

A DECOMPOSITION OF $(\lambda K_v)^+$ WITH EXTENDED TRIANGLES

WEN-CHUNG HUANG* — C. A. RODGER**

Dedicated to Professor Alex Rosa on the occasion of his 70th birthday

(Communicated by Peter Horák)

ABSTRACT. By an extended triangle, we mean a loop, a loop with an edge attached (known as a lollipop), or a copy of K_3 (known as a triangle). In this paper, we completely solve the problem of decomposing the graph $(\lambda K_v)^+$ into extended triangles for all possible number of loops.

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1. Introduction

Over 40 years ago, Alex Rosa showed great insight by pursuing his interest in cycle decompositions of complete graphs, and subsequent generalizations to other families of graphs. His seminal work has resulted in a wealth of mathematical literature in both pure mathematics, as discussed in this article, and applied mathematics, such as the work on neighbor designs and on optimal networks (see [1], [10], [7], for example). For this, and for his devotion to developing both this captivating area of mathematics and young mathematicians (count the second author among those in this second category — at least one time young!) we owe Alex a great debt of gratitude.

Probably the area for which Alex is most renowned is for his work on Steiner triple systems. One has only to flip through the tome he and Charlie Colbourn wrote [6] to see the richness and depth of results that have appeared over the years, many being discovered and/or inspired by Alex. In this article, we delve into a pure mathematical connection between triple systems, graph designs, and universal algebra.

2000 Mathematics Subject Classification: Primary 05B07, 05C70.

Keywords: Steiner triple system, extended triple system, decomposition, quasigroup.

A Steiner triple system (STS) is an ordered pair (S, T) , where S is a set of symbols and T is a set of 3-element subsets of S such that each pair of elements on S is a subset of exactly one element of T ; this is the definition that places STSs in the realm of experimental designs, and is one in which statisticians have been interested for as long as Alex has been alive! One can also view STSs graph theoretically, where S is the vertex set of the complete graph K_n and T is a set of cycles of length 3, the edges of which partition the edges of K_n . Viewing STSs in this way has led to the development of a vast array of literature on graph decompositions. A third definition is algebraic — one can define a quasigroup (S, \circ) from (S, T) by defining for all $x, y \in S$:

$$x \circ x = x, \tag{1}$$

and

$$x \circ y = y \circ x = z \quad \text{iff} \quad \{x, y, z\} \in T. \tag{2}$$

Such a quasigroup is clearly idempotent (by (1)) and symmetric (by (2)), but also satisfies the identity

$$(y \circ x) \circ x = y. \tag{3}$$

It can be shown that the reverse is also true: any quasigroup satisfying these three identities can be used to construct a corresponding STS. In this sense they are equivalent structures. So this relationship between such algebraic structures and graph decompositions is now ripe to be explored. For an excellent survey on this topic see [12].

Here we consider one such avenue of research: what graph decompositions, if any, does one find equivalent to the algebraic structure one obtains by dropping the idempotent requirement (1)? Such structures are known as totally symmetric quasigroups. It turns out that they correspond to decompositions of the complete graph with exactly one loop on each vertex, denoted by $K_{|S|}^+$, into loops, lollipops and 3-cycles (often called triangles in this setting) — see Figure 1. Furthermore, the graph formed by the edges in the lollipops is made up of components that are: cycles when $|S|$ is odd; and unicyclic graphs in which all vertices have odd degree when $|S|$ is even (see [9], for example). In the milieu of graph decompositions, these are known as extended triple systems, where again they have been extensively studied. For example, the embedding problem asks if it is possible for each totally symmetric quasigroup (S, \circ_1) to find a totally symmetric quasigroup (T, \circ_2) with $S \subseteq T$ such that $s_1 \circ_1 s_2 = s_1 \circ_2 s_2$ for all $s_1, s_2 \in S$? For results on this and related questions, see [15] for example. Of course, one can also ask the more basic question of just when do they exist?

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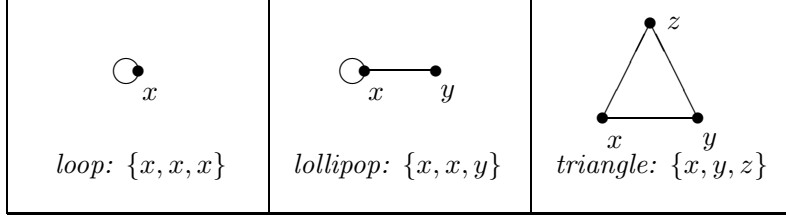


FIGURE 1. The graphs of the extended triples: loop, lollipop and triangle

More formally, an extended triple system of order v ($\text{ETS}(v)$) is a pair (V, B) , where V is a v -set and B is a collection of triples of elements in V (each triple may contain repeated elements), such that every pair of the elements of V (not necessarily distinct) belongs to exactly one triple. The elements of B are called extended triples. There are three types of extended triples:

- (1) $\{x, y, z\}$
- (2) $\{x, x, y\}$
- (3) $\{x, x, x\}$

(denoted by xyz , xyx and xxx for brevity) called a *triangle*, a *lollipop* and a *loop*, respectively. If (V, B) is an extended triple system on v elements and with a loops, then we say B is an $\text{ETS}(v, a)$. We say an $\text{ETS}(v, a)$ exists if there exists an extended triple system with parameters v and a .

This concept of an extended triple system was first introduced by D. M. Johnson and N. S. Mendelsohn [11]. They established necessary conditions for the existence of $\text{ETS}(v, a)$, and F. E. Bennet and N. S. Mendelsohn [2] showed that these necessary conditions were also sufficient.

THEOREM 1.1. ([2], [11]) *There exists an $\text{ETS}(v, a)$, if and only if, $0 \leq a \leq v$ and*

- (i) *if $v \equiv 0 \pmod{3}$ then $a \equiv 0 \pmod{3}$;*
- (ii) *if $v \not\equiv 0 \pmod{3}$ then $a \equiv 1 \pmod{3}$;*
- (iii) *if v is even then $a \leq v/2$;*
- (iv) *if $a = v - 1$ then $v = 2$.*

For any graph G , let G^+ denote the graph formed from G by attaching a single loop at each vertex, and let λK_n be the graph in which each pair of vertices is joined by exactly λ edges. We now address a related question: can the graph $(\lambda K_v)^+$ be decomposed into extended triangles with a loops? Denote such a decomposition by $\mathcal{ETS}(v, a, \lambda)$; λ is referred to as the index of the decomposition. Recently, some papers investigated the structure of these and other generalized triple systems. For example, M. E. Raines and C. A. Rodger [14], [15],

[16] considered the embedding problem for extended triple systems of arbitrary index, and see V. E. Castellana and M. E. Raines [3] for results in the directed version, namely extended Mendelsohn triple systems. In this paper we completely solve the existence problem for $\mathcal{ETS}(v, a, \lambda)$ s with the following result.

THE MAIN THEOREM. *The graph $(\lambda K_v)^+$ can be decomposed into triangles, lollipops and a loops, if and only if,*

- (1) *if $v \equiv 0 \pmod{3}$, or if $v \equiv 2 \pmod{3}$ and $\lambda \equiv 2 \pmod{3}$, then $a \equiv 0 \pmod{3}$;*
- (2) *if $v \equiv 1 \pmod{3}$, or if $v \equiv 2 \pmod{3}$ and $\lambda \equiv 1 \pmod{3}$, then $a \equiv 1 \pmod{3}$;*
- (3) *if $v \equiv 2 \pmod{3}$ and $\lambda \equiv 0 \pmod{3}$, then $a \equiv 2 \pmod{3}$;*
- (4) *if v is even and λ is odd then $a \leq v/2$; and*
- (5) *$0 \leq a \leq v$, and if $(v, \lambda) \neq (2, 1)$ then $a \neq v - 1$.*

Conditions (1)–(3) are necessary, since the number of edges in triangles, $\lambda v(v-1)/2 - l$, must be divisible by 3, where $l = v - a$ is the number of lollipops. Condition (4) is necessary since each triangle uses an even number of edges at each vertex, so each vertex is incident with an odd number of lollipops, implying that $l \geq a$. Clearly $0 \leq a \leq v$, and removing a lone lollipop could not leave a graph in which all vertices have even degree unless $(v, \lambda) = (2, 1)$. To prove the sufficiency, the problem naturally divides itself into 36 cases for λ and $v \pmod{6}$. The following table summarizes our approach, where I, II and III refer to the cases $a \equiv 1, 2$ and $0 \pmod{3}$, respectively.

TABLE 1

$\lambda \pmod{6}$							
$v \pmod{6}$		1	2	3	4	5	6
	1	I	I	I	I	I	I
	2	I	III	II	I	III	II
	3	III	III	III	III	III	III
	4	I	I	I	I	I	I
	5	I	III	II	I	III	II
	6	III	III	III	III	III	III

To prove the Main Theorem, we will make use of the following results in the literature.

A λ -fold triple system of order v , $\text{TS}(v, \lambda)$, is a pair (S, T) , where S is a v -set and T is a collection of 3-element subsets of S called triples such that each pair of the distinct elements of S belongs to exactly λ triples of T . Just as with Steiner triple systems, we can think of a λ -fold triple system as a decomposition

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of λK_v into triangles. The spectrum for λ -fold triple systems of order v (i.e. the values of v and λ for which a $\text{TS}(v, \lambda)$ exists) is as follows (see [13] for a proof).

THEOREM 1.2. *The following table gives the necessary and sufficient conditions for the existence of a λ -fold triple system of order v .*

TABLE 2. *The spectrum of λ -fold triple systems*

λ	<i>the spectrum of λ-fold triple systems</i>
0 (mod 6)	<i>all $v \neq 2$</i>
1 or 5 (mod 6)	<i>all $v \equiv 1$ or 3 (mod 6)</i>
2 or 4 (mod 6)	<i>all $v \equiv 0$ or 1 (mod 3)</i>
3 (mod 6)	<i>all odd v</i>

In some constructions we use packings of the complete graph K_v . A packing of the graph G with triangles (or a packing of G for brevity) is a set M of edge-disjoint triangles in G ; the leave of the packing M is the set of edges of G not belonging to any triangles in M . When the cardinality of the triangle set M is a maximum, the packing is called the maximum packing of G , and the corresponding minimum leave is denoted by $L(G)$.

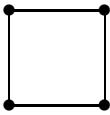
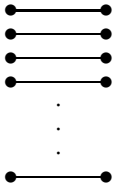
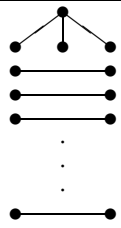
$v \equiv 1$ or 3 (mod 6)	$v \equiv 5$ (mod 6)	$v \equiv 0$ or 2 (mod 6)	$v \equiv 4$ (mod 6)
\emptyset			
STS	4-cycle	1-factor	tripole

FIGURE 2. Leaves of maximum packing

- The set $L(K_v)$ is (see Figure 2, and [13], for example),
- (i) an empty set if $v \equiv 1$ or 3 (mod 6);
 - (ii) a 4-cycle if $v \equiv 5$ (mod 6);
 - (iii) a 1-factor if $v \equiv 0$ or 2 (mod 6);
 - (iv) a tripole if $v \equiv 4$ (mod 6).

Another memorable result of Rosa and Colbourn is the following. Not only is the result a precursor of several more in the literature (see [5]), but also

the proof is an intricate, natural application of difference methods, and well worth reading. Throughout this paper, if H is a subgraph of G then let $G \setminus H$ denote the graph formed from G by removing the edges and loops in H .

LEMMA 1.3. ([5]) *Let t be an odd positive integer. Let H be a 2-regular subgraph of K_t . Then $C_3|(K_t \setminus H)$ if and only if the number of edges in $K_t \setminus H$ is a multiple of 3, where $H \neq C_4 \cup C_5$ for $t = 9$.*

The next two results are also very useful. The first is easy to obtain using the direct product of a commutative quasigroup with holes of size 2 and a latin square of size 2. The second result is surprisingly complicated to obtain, being recently proved in [8].

LEMMA 1.4. *For all $n \geq 3$, there exists a commutative quasigroup of order $4n$ with holes $H = \{ \{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \dots, \{4n-3, 4n-2, 4n-1, 4n\} \}$.*

LEMMA 1.5. ([8]) *For all $n \geq 4$, there exists a commutative quasigroup of order $4n+2$ with holes $H = \{ \{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \dots, \{4n-7, 4n-6, 4n-5, 4n-4\}, \{4n-3, 4n-2, 4n-1, 4n, 4n+1, 4n+2\} \}$.*

Finally, we will also use the next result which has been proved in more generality, but this will suffice for our purposes.

LEMMA 1.6. ([4]) *Let t be nonnegative integer. Then there exists a 3-GDD of type $3^t 1^1$ if and only if t is even, $t \geq 4$.*

2. Holey constructions and small cases

In this section, some decompositions of small order are presented. We generalize notation in a natural way, defining an extended triple system of a graph G (G may have multiple edges and loops) to be a partition of the edges and loops of G into sets, each of which induces an extended triple; we denote such a decomposition by \mathcal{ETS} or $\mathcal{ETS}(G)$. Several of these constructions make use of extended triple systems of complete graphs with holes (that is, of $K_v^+ \setminus K_w^+$; so there are no loops on the w vertices in the hole), which we now construct. The following result is essentially the observation of Stern and Lenz (see [13], for example), as can be seen by removing the 1-factor F_0 before applying their result; here we require the special 1-factor F_0 to be a part of the 1-factorization. A proof is provided in this simpler setting for the interested reader.

LEMMA 2.1 (Stern-Lenz Lemma). *Let G be a regular graph on the vertex set Z_n , let G_0 be formed from G by renaming each vertex $j \in Z_n$ with $(j, 0)$, and let G_1 be formed from G by renaming each vertex $j \in Z_n$ with $(j + \alpha, 1)$ for some $\alpha \in \{1, \dots, n-1\}$. Let B be a regular bipartite graph with bipartition*

$\{Z_n \times \{0\}, Z_n \times \{1\}\}$ that contains the edges in $\{\{(j, 0), (j, 1)\}, \{(j, 0), (j + \alpha, 1)\} : j \in Z_n\}$. Then $G_0 \cup G_1 \cup B$ has a 1-factorization, one 1-factor in which is $F_0 = \{\{(j, 0), (j, 1)\} : j \in Z_n\}$.

Proof. Let G be x -regular. Give G_0 a proper $(x + 1)$ -edge-coloring f (using Vizing's Theorem). Color each edge $\{(u + \alpha, 1)(v + \alpha, 1)\}$ of G_1 with $f(\{(u, 0), (v, 0)\})$. At each vertex $(v, 0) \in V(G_0)$ exactly one color is missing (does not occur on an edge incident with $(v, 0)$); color the edge $\{(v, 0), (v + \alpha, 1)\}$ with this missing color. The remaining uncolored edges in $B \setminus E(F_0)$ induce a y -regular bipartite graph, so can be properly colored with y colors. The resulting $x + y + 1$ -edge-coloring together with the 1-factor F_0 gives the desired 1-factorization. \square

LEMMA 2.2. *There exists an extended triple system of $K_{16}^+ \setminus K_4^+$ and of $K_{17}^+ \setminus K_5^+$, each of which contains a set T of 4 triangles (dotted lines in Figure 3) partitioning the 12 vertices not in the hole. Furthermore, the first graph contains 6 lollipops (the solid lines in Figure 3) and contains 6 loops that occur on the vertices in 2 of the triangles in T , and the second graph has 12 loops.*

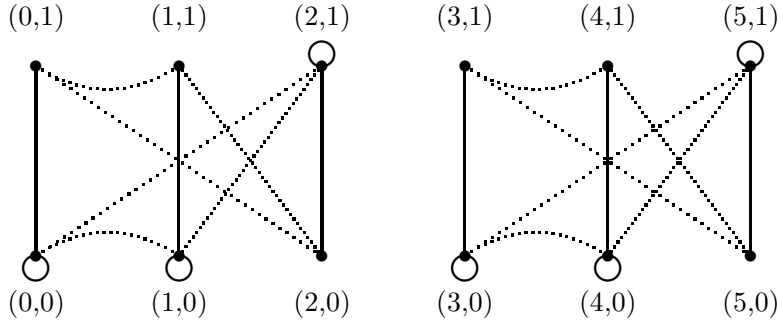


FIGURE 3. Position of triangles and lollipops in $K_{16}^+ \setminus K_4^+$

Proof. We begin with a useful decomposition of K_{12} . Let G be a graph on the vertex set $Z_6 \times Z_2$ whose edge set is the union of the edges of the triangles in $C = \{\{(j, i), (j + 1, i), (j + 2, i + 1)\} : j \in Z_6, i \in Z_2\}$. Note that C contains the 4 vertex-disjoint triangles in $T = \{\{(0, i), (1, i), (2, i + 1)\}, \{(3, i), (4, i), (5, i + 1)\} : i \in Z_2\}$. Then G is a 6-regular graph, and its 5-regular complement satisfies the conditions of the Stern-Lenz Lemma (with $\alpha = 3$), so has a 1-factorization $F = \{F_i : i \in Z_5\}$ in which $F_0 = \{\{(j, 0), (j, 1)\} : j \in Z_6\}$. \square

Let $n \in \{16, 17\}$. Add $n - 12$ new vertices, and for $1 \leq i \leq n - 12$ add the i th new vertex to each edge in F_{5-i} to form a set of triangles. If $n = 16$ then add lollipops containing the edges in F_0 , and add a loop on the other end of each lollipop; do this so that the 6 loops are on the vertices of two of 4 disjoint

triangles in T , say on the vertices in $\{(0, 1), (1, 1), (2, 0), (3, 1), (4, 1), (5, 0)\}$. If $n = 17$ then add 12 loops to the vertices in $\{(i, j) : i \in \mathbb{Z}_6, j \in \mathbb{Z}_2\}$. Combining all these extended triangles forms an \mathcal{ESTS} of $K_{16}^+ \setminus K_4^+$ or of $K_{17}^+ \setminus K_5^+$ when n is 16 or 17 respectively.

LEMMA 2.3. *There exists an extended triple system of $K_{22}^+ \setminus K_4^+$ and of $K_{23}^+ \setminus K_5^+$, each of which contains a set T of 6 triangles (dotted lines in Figure 4) partitioning the 18 vertices not in the hole. Furthermore, the first graph contains 9 lollipops (the solid lines in Figure 4) and contains 9 loops that occur on the vertices in 3 of the triangles in T , and the second graph has 18 loops.*

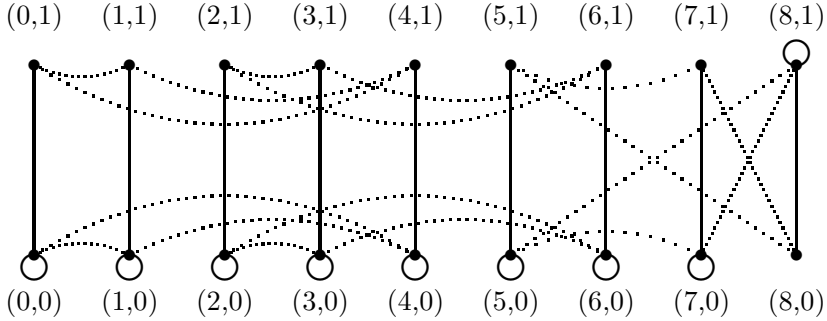


FIGURE 4. Position of triangles and lollipops in $K_{22}^+ \setminus K_4^+$

Proof. We begin with a useful decomposition of K_{18} . Let G be a graph on the vertex set $\mathbb{Z}_9 \times \mathbb{Z}_2$ whose edge set is the union of the edges of the triangles in $C = \{\{(j, i), (j+1, i), (j+4, i)\}, \{(j, i), (j+2, i), (j+3, i+1)\} : j \in \mathbb{Z}_9, i \in \mathbb{Z}_2\}$. Note that C contains the 6 disjoint triangles in $T = \{\{(0, i), (1, i), (4, i)\}, \{(2, i), (3, i), (6, i)\}, \{(5, i), (7, i), (8, i+1)\} : i \in \mathbb{Z}_2\}$. Then G is a 12-regular graph and its 5-regular complement satisfies the conditions of the Stern-Lenz Lemma (with $\alpha = 2$ for example), so has a 1-factorization $F = \{F_i : i \in \mathbb{Z}_5\}$ in which $F_0 = \{\{(j, 0), (j, 1)\} : j \in \mathbb{Z}_9\}$.

Let $n \in \{22, 23\}$. Add $n - 18$ new vertices, and for $1 \leq i \leq n - 18$ add the i th new vertex to each edge in F_{5-i} to form a set of triangles. If $n = 22$ then add lollipops containing the edges in F_0 , and add a loop on the other end of each lollipop; do this so that the 9 loops are on the vertices of three of 6 disjoint triangles in T , say on the vertices in $\{(0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (7, 1), (8, 2)\}$. If $n = 23$ then add 18 loops to the vertices in $\{(i, j) : i \in \mathbb{Z}_9, j \in \mathbb{Z}_2\}$. Combining all these extended triangles forms an \mathcal{ESTS} of $K_{22}^+ \setminus K_4^+$ or of $K_{23}^+ \setminus K_5^+$ when n is 22 or 23 respectively. \square

LEMMA 2.4. *There exists an extended triple system of $K_{28}^+ \setminus K_4^+$ and of $K_{29}^+ \setminus K_5^+$, each of which contains a set T of 8 triangles partitioning the 24 vertices not in the hole. Furthermore, the first graph contains 12 lollipops and contains 12 loops that occur on the vertices in 4 of the triangles in T , and the second graph has 24 loops.*

Proof. We begin with a useful decomposition of K_{24} . Let G be a graph on the vertex set $Z_{12} \times Z_2$ whose edge set is the union of the edges of the triangles in $C = \{\{(j, i), (j+1, i), (j+4, i)\}, \{(j, i), (j+5, i), (j+4, i+1)\}, \{(j, i), (j+2, i), (j+5, i+1)\} : j \in Z_{12}, i \in Z_2\}$. Note that C contains the 8 disjoint triangles in $T = \{\{(0, i), (1, i), (4, i)\}, \{(2, i), (3, i), (6, i)\}, \{(7, i), (8, i), (11, i)\}, \{(5, i), (10, i), (9, i+1)\} : i \in Z_2\}$. Then G is an 18-regular graph, and its 5-regular complement satisfies the conditions of the Stern-Lenz Lemma (with $\alpha = 2$ for example), so has a 1-factorization $F = \{F_i : i \in Z_5\}$ in which $F_0 = \{\{(j, 0), (j, 1)\} : j \in Z_{12}\}$.

Now proceed as in the two previous lemmas to produce an \mathcal{ESTS} of $K_{28}^+ \setminus K_4^+$ and an \mathcal{ESTS} of $K_{29}^+ \setminus K_5^+$ as required. \square

LEMMA 2.5. *There exists an extended triple system of $K_{34}^+ \setminus K_4^+$ and of $K_{35}^+ \setminus K_5^+$, each of which contains a set T of 10 triangles partitioning the 30 vertices not in the hole. Furthermore, the first graph contains 15 lollipops and contains 15 loops that occur on the vertices in 5 of the triangles in T , and the second graph has 30 loops.*

Proof. We begin with a useful decomposition of K_{30} . Let G be a graph on the vertex set $Z_{15} \times Z_2$ whose edge set is the union of the edges of the triangles in $C = \{\{(j, i), (j+1, i), (j+5, i)\}, \{(j, i), (j+2, i), (j+8, i+1)\}, \{(j, i), (j+3, i), (j+5, i+1)\}, \{(j, i), (j+7, i), (j+4, i+1)\} : j \in Z_{15}, i \in Z_2\}$. Note that C contains the 10 disjoint triangles in $T = \{\{(0, i), (1, i), (5, i)\}, \{(2, i), (3, i), (7, i)\}, \{(9, i), (10, i), (14, i)\}, \{(4, i), (6, i), (12, i+1)\}, \{(8, i), (11, i), (13, i+1)\} : i \in Z_2\}$. Then G is a 24-regular graph, and its 5-regular complement satisfies the conditions of the Stern-Lenz Lemma (with $\alpha = 1$), so has a 1-factorization $F = \{F_i : i \in Z_5\}$ in which $F_0 = \{\{(j, 0), (j, 1)\} : j \in Z_{15}\}$.

Now proceed as before to produce an \mathcal{ESTS} of $K_{34}^+ \setminus K_4^+$ and an \mathcal{ESTS} of $K_{35}^+ \setminus K_5^+$ as required. \square

LEMMA 2.6. *There exists an extended triple system of $K_{46}^+ \setminus K_4^+$ and of $K_{47}^+ \setminus K_5^+$, each of which contains a set T of 14 triangles partitioning the 42 vertices not in the hole. Furthermore, the first graph contains 21 lollipops and contains 21 loops that occur on the vertices in 7 of the triangles in T , and the second graph has 42 loops.*

Proof. We begin with a useful decomposition of K_{42} . Let G be a graph on the vertex set $Z_{21} \times Z_2$ whose edge set is the union of the edges of the triangles in

$C = \{ \{(j, i), (j+1, i), (j+5, i)\}, \{(j, i), (j+2, i), (j+6, i+1)\}, \{(j, i), (j+3, i), (j+1, i+1)\}, \{(j, i), (j+6, i), (j+13, i)\}, \{(j, i), (j+9, i), (j+14, i+1)\}, \{(j, i), (j+10, i), (j+13, i+1)\} : j \in Z_{21}, i \in Z_2 \}$. Note that C contains the 14 disjoint triangles in $T = \{ \{(0, i), (1, i), (5, i)\}, \{(2, i), (3, i), (7, i)\}, \{(8, i), (9, i), (13, i)\}, \{(11, i), (12, i), (16, i)\}, \{(14, i), (15, i), (19, i)\}, \{(4, i), (6, i), (10, i+1)\}, \{(17, i), (20, i), (18, i+1)\} : i \in Z_2 \}$. Then G is a 36-regular graph, and its 5-regular complement satisfies the conditions of the Stern-Lenz Lemma (with $\alpha = 9$ for example), so has a 1-factorization $F = \{F_i : i \in Z_5\}$ in which $F_0 = \{ \{(j, 0), (j, 1)\} : j \in Z_{21} \}$.

Now proceed as before to produce an \mathcal{ESTS} of $K_{46}^+ \setminus K_4^+$ and an \mathcal{ESTS} of $K_{47}^+ \setminus K_5^+$ as required. \square

The following special examples, and the small cases obtained in Lemma 2.8, will all be used in the proof of The Main Theorem provided in Section 3.

Example 2.7.

- (a) $\{336, 448, 557, 666, 777, 888, 147, 258, 345, 678, 156, 237, 246, 138\}$ is an $\mathcal{ESTS}(K_8^+ \setminus K_2^+)$ with $V(K_2) = \{1, 2\}$ containing the triangle 678 with loops on each vertex.
- (b) $\{345, 678, 245, 256, 136, 267, 478, 157, 147, 158, 138, 237, 146, 238, 357, 248, 346, 568\} \cup \{iii : i = 3, 4, \dots, 8\}$ is an $\mathcal{ESTS}((2K_8)^+ \setminus (2K_2)^+)$ with $V(2K_2) = \{1, 2\}$ containing vertex disjoint triangles 345 and 678, each of which has a loop on each of its vertices.

From the proof of Lemmas 2.2–2.6, the decompositions $\mathcal{ESTS}(K_i^+ \setminus K_5^+)$ exist, where $i = 17, 23, 29, 35, 47$. For convenience, we will write t_i for $10+i$ and t for 10 , and define $A+B = \{a+b : a \in A, b \in B\}$, where A and B are two sets of integers. We now turn to constructing $\mathcal{ESTS}(v, a, \lambda)$'s when v is small.

LEMMA 2.8. *Let $v \leq 47$ with $v \notin \{40, 41\}$, let $\lambda = 1$ if $v \equiv 4 \pmod{6}$, and let $\lambda \in \{1, 2\}$ if $v \in \{6, 8, 14\}$ or if $v \equiv 5 \pmod{6}$. There exists an $\mathcal{ESTS}(v, a, \lambda)$ if Conditions 1–5 of The Main Theorem are satisfied.*

Proof.

- $v = 4$. $\mathcal{ESTS}(4, 1, 1)$: $\{111, 122, 133, 144, 234\}$.
- $v = 5$. $\mathcal{ESTS}(5, 1, 1)$: $\{123, 145, 224, 443, 335, 552, 111\}$;
 $\mathcal{ESTS}(5, 0, 2)$: $\{112, 223, 334, 445, 551, 135, 352, 524, 241, 413\}$;
 $\mathcal{ESTS}(5, 3, 2)$: $\{112, 221, 333, 444, 555, 234, 134, 135, 245, 235, 145\}$.
- $v = 6$. $\mathcal{ESTS}(6, 0, 1)$: $\{112, 223, 331, 441, 552, 663, 156, 246, 345\}$;
 $\mathcal{ESTS}(6, 3, 1)$: $\{113, 226, 445, 333, 666, 555, 124, 235, 346, 156\}$;
 $\mathcal{ESTS}(6, 0, 2)$: $\{124, 235, 346, 156, 124, 235, 346, 156\}$
 $\cup \{113, 331, 226, 662, 445, 554\}$;

$\mathcal{ETS}(6, 3, 2)$: $\{333, 444, 666, 112, 225, 551, 261, 641, 456, 531, 245, 234, 356, 236, 134\}$;
 $\mathcal{ETS}(6, 6, 2)$: $\{125, 261, 641, 456, 531, 245, 234, 356, 236, 134\}$
 $\cup \{iii : i = 1, 2, \dots, 6\}$;
 $v = 8$. $\mathcal{ETS}(8, 1, 1)$: $\{111, 122, 133, 144, 156, 178, 235, 246, 277, 288, 347, 368, 458, 557, 667\}$;
 $\mathcal{ETS}(8, 4, 1)$: $\{114, 126, 137, 158, 223, 248, 257, 333, 345, 368, 444, 467, 555, 566, 777, 788\}$.
 $\{671, 682, 124, 258, 834, 815, 486, 537, 543, 138, 367, 156, 147, 247, 257, 236, 778, 887\} \cup S$ is an $\mathcal{ETS}(8, 6, 2)$, where $S = \{123, 456\} \cup \{iii : i = 1, 2, \dots, 6\}$.
 Replacing S with $\{112, 223, 331, 445, 556, 664\}$ or $\{112, 223, 331, 456, 444, 555, 666\}$, we obtain the decompositions $\mathcal{ETS}(8, 0, 2)$ and $\mathcal{ETS}(8, 3, 2)$.
 $v = 10$. $\mathcal{ETS}(10, 1, 1)$: $\{113, 122, 144, 15t, 167, 189, 233, 247, 255, 269, 28t, 348, 357, 366, 39t, 459, 46t, 568, 777, 788, 799, 7tt\}$;
 $\mathcal{ETS}(10, 4, 1)$: $\{444, 555, 777, 999, 221, 331, 114, 665, 887, tt9, 345, 157, 169, 18t, 238, 249, 25t, 267, 36t, 379, 468, 47t, 589\}$.
 $v = 11$. $\mathcal{ETS}(11, 1, 1)$: $\{111, 127, 138, 149, 15t, 16t_1, 223, 24t, 258, 266, 29t_1, 334, 35t_1, 369, 37t, 445, 467, 48t_1, 556, 579, 68t, 778, 7t_1t_1, 889, 99t, ttt_1\}$;
 $\mathcal{ETS}(11, 4, 1)$: $\{113, 122, 145, 167, 18t_1, 19t, 235, 244, 268, 279, 2tt_1, 334, 369, 37t_1, 38t, 46t, 478, 49t_1, 556, 57t, 589, 5t_1t_1, 66t_1, 777, 888, 999, ttt\}$;
 $\mathcal{ETS}(11, 7, 1)$: $\{111, 123, 145, 167, 18t_1, 19t, 225, 244, 268, 279, 2tt_1, 334, 355, 369, 37t_1, 38t, 46t, 478, 49t_1, 56t_1, 57t, 589, 666, 777, 888, 999, ttt, t_1t_1t_1\}$;
 $\{682, 127, 14t, 249, 2t8, 834, 815, 4t_16, 637, 543, 138, 369, 156, 147, 24t, 257, 23t_1, 39t_1, 1t9, 269, 52t_1, 87t_1, 1t_16, 57t_1, 37t, 59t, 35t, 67t, 589, 48t_1, 68t, 1t_19, 479, ttt_1, t_1t_1t\} \cup S$ is an $\mathcal{ETS}(11, 9, 2)$, where $S = \{123, 456, 789\} \cup \{iii : i = 1, 2, \dots, 9\}$.
 Replacing S with $\{112, 223, 331, 445, 556, 664, 778, 889, 997\}$, $\{112, 223, 331, 445, 556, 664, 789, 777, 888, 999\}$ or $\{112, 223, 331, 456, 789, 444, 555, 666, 777, 888, 999\}$, we obtain the decompositions $\mathcal{ETS}(11, 0, 2)$, $\mathcal{ETS}(11, 3, 2)$ and $\mathcal{ETS}(11, 6, 2)$.
 $v = 14$. $\mathcal{ETS}(14, 1, 1)$: $\{t_4t_4t_4\} \cup \{iit_4 : i = 1, 2, \dots, 13\} \cup S$, where S is a Steiner triple system of order 13 on the set $\{1, 2, \dots, 13\}$;
 $\mathcal{ETS}(14, 4, 1)$: $\{112, 133, 144, 157, 168, 19t, 1t_1t_4, 1t_2t_3, 222, 234, 256, 27t_1, 28t, 29t_3, 2t_2t_4, 35t_1, 36t_3, 37t_2, 389, 3tt_4, 455, 466, 47t_3, 48t_4, 49t_2, 4tt_1, 58t_2, 59t_4, 5tt_3, 67t_4, 69t_1, 6tt_2, 778, 799, 7tt, 888, 8t_1t_3, t_1t_1t_2, t_2t_2t_2, t_3t_3t_4, t_4t_4t_4\}$;
 $\mathcal{ETS}(14, 7, 1)$: Removing an element from Steiner triple system of order 15, the resulting system contains a 1-factor. Replacing the 1-factor by 7 lollipops and 7 loops, we can obtain the decomposition $\mathcal{ETS}(14, 7, 1)$.
 $\{68t_2, 127, t_34t, 249, t_298, 835, 815, 4t_16, 6t_47, 54t_2, 438, t_369, 156, 147, 2t_4t, 2t_27, 23t_1, 39t_1, 1tt_2, 269, 52t_1, 87t_1, 1t_16, 5t_2t_1, 37t, 59t, 35t, 67t, 52t_3, 48t_1,$

$68t, t_4t_19, 479, 1t_2t_3, 1t_44, 24t, 2t_26, 28t_3, 37t_2, 3t_2t_4, 4t_2t_4, 34t_3, 36t_4, 1t_9, 82t_4, tt_1t_3, 57t_3, 7t_1t_3, 57t_4, 18t_3, 139, 59t_4, 1t_1t_4, 8tt_4, 9t_2t_3, 36t_3, t_3t_3t_4, t_4t_4t_3\} \cup S$ is an $\mathcal{ETS}(14, 12, 2)$, where $S = \{123, 456, 789, tt_1t_2\} \cup \{iii, ttt, t_1t_1t_1, t_2t_2t_2 : i = 1, 2, \dots, 9\}$. Replacing S with $\{112, 223, 331, 445, 556, 664, 778, 889, 997, ttt_1, t_1t_1t_2, t_2t_2t_2\}$, $\{112, 223, 331, 445, 556, 664, 778, 889, 997, tt_1t_2, ttt, t_1t_1t_1, t_2t_2t_2\}$, $\{112, 223, 331, 445, 556, 664, 789, tt_1t_2, 777, 888, 999, ttt, t_1t_1t_1, t_2t_2t_2\}$ or $\{112, 223, 331, 456, 789, tt_1t_2, 444, 555, 666, 777, 888, 999, ttt, t_1t_1t_1, t_2t_2t_2\}$, we obtain the decompositions $\mathcal{ETS}(14, 0, 2)$, $\mathcal{ETS}(14, 3, 2)$, $\mathcal{ETS}(14, 6, 2)$ and $\mathcal{ETS}(14, 9, 2)$.

$v = 16, 17$. From the extended triple systems formed in Lemma 2.2, other extended triples systems can be formed by replacing triples that have three incident loops with three lollipops. So from Lemma 2.2 and the systems $\mathcal{ETS}(4, 1, 1)$, $\mathcal{ETS}(5, 1, 1)$, we can obtain the decompositions $\mathcal{ETS}(16, s, 1)$ and $\mathcal{ETS}(17, t, 1)$, for $s = 1, 4, 7$ and $t = 1, 4, 7, 10, 13$, respectively. Since $(2K_{17})^+ = (K_{17}^+ \setminus K_5^+) \cup (K_{17} \setminus K_5) \cup (2K_5)^+$, since \mathcal{ETS} of graphs $K_{17}^+ \setminus K_5^+$, $K_{17} \setminus K_5$, $(2K_5)^+$ exist, and noting that the number of loops in each desired system is in $\{0, 3, 6, \dots, 12\} + \{0, 3\} = \{0, 3, 6, \dots, 12, 15\} = A$, we can obtain the decompositions $\mathcal{ETS}(17, t, 2)$, for all $t \in A$.

$v = 22, 23$. From Lemma 2.3 and the systems $\mathcal{ETS}(4, 1, 1)$, $\mathcal{ETS}(5, 1, 1)$, we can obtain the decompositions $\mathcal{ETS}(22, s, 1)$ and $\mathcal{ETS}(23, t, 1)$, for $s = 1, 4, 7, 10$ and $t = 1, 4, 7, 10, 13, 16, 19$, respectively. Since $(2K_{23})^+ = (K_{23}^+ \setminus K_5^+) \cup (K_{23} \setminus K_5) \cup (2K_5)^+$, since \mathcal{ETS} of graphs $K_{23}^+ \setminus K_5^+$, $K_{23} \setminus K_5$, $(2K_5)^+$ exist, and noting that the number of loops in each desired system is in $\{0, 3, 6, \dots, 18\} + \{0, 3\} = \{0, 3, 6, \dots, 18, 21\} = A$, we can obtain the decompositions $\mathcal{ETS}(23, t, 2)$, for all $t \in A$.

$v = 28, 29$. From Lemma 2.4 and the systems $\mathcal{ETS}(4, 1, 1)$, $\mathcal{ETS}(5, 1, 1)$, we can obtain the decompositions $\mathcal{ETS}(28, s, 1)$ and $\mathcal{ETS}(29, t, 1)$, for $s = 1, 4, 7, 10, 13$ and $t = 1, 4, 7, 10, 13, 16, 19, 22, 25$, respectively. Since $(2K_{29})^+ = (K_{29}^+ \setminus K_5^+) \cup (K_{29} \setminus K_5) \cup (2K_5)^+$, since \mathcal{ETS} of graphs $K_{29}^+ \setminus K_5^+$, $K_{29} \setminus K_5$, $(2K_5)^+$ exist, and noting that the number of loops in each desired system is in $\{0, 3, 6, \dots, 24\} + \{0, 3\} = \{0, 3, 6, \dots, 24, 27\} = A$, we can obtain the decompositions $\mathcal{ETS}(29, t, 2)$, for all $t \in A$.

$v = 34, 35$. From Lemma 2.5 and the systems $\mathcal{ETS}(4, 1, 1)$, $\mathcal{ETS}(5, 1, 1)$, we can obtain the decompositions $\mathcal{ETS}(34, s, 1)$ and $\mathcal{ETS}(35, t, 1)$, for $s = 1, 4, 7, 10, 13, 16$ and $t = 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31$, respectively. Since $(2K_{35})^+ = (K_{35}^+ \setminus K_5^+) \cup (K_{35} \setminus K_5) \cup (2K_5)^+$, since \mathcal{ETS} of graphs $K_{35}^+ \setminus K_5^+$, $K_{35} \setminus K_5$, $(2K_5)^+$ exist, and noting that the number of loops in each desired system is in $\{0, 3, 6, \dots, 30\} + \{0, 3\} = \{0, 3, 6, \dots, 30, 33\} = A$, we can obtain the decompositions $\mathcal{ETS}(35, t, 2)$, for all $t \in A$.

$v = 46, 47$. From Lemma 2.6 and the systems $\mathcal{ETS}(4, 1, 1)$, $\mathcal{ETS}(5, 1, 1)$, we can obtain the decompositions $\mathcal{ETS}(46, s, 1)$ and $\mathcal{ETS}(47, t, 1)$, for $s = 1, 4, 7, 10, 13, 16, 19, 22$ and $t = 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43$, respectively. Since $(2K_{47})^+ = (K_{47}^+ \setminus K_5^+) \cup (K_{47} \setminus K_5) \cup (2K_5)^+$, since \mathcal{ETS} of graphs $K_{47}^+ \setminus K_5^+$, $K_{47} \setminus K_5$, $(2K_5)^+$ exist, and the number of loops in each desired system is in $\{0, 3, 6, \dots, 42\} + \{0, 3\} = \{0, 3, 6, \dots, 42, 45\} = A$, we can obtain the decompositions $\mathcal{ETS}(47, t, 2)$, for all $t \in A$. \square

3. Decompositions of $(\lambda K_v)^+$

We now provide some neat methods which together will prove the sufficiency of the Main Theorem. The constructions are modifications of the Bose and Skolem Constructions (see [13] for example). Since $L(\lambda K_v) = \emptyset$ for $\lambda \equiv 0 \pmod{6}$, we need only consider $\lambda = 1, 2, 3, 4, 5, 6$.

PROPOSITION 3.1. *For $v \equiv 0 \pmod{6}$, the graph $(\lambda K_v)^+$ can be decomposed into triangles, lollipops and a loops, where $a \equiv 0 \pmod{3}$ and if λ is odd then $a \leq v/2$.*

Proof. Let $v = 6n$, for some integer n . Since the cases with $v = 6$ and $\lambda \leq 2$ are considered in Lemma 2.8, and since there exists a $TS(6, 2)$, we can assume that $n \geq 2$.

Case 1: $\lambda \in \{1, 3, 5\}$.

Let (Q, \circ) be a half-idempotent commutative quasigroup of order $2n$, where $Q = \{1, 2, \dots, 2n\}$ and $n \geq 2$. Let $X = Q \times \{1, 2, 3\}$, and define the $\mathcal{ETS}(v, v/2, 1)$ (X, B_1) by the following four types of blocks.

Type 1: For $1 \leq i \leq n$, $\{(i, 1), (i, 2), (i, 3)\} \in B_1$.

Type 2: For $1 \leq i < j \leq 2n$, $\{(i, 1), (j, 1), (i \circ j, 2)\}$, $\{(i, 2), (j, 2), (i \circ j, 3)\}$, $\{(i, 3), (j, 3), (i \circ j, 1)\} \in B_1$.

Type 3: For $1 \leq i \leq n$, $\{(i+n, 1), (i+n, 1), (i, 2)\}$, $\{(i+n, 2), (i+n, 2), (i, 3)\}$, $\{(i+n, 3), (i+n, 3), (i, 1)\} \in B_1$.

Type 4: For $1 \leq i \leq n$ and $1 \leq j \leq 3$, $\{(i, j), (i, j), (i, j)\} \in B_1$.

Let (X, B_2) be a $TS(v, \lambda-1)$ (see Table 2). Then $B_1 \cup B_2$ is an $\mathcal{ETS}(v, v/2, \lambda)$ with a half parallel class T of Type 1. Since $|T| = v/6$, let $T' \subseteq T$ with $|T'| = v/6 - a/3$. Removing the blocks $\{xyz, xxx, yyy, zzz : xyz \in T'\}$ and replacing them with $\{xxy, yyz, zzx : xyz \in T'\}$ produces an $\mathcal{ETS}(v, a, \lambda)$.

Case 2: $\lambda \in \{2, 4, 6\}$.

Let (Q, \circ) be an idempotent (non-commutative) quasigroup of order $2n$, where $Q = \{1, 2, \dots, 2n\}$ and $n \geq 2$. Let $X = Q \times \{1, 2, 3\}$, and let B contain the following three types of blocks.

Type 1: For $1 \leq i \leq 2n$, λ copies of $\{(i, 1), (i, 2), (i, 3)\}$.

Type 2: For $1 \leq i \neq j \leq 2n$, $\lambda/2$ copies of $\{(i, 1), (j, 1), (i \circ j, 2)\}$, $\{(i, 2), (j, 2), (i \circ j, 3)\}$, $\{(i, 3), (j, 3), (i \circ j, 1)\}$.

Type 3: For $1 \leq i \leq 2n$ and $1 \leq j \leq 3$, $\{(i, j), (i, j), (i, j)\} \in B$.

Then (X, B) is an $\mathcal{ESTS}(v, v, \lambda)$ with a parallel class T of Type 1. Since $|T| = v/3$, let $T' \subseteq T$ with $|T'| = (v - a)/3$. Removing the blocks $\{xyz, xxx, yyy, zzz : xyz \in T'\}$ and replacing them with $\{xxy, yyz, zzz : xyz \in T'\}$ produces an $\mathcal{ESTS}(v, a, \lambda)$. \square

PROPOSITION 3.2. *For $v \equiv 1 \pmod{6}$, the graph $(\lambda K_v)^+$ can be decomposed into triangles, lollipops and a loops, where $a \equiv 1 \pmod{3}$ and $0 \leq a \leq v$.*

Proof. By Lemma 1.6, there exists a 3-GDD of type $3^{(v-1)/3}1^1$, so we have a $\text{TS}(v, \lambda)$ with a near parallel class T of size $(v - 1)/3$. Let $T' \subseteq T$ with $|T'| = (v - a)/3$. Adding a loop to each vertex, then removing the blocks in $\{xyz, xxx, yyy, zzz : xyz \in T'\}$ and replacing them with the blocks in $\{xxy, yyz, zzz : xyz \in T'\}$ produces an $\mathcal{ESTS}(v, a, \lambda)$. \square

When $v \equiv 2 \pmod{6}$ we will make use of packings of λK_v with triangles, so we now easily obtain these packings using Figure 2. Let the underlying vertex set of λK_{6k+2} be $S = \{x_i, y_i : 1 \leq i \leq 3k\} \cup \{a, b\}$. Let T_1 be the set of triples in a Steiner triple system on the vertex set $S \setminus \{a\}$, and let T_2 be the set of triples in a maximum packing with triples on the vertex set S with leave the 1-factor $\{\{x_i, y_i\} : 1 \leq i \leq 3k\} \cup \{\{a, b\}\}$. Then $T_1 \cup T_2 \cup \{ax_iy_i : 1 \leq i \leq 3k\}$ is a maximum packing of $2K_{6k+2}$ with leave $2K_2$ on the vertex set $\{a, b\}$. Taking 2 copies of this, we can obtain a maximum packing of $4K_{6k+2}$ with leave $D = 2K_2 \cup 2K_2$ on 4 vertices, and a maximum packing of $3K_{6k+2}$ with leave $E = \{\{a, b\}, \{a, b\}\} \cup F$ where F is a one-factor of the form $\{\{a, x_1\}, \{b, y_1\}\} \cup \{\{x_i, y_i\} : 2 \leq i \leq 3k\}$. Combining a maximum packing of $2K_{6k+2}$ with leave $G = \{\{a, y_1\}, \{a, y_1\}\}$, with the maximum packing of $3K_{6k+2}$ with leave E and adding a triple aby_1 produces a maximum packing of $5K_{6k+2}$ with leave the tripole $T = \{\{a, b\}, \{a, x_1\}, \{a, y_1\}\} \cup \{\{x_i, y_i\} : 2 \leq i \leq 3k\}$.

PROPOSITION 3.3. *For $v \equiv 2 \pmod{6}$, the graph $(\lambda K_v)^+$ can be decomposed into triangles, lollipops and a loops, where $a \equiv 1 \pmod{3}$ if $\lambda = 1$ or 4; $a \equiv 0 \pmod{3}$ if $\lambda = 2$ or 5; $a \equiv 2 \pmod{3}$ if $\lambda = 3$ or 6; $a \neq v - 1$ if $\lambda = 4$; and $a \leq v/2$ if $\lambda = 1, 3, 5$.*

Proof. Let $v = 6n + 2$ for some integer n .

Case 1: $\lambda = 1$.

Since the cases where $v \in \{8, 14\}$ with $\lambda = 1$ are considered in Lemma 2.8, we can assume that $n \geq 3$.

Let (Q, \circ) be a commutative quasigroup of order $2n$ with holes $H = \{\{1, 2\}, \{3, 4\}, \dots, \{2n - 1, 2n\}\}$, where $Q = \{1, 2, \dots, 2n\}$ and $n \geq 3$. Let $X = \{\infty_1, \infty_2\} \cup (Q \times \{1, 2, 3\})$, and define B as follows.

Type 1: For each hole $h = \{2i - 1, 2i\} \in H$, let $(\{\infty_1, \infty_2\} \cup (h \times \{1, 2, 3\}), B_h)$ be an \mathcal{ESTS} of $K_8^+ \setminus K_2^+$, the vertex set of K_2^+ being $\{\infty_1, \infty_2\}$, which contains the triple $T_h = \{(2i, 1), (2i, 2), (2i, 3)\}$ that has a loop on each vertex (as in Example 2.7(a)); let $B_h \subset B$.

Type 2: For $1 \leq i < j \leq 2n$ and $\{i, j\} \notin H$, $\{(i, 1), (j, 1), (i \circ j, 2)\}$, $\{(i, 2), (j, 2), (i \circ j, 3)\}$, $\{(i, 3), (j, 3), (i \circ j, 1)\} \in B$.

Type 3: $\{\infty_1, \infty_1, \infty_2\}, \{\infty_2, \infty_2, \infty_2\} \in B$.

Then B is an $\mathcal{ESTS}(v, v/2, 1)$. Let $T = \bigcup T_h$. Since $|T| = (v-2)/6$, let $T' \subseteq T$ with $|T'| = (v-2)/6 - (a-1)/3$. Removing the blocks $\{xyz, xxx, yyy, zzz : xyz \in T'\}$ and replacing them with $\{xyx, xxy, zzz : xyz \in T'\}$ produces an $\mathcal{ESTS}(v, a, 1)$.

Case 2: $\lambda = 3, 5$.

When $\lambda = 3$, consider $(3K_v)^+ = K_v^+ \cup 2K_v$. By Case 1, the first graph K_v^+ can be decomposed into triangles, lollipops and $3k + 1$ loops, where $k = 1, 2, \dots, n$. So, there are at least four loops in those decompositions, say $\{111, 222, 333, 444\}$. Since $L(2K_v) = 2K_2$, let the vertex set of the leave be $\{3, 4\}$. Combining the 2 loops $\{333, 444\}$ and the leave $2K_2$, the graph $(3K_v)^+$ can be decomposed into triangles, lollipops and $3k - 1$ loops, where $k = 1, \dots, n$. When $\lambda = 5$, consider $(5K_v)^+ = K_v^+ \cup 4K_v$. By Case 1, the first graph K_v^+ can be decomposed into triangles, lollipops and $3k + 1$ loops, where $k = 1, 2, \dots, n$. So, there are at least four loops in those decompositions, say $\{111, 222, 333, 444\}$. Since $L(4K_v) = 2K_2 \cup 2K_2$, let the vertex set of the leave be $\{1, 2\} \cup \{3, 4\}$. Combining the 4 loops $\{111, 222, 333, 444\}$ and the leave $2K_2 \cup 2K_2$, the graph $(5K_v)^+$ can be decomposed into triangles, lollipops and $3k - 3$ loops, where $k = 1, \dots, n$. As for the last case where $a = 3n$, the $\mathcal{ESTS}(v, a, 5)$ exists because $L(5K_v)$ is a tripole.

Case 3: $\lambda = 2$.

Since the cases where $v \in \{8, 14\}$ with $\lambda = 2$ are considered in Lemma 2.8, we can assume that $n \geq 3$. Let (Q, \circ) be a commutative quasigroup of order $2n$ with holes $H = \{\{1, 2\}, \{3, 4\}, \dots, \{2n-1, 2n\}\}$, where $Q = \{1, 2, \dots, 2n\}$ and $n \geq 3$. Let $X = \{\infty_1, \infty_2\} \cup (Q \times \{1, 2, 3\})$, and let B contain the following three types of blocks.

Type 1: For each hole $h \in H$, let $(\{\infty_1, \infty_2\} \cup (h \times \{1, 2, 3\}), B_h)$ be an \mathcal{ESTS} of $(2K_8)^+ \setminus (2K_2)^+$, the vertex set of $2K_2^+$ being $\{\infty_1, \infty_2\}$, which contains the vertex-disjoint triples in $T_h = \{ \{(i, 1), (i, 2), (i, 3)\} : i \in h \}$, each of which contains a loop on each of its three vertices (as in Example 2.7(b)).

Type 2: For $1 \leq i < j \leq 2n$ and $\{i, j\} \notin H$, take 2-copies of the following blocks $\{(i, 1), (j, 1), (i \circ j, 2)\}$, $\{(i, 2), (j, 2), (i \circ j, 3)\}$, $\{(i, 3), (j, 3), (i \circ j, 1)\}$.

Type 3: $\{\infty_1, \infty_1, \infty_2\}, \{\infty_2, \infty_2, \infty_1\} \in B$.

Then B is an $\mathcal{ESTS}(v, v-2, 2)$. Let $T = \bigcup T_h$. Since $|T| = (v-2)/3$, let $T' \subseteq T$ with $|T'| = (v-2)/3 - a/3$. Removing the blocks $\{xyz, xxx, yyy, zzz : xyz \in T'\}$ and replacing them with $\{xyx, yzy, zzz : xyz \in T'\}$ produces an $\mathcal{ESTS}(v, a, 2)$.

Case 4: $\lambda = 4, 6$.

When $\lambda = 4$, consider $(4K_v)^+ = (2K_v)^+ \cup 2K_v$. By Case 3, the first graph $(2K_v)^+$ can be decomposed into triangles, lollipops and $3k$ loops, where $k = 1, 2, \dots, 2n$. So, there are at least two loops in those decompositions, say $\{111, 222\}$. Since $L(2K_v) = 2K_2$, let the vertex set of the leave be $\{1, 2\}$. Combining the 2 loops $\{111, 222\}$ and the leave $2K_2$, the graph $(4K_v)^+$ can be decomposed into triangles, lollipops and $3k-2$ loops, where $k = 1, \dots, 2n$. When $\lambda = 6$, consider $(6K_v)^+ = (2K_v)^+ \cup 4K_v$. By Case 3, the first graph $(2K_v)^+$ can be decomposed into triangles, lollipops and $3k$ loops, where $k = 2, 3, \dots, 2n$. So, there are at least four loops in those decompositions, say $\{111, 222, 333, 444\}$. Since $L(4K_v) = 2K_2 \cup 2K_2$, let the vertex set of the leave be $\{1, 2\} \cup \{3, 4\}$. Combining the 4 loops $\{111, 222, 333, 444\}$ and the leave $2K_2 \cup 2K_2$, the graph $(6K_v)^+$ can be decomposed into triangles, lollipops and $3k-4$ loops, where $k = 2, \dots, 2n$. As for the last two cases where $a = 6n+2$ or $6n-1$, an $\mathcal{ESTS}(v, 6n+2, 6)$ exists since $L(6K_v) = \emptyset$; this can be used to produce an $\mathcal{ESTS}(v, 6n-1, 6)$ by replacing one triangle with 3 lollipops. \square

PROPOSITION 3.4. *For $v \equiv 3 \pmod{6}$, the graph $(\lambda K_v)^+$ can be decomposed into triangles, lollipops and a loops, where $a \equiv 0 \pmod{3}$ and $0 \leq a \leq v$.*

Proof. From the existence of Kirkman triple system of order v , we have an $\mathcal{ESTS}(v, v, \lambda)$ with a parallel class T . Since $|T| = v/3$, let $T' \subseteq T$ with $|T'| = (v-a)/3$. Removing the blocks $\{xyz, xxx, yyy, zzz : xyz \in T'\}$ and replacing them with $\{xyx, yzy, zzz : xyz \in T'\}$ produces an $\mathcal{ESTS}(v, a, \lambda)$. \square

PROPOSITION 3.5. *For $v \equiv 4 \pmod{6}$, the graph $(\lambda K_v)^+$ can be decomposed into triangles, lollipops and a loops, where $a \equiv 1 \pmod{3}$ and if λ is odd then $a \leq v/2$.*

Proof. Let $v = 6n+4$, for some integer n .

Case 1: $\lambda \in \{1, 3, 5\}$.

Since the cases $n \in \{0, 1, 2, 3, 4, 5, 7\}$ with $\lambda = 1$ are considered in Lemma 2.8, and since there exists a $\text{TS}(6n+4, 2)$ for each such value of n , we can assume that $n = 6$ or $n \geq 8$.

If n is even then $v = 12m+4$ for some integer $m \geq 3$.

Let (Q, \circ) be a commutative quasigroup of order $4m$ with holes $H = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \dots, \{4m-3, 4m-2, 4m-1, 4m\}\}$ of size 4, where $Q = \{1, 2, \dots, 4m\}$. Let $X = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (Q \times \{1, 2, 3\})$, and let B_1 contain the following three types of blocks.

Type 1: For each hole $h \in H$, let $(\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (h \times \{1, 2, 3\}), B_h)$ be an \mathcal{ETS} of $K_{16}^+ \setminus K_4^+$ containing a set T_h of 2 vertex-disjoint triangles defined on $h \times \{1, 2, 3\}$ (see Lemma 2.2 and Figure 5).

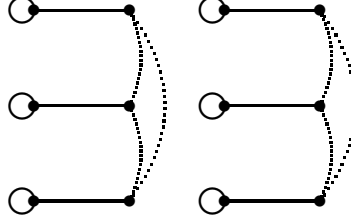


FIGURE 5. Position of triangles and lollipops in $K_{16}^+ \setminus K_4^+$

Type 2: The blocks in an \mathcal{ETS} of K_4^+ on $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ as in Section 2.

Type 3: For $1 \leq i < j \leq 4m$ and $\{i, j\} \not\subseteq h$, for all $h \in H$, $\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{(i, 3), (j, 3), (i \circ j, 1)\} \in B_1$.

If n is odd then $v = 12m + 10$ for some integer $m \geq 4$.

Let (Q, \circ) be a commutative quasigroup of order $4m + 2$ with holes $H = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \dots, \{4m - 7, 4m - 6, 4m - 5, 4m - 4\}, \{4m - 3, 4m - 2, 4m - 1, 4m, 4m + 1, 4m + 2\}\}$, where $Q = \{1, 2, \dots, 4m + 2\}$. Let $X = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (Q \times \{1, 2, 3\})$, and let B_1 contain the following four types of blocks.

Type 1: For each hole h in H of size 4, let $(\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (h \times \{1, 2, 3\}), B_h)$ be an \mathcal{ETS} of $K_{16}^+ \setminus K_4^+$, the vertex set of K_4^+ being $\{\infty_1, \infty_2, \infty_3, \infty_4\}$, containing a set T_h of 2 vertex-disjoint triangles defined on $h \times \{1, 2, 3\}$, each of which contains a loop on each of its three vertices (see Lemma 2.2 and Figure 5).

Type 2: For the hole h in H of size 6, let $(\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (h \times \{1, 2, 3\}), B_h)$ be an \mathcal{ETS} of $K_{22}^+ \setminus K_4^+$ containing a set T_h of 3 vertex-disjoint triangles defined on $h \times \{1, 2, 3\}$ (see Lemma 2.3 and Figure 6).

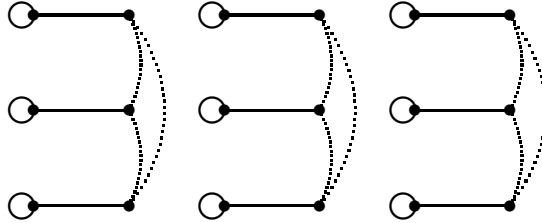


FIGURE 6. Position of triangles and lollipops in $K_{22}^+ \setminus K_4^+$

Type 3: The blocks in an \mathcal{ETS} of K_4^+ on $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ as in Lemma 2.8.

Type 4: For $1 \leq i < j \leq 4m + 2$ and $\{i, j\} \not\subseteq h$, for all $h \in H$, $\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{(i, 3), (j, 3), (i \circ j, 1)\} \in B$.

In the above two situations, let (X, B_2) be a $TS(v, \lambda - 1)$; this exists since $L((\lambda - 1)K_v) = \emptyset$. Then $B_1 \cup B_2$ is an $\mathcal{ETS}(v, v/2 - 1, \lambda)$. Let $T = \bigcup_{h \in H} T_h$. Since $|T| = (v - 4)/6$, let $T' \subseteq T$ with $|T'| = (v - 4)/6 - (a - 1)/3$. Removing the blocks $\{xyz, xxx, yyy, zzz : xyz \in T'\}$ and replacing them with $\{xxy, yyz, zzx : xyz \in T'\}$ produces an $\mathcal{ETS}(v, a, \lambda)$.

Case 2: $\lambda \in \{2, 4, 6\}$.

Let (Q, \circ) be an idempotent quasigroup of order $2n + 1$, where $Q = \{1, 2, \dots, 2n + 1\}$. Let $X = \{\infty\} \cup (Q \times \{1, 2, 3\})$, and let B contain a loop on each vertex and the blocks of the following three types.

Type 1: For $1 \leq i \leq 2n + 1$, take $\lambda/2$ -copies of a $TS(v, 2)$ on $\{\infty, (i, 1), (i, 2), (i, 3)\}$.

Type 2: For $1 \leq i \neq j \leq 2n + 1$, take $\lambda/2$ -copies of the following blocks $\{(i, 1), (j, 1), (i \circ j, 2)\}$, $\{(i, 2), (j, 2), (i \circ j, 3)\}$, $\{(i, 3), (j, 3), (i \circ j, 1)\}$.

Type 3: For $x \in X$, $\{x, x, x\} \in B$.

Then B is an $\mathcal{ETS}(v, v, \lambda)$ with a near parallel class $T = \{\{(i, 1), (i, 2), (i, 3)\} : 1 \leq i \leq 2n + 1\}$ of Type 1. Since $|T| = (v - 1)/3$, let $T' \subseteq T$ with $|T'| = (v - 1)/3 - (a - 1)/3$. Removing the blocks $\{xyz, xxx, yyy, zzz : xyz \in T'\}$ and replacing them with $\{xxy, yyz, zzx : xyz \in T'\}$ produces an $\mathcal{ETS}(v, a, \lambda)$. \square

According to Figure 2, $L(K_{6k+5})$ is a 4-cycle. If we take the two 4-cycles (a, b, c, d) and (a, b, d, c) to be the leaves of two maximum packings of K_{6k+5} and then replace the two 4-cycles with $\{bcd, acd\} \cup \{\{a, b\}, \{a, b\}\}$, a maximum packing of $2K_{6k+5}$ is produced with leave $2K_2$. So, $L(5K_{6k+5}) = 2K_2$ follows since $L(3K_{6k+5}) = \emptyset$.

PROPOSITION 3.6. *For $v \equiv 5 \pmod{6}$, the graph $(\lambda K_v)^+$ can be decomposed into triangles, lollipops and a loops, where $a \equiv 1 \pmod{3}$ and $a \neq v - 1$ if $\lambda = 1$ or 4; $a \equiv 0 \pmod{3}$ if $\lambda = 2$ or 5; and $a \equiv 2 \pmod{3}$ if $\lambda = 3$ or 6.*

Proof. Let $v = 6n + 5$, for some integer n .

Case 1: $\lambda = 1$.

Since the cases $n \in \{0, 1, 2, 3, 4, 5, 7\}$ with $\lambda = 1$ are considered in Lemma 2.8, we can assume that $n = 6$ or $n \geq 8$.

If n is even then $v = 12m + 5$ for some integer $m \geq 3$.

Let (Q, \circ) be a commutative quasigroup of order $4m$ with holes $H = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \dots, \{4m - 3, 4m - 2, 4m - 1, 4m\}\}$, where $Q = \{1, 2, \dots, 4m\}$. Let $X = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (Q \times \{1, 2, 3\})$, and let B contain the following three types of blocks.

Type 1: For each hole $h \in H$, let $(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (h \times \{1, 2, 3\}), B_h)$ be an \mathcal{ETS} of $K_{17}^+ \setminus K_5^+$ containing a set $T_h = \{\{(i, 1), (i, 2), (i, 3)\} : i \in h\}$ of 4 vertex-disjoint triangles defined on $h \times \{1, 2, 3\}$ (see Lemma 2.2).

Type 2: The blocks in an \mathcal{ETS} of K_5^+ on $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ as in Section 2.

Type 3: For $1 \leq i < j \leq 4m$ and $\{i, j\} \not\subseteq h$, for all $h \in H$, $\{(i, 1), (j, 1), (i \circ j, 2)\}$, $\{(i, 2), (j, 2), (i \circ j, 3)\}$, $\{(i, 3), (j, 3), (i \circ j, 1)\} \in B$.

If n is odd, then $v = 12m + 11$ for some integer $m \geq 4$.

Let (Q, \circ) be a commutative quasigroup of order $4m + 2$ with holes $H = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \dots, \{4m - 7, 4m - 6, 4m - 5, 4m - 4\}, \{4m - 3, 4m - 2, 4m - 1, 4m, 4m + 1, 4m + 2\}\}$, where $Q = \{1, 2, \dots, 4m + 2\}$. Let $X = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (Q \times \{1, 2, 3\})$, and let B contain the following four types of blocks.

Type 1: For each hole h in H of size 4, let $(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (h \times \{1, 2, 3\}), B_h)$ be an \mathcal{ETS} of $K_{17}^+ \setminus K_5^+$ containing a set $T_h = \{\{(i, 1), (i, 2), (i, 3)\} : i \in h\}$ of 4 vertex-disjoint triangles defined on $h \times \{1, 2, 3\}$ (see Lemma 2.2).

Type 2: For the hole h in H of size 6, let $(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (h \times \{1, 2, 3\}), B_h)$ be an \mathcal{ETS} of $K_{23}^+ \setminus K_5^+$ containing a set $T_h = \{\{(i, 1), (i, 2), (i, 3)\} : i \in h\}$ of 6 vertex-disjoint triangles defined on $h \times \{1, 2, 3\}$ (see Lemma 2.3).

Type 3: The blocks in an \mathcal{ETS} of K_5^+ on $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ as in Section 2.

Type 4: For $1 \leq i < j \leq 4m + 2$ and $\{i, j\} \not\subseteq h$, for all $h \in H$, $\{(i, 1), (j, 1), (i \circ j, 2)\}$, $\{(i, 2), (j, 2), (i \circ j, 3)\}$, $\{(i, 3), (j, 3), (i \circ j, 1)\} \in B$.

In the above two situations, B is an $\mathcal{ETS}(v, v - 4, 1)$. Let $T = \bigcup T_h$. Since $|T| = (v - 5)/3$, let $T' \subseteq T$ with $|T'| = (v - 5)/3 - (a - 1)/3$. Removing the blocks $\{xyz, xxx, yyy, zzz : xyz \in T'\}$ and replacing them with $\{xyx, yzy, zzz : xyz \in T'\}$ produces an $\mathcal{ETS}(v, a, \lambda)$.

Case 2: $\lambda = 3, 4, 5, 6$.

For $\lambda = 3, 5$, consider $(\lambda K_v)^+ = K_v^+ \cup (\lambda - 1)K_v$. By Case 1, the first graph K_v^+ can be decomposed into triangles, lollipops and $3k + 1$ loops, where $k = 1, 2, \dots, 2n$. So, there are at least four loops in those decompositions, say $\{111, 222, 333, 444\}$. When $\lambda = 3$, $L((\lambda - 1)K_v) = 2K_2$. Let the vertex set of the leave be $\{3, 4\}$. Combining the 2 loops $\{333, 444\}$ and the leave $2K_2$, the graph $(3K_v)^+$ can be decomposed into triangles, lollipops and $3k - 1$ loops, where $k = 1, 2, \dots, 2n$. As for the cases where $a \in \{6n + 2, 6n + 5\}$, an $\mathcal{ETS}(v, 6n + 5, 3)$ exists since $L(3K_v) = \emptyset$; this can be used to produce an $\mathcal{ETS}(v, 6n + 2, 3)$ by replacing one triangle with 3 lollipops. When $\lambda = 5$, $L((\lambda - 1)K_v) = 2K_2 \cup 2K_2$. Let the vertex set of the leave be $\{1, 2\} \cup \{3, 4\}$. Combining the 4 loops $\{111, 222, 333, 444\}$ and the leave $2K_2 \cup 2K_2$, the graph $(5K_v)^+$ can be decomposed into triangles, lollipops and $3k - 3$ loops, where $k = 1, 2, \dots, 2n$. As for the cases where $a \in \{6n, 6n + 3\}$, it follows by $L(5K_v) = 2K_2$. When $\lambda = 4$ or 6, consider $(\lambda K_v)^+ = ((\lambda - 3)K_v)^+ \cup 3K_v$. Since $L(3K_v) = \emptyset$, the results follow by the above cases.

Case 3: $\lambda = 2$.

Since the cases $n \in \{0, 1, 2, 3, 4, 5, 7\}$ with $\lambda = 2$ are considered in Lemma 2.8, we can assume that $n = 6$ or $n \geq 8$.

If n is even then $v = 12m + 5$ for some integer $m \geq 3$.

Let (Q, \circ) be a commutative quasigroup of order $4m$ with holes $H = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \dots, \{4m-3, 4m-2, 4m-1, 4m\}\}$, where $Q = \{1, 2, \dots, 4m\}$. Let $X = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (Q \times \{1, 2, 3\})$, and let B contain the following three types of blocks.

Type 1: For each hole $h \in H$, let $(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (h \times \{1, 2, 3\}), B_h)$ be the union of an $\mathcal{ESTS}(K_{17}^+ \setminus K_5^+)$ and an $\mathcal{ESTS}(K_{17} \setminus K_5)$, where the system $\mathcal{ESTS}(K_{17}^+ \setminus K_5^+)$ contains a set $T_h = \{(i, 1), (i, 2), (i, 3) : i \in h\}$ of 4 vertex-disjoint triangles defined on $h \times \{1, 2, 3\}$ (see Lemma 2.2).

Type 2: The blocks in an $\mathcal{ESTS}(5, 3, 2)$ on $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ as in Section 2.

Type 3: For $1 \leq i < j \leq 4m$ and $\{i, j\} \not\subseteq h$, for all $h \in H$, take 2-copies of the following blocks $\{(i, 1), (j, 1), (i \circ j, 2)\}$, $\{(i, 2), (j, 2), (i \circ j, 3)\}$, $\{(i, 3), (j, 3), (i \circ j, 1)\}$.

If n is odd then $v = 12m + 11$ for some integer $m \geq 4$.

Let (Q, \circ) be a commutative quasigroup of order $4m + 2$ with holes $H = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \dots, \{4m-7, 4m-6, 4m-5, 4m-4\}, \{4m-3, 4m-2, 4m-1, 4m, 4m+1, 4m+2\}\}$, where $Q = \{1, 2, \dots, 4m+2\}$. Let $X = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (Q \times \{1, 2, 3\})$, and let B contain the following four types of blocks.

Type 1: For each hole h in H of size 4, let $(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (h \times \{1, 2, 3\}), B_h)$ be the union of an $\mathcal{ESTS}(K_{17}^+ \setminus K_5^+)$ and an $\mathcal{ESTS}(K_{17} \setminus K_5)$, where the system $\mathcal{ESTS}(K_{17}^+ \setminus K_5^+)$ contains a set $T_h = \{(i, 1), (i, 2), (i, 3) : i \in h\}$ of 4 vertex-disjoint triangles defined on $h \times \{1, 2, 3\}$ (see Lemma 2.2).

Type 2: For the hole h in H of size 6, let $(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (h \times \{1, 2, 3\}), B_h)$ be the union of an $\mathcal{ESTS}(K_{23}^+ \setminus K_5^+)$ and an $\mathcal{ESTS}(K_{23} \setminus K_5)$, where the system $\mathcal{ESTS}(K_{23}^+ \setminus K_5^+)$ contains a set $T_h = \{(i, 1), (i, 2), (i, 3) : i \in h\}$ of 6 vertex-disjoint triangles defined on $h \times \{1, 2, 3\}$ (see Lemma 2.3).

Type 3: The blocks in an $\mathcal{ESTS}(5, 3, 2)$ on $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ as in Section 2.

Type 4: For $1 \leq i < j \leq 4m + 2$ and $\{i, j\} \not\subseteq h$, for all $h \in H$, take 2-copies of the following blocks $\{(i, 1), (j, 1), (i \circ j, 2)\}$, $\{(i, 2), (j, 2), (i \circ j, 3)\}$, $\{(i, 3), (j, 3), (i \circ j, 1)\}$.

In the above two situations, B is an $\mathcal{ESTS}(v, v-2, 2)$ with a parallel class T , where $T = \bigcup T_h$. Since $|T| = (v-5)/3$, let $T' \subseteq T$ with $|T'| = (v-5)/3 - a/3$. Removing the blocks $\{xyz, xxx, yyy, zzz : xyz \in T'\}$ and replacing them with $\{xxy, yyz, zzz : xyz \in T'\}$ produces an $\mathcal{ESTS}(v, a+3, \lambda)$.

The existence of an $\mathcal{ESTS}(v, 0, 2)$ follows from a decomposition of $2K_{6n+5}$ into one 4-cycle, one $(6n+1)$ -cycle, and 3-cycles. This decomposition can be obtained by taking $H = C_{6n+1}$ in Lemma 1.3 and $L(K_{6n+5}) = C_4$. \square

4. Conclusions

From Propositions 3.1–3.6, we obtain the following result:

THE MAIN THEOREM. *The graph $(\lambda K_v)^+$ can be decomposed into triangles, lollipops and a loops, if and only if,*

- (1) *if $v \equiv 0 \pmod{3}$, or if $v \equiv 2 \pmod{3}$ and $\lambda \equiv 2 \pmod{3}$, then $a \equiv 0 \pmod{3}$;*
- (2) *if $v \equiv 1 \pmod{3}$, or if $v \equiv 2 \pmod{3}$ and $\lambda \equiv 1 \pmod{3}$, then $a \equiv 1 \pmod{3}$;*
- (3) *if $v \equiv 2 \pmod{3}$ and $\lambda \equiv 0 \pmod{3}$, then $a \equiv 2 \pmod{3}$;*
- (4) *if v is even and λ is odd then $a \leq v/2$; and*
- (5) *$0 \leq a \leq v$, and if $(v, \lambda) \neq (2, 1)$, then $a \neq v - 1$.*

Acknowledgement. The authors would like to thank the referees for their diligence and excellent suggestions for improving this paper.

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Received 10. 5. 2007

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