G is a *basis* of G if

(i) B is a maximal disjoint subset of G,

(ii) each element of B is basic.

Lemma 4.1. *Let B be a basis of G. Then B is a basis of G_{ω_1}.*

Proof. Let B be a basis of G. Then B is a disjoint subset of G_{ω_1}, because G is an l-subgroup of G_{ω_1}. Assume that B is not a maximal disjoint subset of G_{ω_1}. Then there exists $0 < x \in G_{\omega_1}$, $x \notin G$ with $x \wedge b = 0$ for each $b \in B$. By 1.21, there exists $g \in G$, $0 < g \leq x$. Hence $g \wedge b = 0$ for each $b \in B$. This contradicts the maximality of B in G.

Let $b \in B$. It remains to show that the set $\{x \in G_{\omega_1} : \ 0 \leq x \leq b\}$ is a chain. Suppose that there are $x_1, x_2 \in G_{\omega_1}$, $0 \leq x_1 \leq b$, $0 \leq x_2 \leq b$, $x_1 \parallel x_2$. By 1.21, there are $g_1, g_2 \in G$, $0 < g_1 \leq x_1$, $0 < g_2 \leq x_2$. If $x_1 \wedge x_2 = 0$, then $g_1 \wedge g_2 = 0$, a contradiction. If $x_1 \wedge x_2 = x > 0$, then $y_1 = x_1 - x > 0$, $y_2 = x_2 - x > 0$ are elements of G_{ω_1} and $y_1 \wedge y_2 = 0$. Repeating the previous procedure with elements y_1 and y_2 we finish the proof. $\square$

P. Conrad [6] studied the following condition for a lattice ordered group G:

(F) Every disjoint upper bounded subset of G is finite.

Theorem 4.2. ([11, Theorem 5.3]) *Let G be an Archimedean lattice ordered group fulfilling the condition (F). Then G is isomorphic to a completely subdirect product of its l-subgroups G_i $(i \in I)$ which are o-isomorphic to subgroups of $\mathbb{R}$.*

Theorem 4.3. *Let G satisfy the condition (F). Then so does G_{ω_1}.*

Proof. Assume, by way of contradiction, that there exists a disjoint upper bounded subset D of G_{ω_1} that is infinite. Then there exists $x \in G_{\omega_1}$, $d \leq x$ for any $d \in D$. By 1.20, $G_{\omega_1} \subseteq G^{\wedge}$ which together with 1.1 entails that there exists $h \in G$ such that $x \leq h$.

According to 4.2 and 3.5, G_{ω_1} is l-isomorphic to a completely subdirect product of $(G_i)_{\omega_1}$ $(i \in I)$. We can suppose that G_{ω_1} is an l-subgroup of $\prod\limits_{i \in I} G_i^*$. In view of 4.2, $(G_i)_{\omega_1} = G_i^*$ and G_i^* is linearly ordered for any $i \in I$.

Let $I_1 = \{i \in I :\ \text{there exists}\ d \in D\ \text{with}\ d(i) > 0\}$. Let us notice that I_1 is infinite. Otherwise there would exist $d_1, d_2 \in D$, $d_1 \neq d_2$ and $i \in I_1$ with $d_1(i)$, $d_2(i) \in G_i^*$, $d_1(i) > 0$, $d_2(i) > 0$. We have $d_1(i) \wedge d_2(i) = 0$ which is impossible. For any $i \in I_1$ take $d_i \in D$ with $d_i(i) > 0$. By 1.16, for any $i \in I_1$, there exists $g_i \in G_i$ satisfying $0 < g_i \leq d_i(i)$. Let g_i^0 be the element of G with the property $g_i^0(i) = g_i$ and $g_i(j) = 0$ for each $j \in I$, $j \neq i$. Then $0 < g_i^0 \leq d_i$. We conclude that $\{g_i^0 \in G :\ i \in I_1\}$ is a disjoint upper bounded subset of G that is infinite, a contradiction. $\square$

It is well-known that an l-homomorphic image of an Archimedean lattice ordered group need not be Archimedean. If each l-homomorphic image of G is Archimedean then we call G *epiarchimedean*. The convex l-subgroup of G generated by $x \in G$ will be denoted by $c(x)$.

The following result has been obtained by P. Conrad [7].

Theorem 4.4. *A lattice ordered group G is epiarchimedean if and only if $c(x)$ is a direct factor of G for any $x \in G$.*

If G is epiarchimedean then G_{ω_1} need not be epiarchimedean.

Example 4.5. Let G be the set of all eventually constant sequences in $\mathbb{R}$. If the operation $+$ and the partial order are defined componentwise then G is an

Archimedean lattice ordered group. Consider the convex l-subgroup $c((x_n))$ of G generated by the element $(x_n) \in G$. If $x_n \neq 0$ for each $n \in \mathbb{N}$, then $c((x_n)) = G$. Assume that $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}$ and $0 \leq (y_n) \in G$. There exists a greatest element of the set $S = \big\{(z_n) \in c((x_n)) : \ 0 \leq (z_n) \leq (y_n)\big\}$, namely (y'_n) with $y'_n = x_n$ if $x_n = 0$ and $y'_n = y_n$ if $x_n \neq 0$. Therefore $c((x_n))$ is a direct factor of G and by 4.4, G is epiarchimedean.

As observed in Example 1.18, G^* is the lattice ordered group of all convergent sequences in $\mathbb{R}$ and $G^* = G_{\omega_1}$.

Consider the convex l-subgroup $c_1((X_n))$ of G^* generated by the element $(X_n) \in G^*$, $(X_n) = \big(1, \frac{1}{2}, \frac{1}{3}, \dots\big)$. It is easy to see that $c_1((X_n))$ consists only of sequences of G^* converging to 0. If we choose the element $(Y_n) \in G^*$, $(Y_n) = (1, 1, \dots)$, then the set $S_1 = \big\{(Z_n) \in c_1((X_n)) : \ 0 \leq (Z_n) \leq (Y_n)\big\}$ has no greatest element. Consequently, $c_1((X_n))$ fails to be a direct factor of G^*. Whence by 4.4, G^* is not epiarchimedean.

A nonempty subset M of G is said to be a G-ru-*complete subset* of G if for each sequence (x_n) in M such that $(x_n) \in F$ there exists $x \in M$ with $x_n \to x(G)$.

Studying o-convergence in lattice ordered groups, J. Jakubík [17] introduced the concept of a complete subset related to the above. By using the same method as applied in [17], it can be proved:

Lemma 4.6. *Let $a, b \in G$, $0 \leq a, b$. If $[0, a]$ and $[0, b]$ are G-*ru*-complete subsets of G, then $[0, a + b]$ is also a G-*ru*-complete subset of G.*

Lemma 4.7. *The following conditions are equivalent:*

(i) *G is* ru*-complete.*

(ii) *Every interval $[a, b]$ of G is G-*ru*-complete subset of G.*

Proof.

(i) $\implies$ (ii): Let G be ru-complete, (x_n) a sequence in $[a, b]$, $(x_n) \in F$. Then there exists $x \in G$ with $x_n \to x$. By 1.6, $x \in [a, b]$.

(ii) $\implies$ (i): Assume that every interval of G is G-ru-complete and (x_n) is a sequence in G, $(x_n) \in F$. According to 1.9 there exist $a, b \in G$ with $a \leq x_n \leq b$ for all $n \in \mathbb{N}$. Hence $x_n \to x$, $x \in [a, b]$. □

Lemma 4.8. *Let G be* ru*-complete and H a convex l-subgroup of G. Then H is* ru*-complete.*

Proof. Assume that (x_n) is a sequence in H and that (x_n) is u-uniformly Cauchy for some $u \in H$. By 1.9 there are $a, b \in H$, $a < b$, such that $a \leq x_n \leq b$ for each $n \in \mathbb{N}$. Applying 4.7 we get that $[a, b]$ is G-ru-complete. Hence there exists $x \in [a, b]$ such that $x_n \to x(G)$. With respect to 1.12, $x_n \xrightarrow{u} x$. Convexity of H in G completes the proof. $\square$

If a convex l-subgroup H of G is G- ru-complete, then H is ru-complete, but not conversely, in general (see Example 4.9 below).

Consider the following condition for a convex l-subgroup H of G.

(I) If (x_n) is a sequence in H, u-uniformly Cauchy for some $0 < u \in G$, then there exists $0 < u' \in H$ such that (x_n) is u'-uniformly Cauchy.

There are Archimedean lattice ordered groups with all convex l-subgroups satisfying (I) (e.g., $\mathbb{R}$). Example 4.9 shows that this is not valid in all Archimedean lattice ordered groups.

Example 4.9. Let G be the set of all convergent sequences in $\mathbb{R}$ and H the set of all sequences in $\mathbb{R}$ with a finite support. If the operation $+$ and the partial order are performed componentwise, G is an Archimedean lattice ordered group and H is a convex l-subgroup of G. By [11, Theorem 4.11], G is ru-complete. In view of 4.8, H is ru-complete. The sequence (x_n) with $x_n = (1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots)$ for each $n \in \mathbb{N}$ is in H and $(x_n) \in F$ because (x_n) is u-uniformly Cauchy, $u = (1, 1, \dots) \in G$. There is no $x \in H$ with $x_n \to x(G)$. Therefore H is not G- ru-complete. Observe that H does not satisfy (I).

From 4.8 we derive at once:

Lemma 4.10. *Let G be an* ru*-complete lattice ordered group and H a convex l-subgroup of G. If H fulfils (I) then H is G-* ru*-complete.*

Let $\mathscr{C}(G)$ denote the system of all convex l-subgroups of G. If the system $\mathscr{C}(G)$ is partially ordered by the set inclusion, then it is a complete lattice. It is well-known that if $\{G_i : \ i \in I\}$ is a subset of $\mathscr{C}(G)$, then $\bigvee\limits_{i \in I} G_i$ is the subgroup of G generated by the set $\bigcup\limits_{i \in I} G_i$.

Now we are interested in the existence of a greatest G- ru-complete convex l-subgroup of G. As shown by the example, such a convex l-subgroup need not exist.

Example 4.11. Let $K = \prod\limits_{i \in \mathbb{N}} G_i$, $G_i = \mathbb{R}$ for any $i \in \mathbb{N}$. Then K is an Archimedean lattice ordered group. Let G be the set of all elements x of K such that the set $\big\{i \in \mathbb{N} : \ x(i) \ \text{is irrational}\big\}$ is finite. Then G is an l-subgroup of K. For each $i \in \mathbb{N}$, the set $G_i^0 = \big\{g \in G : \ g(j) = 0 \ \text{for any} \ j \in I, \ j \neq i\big\}$ is a convex l-subgroup of G isomorphic to G_i. It is easy to verify that all G_i^0 are G- ru-complete subsets of G.

Assume that there exists a greatest G-ru-complete convex l-subgroup H of G. Hence $G_i^0 \subseteq H$ for any $i \in I$. The convex l-subgroup G_1 of G generated by all G_i^0 consists of all elements of G with a finite support. Whence (x_n) with $x_n = \left(1, \frac{1}{\sqrt{2}}, \dots, \frac{1}{\sqrt{n}}, 0, 0, \dots\right)$ for any $n \in \mathbb{N}$ is a sequence in G_1 and also in H because of $G_1 \subseteq H$. We get $x_n \xrightarrow{u} x$ where $x = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \dots\right) \in K - G$ and $u = (1, 1, \dots) \in G$. Hence $x_n \to x(G)$, so $(x_n) \in F$. However there is no $y \in G$ with $x_n \to y(G)$. We conclude that H is not a G-ru-complete subset of G, a contradiction.

Nevertheless the following result is valid.

Theorem 4.12. *Let $S = \{G_i : i \in I\}$ be the system of all convex l-subgroups of G which are G-*ru*-complete and $H = \bigvee\limits_{i \in I} G_i$. If H satisfies (I), then H is a greatest G-*ru*-complete convex l-subgroup of G.*

P r o o f. It suffices to show that $H = \bigvee\limits_{i \in I} G_i$ is a G-ru-complete subset of G.

Let (x_n) be a sequence in $H, (x_n) \in F$. According to (I), (x_n) is u-uniformly Cauchy for some $0 < u \in H$. This yields that (x_n) is bounded in H. Thus there are $a, b \in H$, $a < b$ with $a \leq x_n \leq b$ for any $n \in \mathbb{N}$. We get $0 < b - a \in H$, $b - a \leq g_1 + \dots + g_n$, $0 < g_1 \in G_{i_1}, \dots, 0 < g_n \in G_{i_n}$ for some $i_1, \dots, i_n \in I$. As every interval of a G-ru-complete subset of G is also G-ru-complete, applying G-ru-completeness of G_{i_k}, we deduce that $[0, g_k]$ is G-ru-complete for $k = 1, \dots, n$. Using 4.6 and induction we obtain that $[0, g_1 + \dots + g_n]$ is G-ru-complete. Since $[0, b - a] \subseteq [0, g_1 + \dots + g_n]$, we infer that $[0, b - a]$ is G-ru-complete. From continuity of the operation $+$ with respect to ru-convergence it follows that $[a, b]$ is a G-ru-complete subset of G. The set inclusion $[a, b] \subseteq H$ completes the proof. □

The idea of the proof of 4.12 is the same as that of [8, 4.12]. J. J a k u b í k [19] studied convergence with a fixed regulator in lattice ordered groups and has obtained an analogous result to 4.12 by using a different procedure from that presented here.

Let H be a convex l-subgroup of G. If H satisfies (I) then H is ru-complete if and only if H is G-ru-complete. Hence we get:

Theorem 4.13. *Let $S = \{G_i : i \in I\}$ be the system of all convex l-subgroups of G which are* ru*-complete and $H = \bigvee\limits_{i \in I} G_i$. If all G_i and H satisfy (I), then H is a greatest* ru*-complete convex l-subgroup of G.*

5. Conclusing remarks and open problems

We focus our attention to a convergence in MV-algebras.

Let A be an MV-algebra and G an abelian lattice ordered group with a strong unit u such that $A = \Gamma(G, u)$ (please, see [4]).

The categorical equivalence of MV-algebras and abelian lattice ordered groups with a strong unit enables to apply the theory of lattice ordered groups to MV-algebras. In particular, R. N. Ball, G. Georgescu, I. Leustean [2] transfered the theory of l-convergence and the Cauchy completion into MV-algebras. By the restriction of an l-convergence in G to A we obtain a convergence on A making the MV-operations continuous. Such a convergence on A is called an MV-convergence. Jakubík [20] translated sequential convergence from lattice ordered groups to MV-algebras.

G. Georgescu, F. Liguori, G. Martini [13] starting from definitions in abelian lattice ordered groups, studied order convergence and the Cauchy completion arising from this convergence in MV-algebras.

Given $0 < v \in A$, the notion of a v-uniform convergence in the MV-algebra A has been defined in [9]. In this definition, the operations of the lattice ordered group G are applied. The notion of a v-Cauchy completion A^* of A is introduced and there is proved that A^* is uniquely determined up to isomorphisms over A. The relation between the Dedekind completion of A and A^* is established.

J. Jakubík has shown (unpublished result) that the definition of v-uniform convergence can be given also merely in terms of the MV-operations. This definition is equivalent to that in [9]. Moreover, v-uniform convergence on A is an MV-convergence.

Open problems

- To describe lattice ordered groups having the property (I).
- To formulate, by means of ru-convergence, conditions under which
  - (a) l-homomorphic image of an Archimedean lattice ordered group is an Archimedean lattice ordered group.
  - (b) ru-completion of an epiarchimedean lattice ordered group is an epiarchimedean lattice ordered group.
- Is the class of all ru-complete lattice ordered groups a radical class of lattice ordered groups? (The notion of a radical class of lattice ordered groups is due to J. Jakubík [16]).

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