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RANDOM FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE RANDOM OPERATORS ON UNBOUNDED DOMAINS

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ABSTRACT. The aim of this paper is to prove some random fixed point theorems for asymptotically nonexpansive random operator defined on an unbounded closed and starshaped subset of a Banach space.

1. Introduction

Random operator theory lie at the heart of probabilistic functional analysis and is needed for the study of various classes of random operator equations (see [5], [11]). Fixed point theorems in connection with the existence of random solution of nonlinear random operator equations are extensively studied and for a survey of random fixed point theory and its applications, we refer to [2], [8], [9], and [17]. Kirk and Ray [13] have shown that if X is an unbounded closed convex subset of a uniformly convex Banach space and $T\colon X \to X$ is a Lipschitzian pseudo contractive mapping for which the set $G(x,Tx;x) = \big\{z \in X : \ \|z-Tx\| \le \|z-x\|\big\}$ is bounded for some $x \in X$, then T has a fixed point in X. Afterward Carbone and Marino [6] examined the structure of some geometric sets in Banach spaces with this property. Penot [16], imposing the condition of asymptotic contractivity on nonexpansive mappings defined on unbounded closed and convex subset of a Banach

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space, established some fixed point theorems. Isac and Nemeth [10] obtained some interesting results for eigenvalues of nonexpansive mappings defined on unbounded sets. Recently, Kaewcharoen and Kirk [12] obtained the existence of fixed point of similar mappings defined on unbounded domains under a weaker condition than given in [16]. On the other hand, Beg and Abbas [3] constructed a random iterative scheme which converges to a random fixed point of an asymptotically nonexpansive random operator which takes values in a closed convex and bounded subset of a Banach space. The aim of this paper is to prove some random fixed point theorems for asymptotically nonexpansive random operators defined on an unbounded closed and starshaped subset of a Banach space. Moreover, from computational point of view, we employ the simplest random iterative process to obtain the existence of random fixed points of such operators. As a consequence, a stochastic generalization and improvements of the comparable results valid for bounded convex sets in the literature ([10] and [16]) are obtained.

2. Preliminaries

We begin with some definitions and state the notations which are used in this paper. Let (Ω, Σ) be a measurable space (Σ — sigma algebra) and C be a nonempty subset of a normed space X. A multivalued mapping $T\colon \Omega \to 2^X$ (or single valued mapping $T\colon \Omega \to X$) is *measurable* if $T^{-1}(U) \in \Sigma$ (or $T^{-1}(U) \in \Sigma$) for each open subset U of X, where $T^{-1}(U) = \{\omega \in \Omega : \ T(\omega) \cap U \neq \emptyset\}$, and 2^X denotes a family of all subsets of X. A multivalued mapping $T\colon \Omega \times C \to 2^X$ (or a single valued mapping $T\colon \Omega \times C \to X$) is a *random operator* if and only if for each fixed $x \in C$, the mapping $T(\cdot\,, x)\colon \Omega \to 2^X$ (or $T(\cdot\,, x)\colon \Omega \to X$) is measurable, and it is *continuous* if for each $\omega \in \Omega$, the mapping $T(\omega, \cdot\,)\colon C \to 2^X$ (or $T(\omega, \cdot\,)\colon C \to X$) is continuous. A measurable mapping $\xi\colon \Omega \to X$ is a *random fixed point* of a random operator $T\colon \Omega \times X \to X$ if and only if $\xi(\omega) = T(\omega, \xi(\omega))$ for each $\omega \in \Omega$. We denote the nth iterate $T(\omega, T(\omega, T(, \dots, T(\omega, x))))$ of T by $T^n(\omega, x)$.

A random operator $T\colon \Omega \times C \to C$ is said to satisfy *condition* (A) if for fixed x_0 in C, we have

$$\limsup_{\substack{\|x\| \to \infty \\ x \in C}} \frac{\|T^n(\omega, x) - T(\omega, x_0)\|}{\|x - x_0\|} < 1$$

for each $\omega \in \Omega$ and $n \in \mathbb{N}$. If we put $n = 1$ in above inequality, we obtain a random version of definition of asymptotic contractive mapping given in [16].

Let $G\colon \Omega \times X \times X \to \mathbb{R}$ (set of real numbers) be a mapping which satisfies $G(\omega, \lambda x, y) = \lambda G(\omega, x, y)$, $G(\omega, x + y, z) = G(\omega, x, z) + G(\omega, y, z)$, $\|x\|^2 \leq G(\omega, x, x)$, and there is $M > 0$ such that $|G(\omega, x, y)| \leq M \|x\| \|y\|$ for any $x, y, z \in X$ and $\omega \in \Omega$. The mapping G defined in [10] in turn helps to obtain a random fixed point of an asymptotically nonexpansive random operators defined on unbounded domain. If $B\colon \Omega \times X \times X \to \mathbb{R}$ is a linear mapping in second and third coordinates, and there is a positive constant k such that $B(\omega, x, x) \geq k \|x\|^2$, then $G\colon \Omega \times X \times X \to \mathbb{R}$ defined by $G(\omega, x, y) = \frac{1}{k} B(\omega, x, y)$ satisfies all the above conditions.

A mapping $T\colon C \to X$ is called *demiclosed with respect to* $y \in X$ if for each sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to $x \in X$ and $\{Tx_n\}$ converges strongly to y imply that $x \in C$ and $Tx = y$.

Definition 2.1. The random operator $T\colon \Omega \times C \to C$ is said to be:

(a) *Nonexpansive random operator* if for any $x, y \in C$ we have

$$\|T(\omega, x) - T(\omega, y)\| \leq \|x - y\|,$$

for each $\omega \in \Omega$.

(b) *Asymptotically nonexpansive random operator* if there exists a sequence of mappings $r_n\colon \Omega \to [0, \infty)$ with $\lim\limits_{n\to\infty} r_n(\omega) = 1$, and for any $x, y \in C$,

$$\|T^n(\omega, x) - T^n(\omega, y)\| \leq r_n(\omega) \|x - y\|,$$

for each $\omega \in \Omega$.

Remark 2.2. Let C be a closed subset of a complete separable metric space X and the sequence of measurable mappings $\{\xi_n\}$ from Ω to C be pointwise convergent, that is, $\xi_n(\omega) \to \xi(\omega)$ for each $\omega \in \Omega$. Then ξ being the limit of the sequence of measurable mappings is measurable and closedness of C implies ξ is a mapping from Ω to C. If T is a continuous random operator from $\Omega \times C$ to C then by [1, Lemma 8.2.3], the map $\omega \mapsto T(\omega, f(\omega))$ is measurable for any measurable mapping f from Ω to C.

Definition 2.3. A random operator $T\colon \Omega \times C \to C$ is said to satisfy *property* (P) if for any bounded sequence $\{x_n\}$ in C with $\lim\limits_{n\to\infty} \|T^n(\omega, x_n) - x_n\| = 0$ for each $\omega \in \Omega$ implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Throughout this paper we assume that T satisfies the property (P). It is noted that for an asymptotically nonexpansive mapping (defined even on unbounded sets), this property implies the bounded approximate fixed point property (that is, for a bounded sequence $\{x_n\}$ in C, $\lim\limits_{n\to\infty} \|T(\omega, x_n) - x_n\| = 0$ for each $\omega \in \Omega$) which in turn is equivalent to the existence of deterministic fixed point of T under some mild compactness conditions.

3. Random fixed points

Theorems established in this section for a nonempty unbounded closed and starshaped subsets generalize comparable results in the existing literature valid for bounded closed convex sets.

In the following theorem we employ the properties of the mapping G ([10]) to obtain the existence of a random fixed point of an asymptotically nonexpansive random operator.

Theorem 3.1. *Let C be a nonempty unbounded closed and starshaped subset with respect to some point u in a separable reflexive Banach space X and $T\colon \Omega \times C \to X$ be an asymptotically nonexpansive random operator with $T(\omega, C) \subseteq C$ and $I - T(\omega, \cdot\,)$ be demiclosed for each $\omega \in \Omega$.*

If $\limsup\limits_{\substack{\|x\|\to\infty \\ x\in C}} \frac{G(\omega, T^n(\omega,x)-u,x)}{\|x\|^2} < 1$, then T has a random fixed point.

Proof. For each n, define the mapping $T_n\colon \Omega \times C \to X$ as, $T_n(\omega, x) = \alpha_n(\omega) T^n(\omega, x) + (1 - \alpha_n(\omega))u$, where $\alpha_n(\omega) = \frac{\lambda_n(\omega)}{r_n(\omega)}$, and $\lambda_n\colon \Omega \to (0,1)$ is a sequence of mappings with $\lim\limits_{n\to\infty} \lambda_n(\omega) = 1$, for each $\omega \in \Omega$. Since $T(\omega, C) \subseteq C$ for each ω in Ω, starshapedness of C with respect to u implies that $T_n(\omega, C) \subseteq C$ for each n. Now

$$\begin{aligned} \|T_n(\omega, x) - T_n(\omega, y)\| &= \alpha_n(\omega) \|T^n(\omega, x) - T^n(\omega, y)\| \\ &\le \lambda_n(\omega) \|x - y\| . \end{aligned}$$

Thus $T_n(\omega, \cdot\,)$ is a contractive random operator, for each $\omega \in \Omega$. Hence we obtain a sequence of measurable mappings $\xi_n\colon \Omega \to X$ with $\xi_n(\omega) = T_n(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$ ([5]). Now we show that $\{\xi_n(\omega)\}$ is a bounded sequence for each $\omega \in \Omega$. If this is not the case, we may assume $\|\xi_n(\omega)\| \to \infty$, for some $\omega \in \Omega$. Let $\alpha \in (0,1)$ and $\beta > 0$ be such that $G(\omega, T^n(\omega, x) - u, x) \le \alpha \|x\|^2$, for each

$\omega \in \Omega$ and $x \in C$ with $\|x\| \geq \beta$. For n large enough, consider

$$\begin{aligned}
\|\xi_n(\omega)\|^2 &\leq G(\omega, \xi_n(\omega), \xi_n(\omega)) \\
&= G\big(\omega, \alpha_n(\omega)[T^n(\omega, \xi_n(\omega)) - u] + u, \xi_n(\omega)\big) \\
&= \alpha_n(\omega) G\big(\omega, T^n(\omega, \xi_n(\omega)) - u, \xi_n(\omega)\big) + G(\omega, u, \xi_n(\omega)) \\
&\leq \alpha_n(\omega)\alpha \|\xi_n(\omega)\|^2 + M \|\xi_n(\omega)\| \|u\| .
\end{aligned}$$

Dividing by $\|\xi_n(\omega)\|^2$ and taking limit $n \to \infty$, we arrive at the conclusion $1 \leq \alpha$, a contradiction. Hence $\{\xi_n(\omega)\}$ is a bounded sequence for each $\omega \in \Omega$. Now, when $n \to \infty$

$$\|\xi_n(\omega) - T^n(\omega, \xi_n(\omega))\| = (1 - \alpha_n(\omega)) \|T^n(\omega, \xi_n(\omega)) - u\| \to 0.$$

Moreover,

$$\begin{aligned}
& \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| \\
\leq\ & \|\xi_n(\omega) - T^n(\omega, \xi_n(\omega))\| + \|T^n(\omega, \xi_n(\omega)) - T(\omega, \xi_n(\omega))\| \\
\leq\ & (1 - \alpha_n(\omega)) \|T^n(\omega, \xi_n(\omega)) - u\| + r_1(\omega) \left\|T^{n-1}(\omega, \xi_n(\omega)) - \xi_n(\omega)\right\| \\
\leq\ & (1 - \alpha_n(\omega)) \|T^n(\omega, \xi_n(\omega)) - u\| \\
& + r_1(\omega)\big[\|T^{n-1}(\omega, \xi_n(\omega)) - T^{n-1}(\omega, \xi_{n-1}(\omega))\| \\
& + \left\|T^{n-1}(\omega, \xi_{n-1}(\omega)) - \xi_{n-1}(\omega)\right\| + \|\xi_{n-1}(\omega) - \xi_n(\omega)\|\big] \\
\leq\ & (1 - \alpha_n(\omega)) \|T^n(\omega, \xi_n(\omega)) - u\| + r_1(\omega)\big[r_{n-1}(\omega) \|\xi_n(\omega) - \xi_{n-1}(\omega)\| \\
& + (1 - \alpha_{n-1}(\omega)) \left\|T^{n-1}(\omega, \xi_{n-1}(\omega)) - u\right\| + \|\xi_{n-1}(\omega) - \xi_n(\omega)\|\big],
\end{aligned}$$

which approaches to zero as $n \to \infty$. $\{\xi_n(\omega)\}$ is a bounded sequence in a reflexive Banach space for each $\omega \in \Omega$. Therefore, for each n, we can define $G_n \colon \Omega \to WK(X)$ by $G_n(\omega) = w\text{-}\operatorname{cl}\big(\operatorname{co}\{\xi_i(\omega) : \ i \geq n\}\big)$, where $w\text{-}\operatorname{cl}(\operatorname{co} C)$ is the weak closure of convex hull of C. Define $G \colon \Omega \to WK(X)$ by $G(\omega) = \bigcap_{n=1}^{\infty} G_n(\omega)$. Since the weak topology on X is a metric topology (see, Dunford and Schwartz [7]) and the mapping G is w-measurable so G has a w-measurable selector ξ ([14], [15]). Since X is separable, ξ is measurable. The map ξ is the required random fixed point. Indeed, for any fixed ω in Ω, we may assume that there exists a subsequence $\left\{\xi_{n_j}(\omega)\right\}$ of $\left\{\xi_n(\omega)\right\}$ weakly convergent to $\xi(\omega)$. Since $I - T(\omega, \cdot)$ is demiclosed, therefore $T(\omega, \xi(\omega)) = \xi(\omega)$, for every $\omega \in \Omega$. $\square$

Corollary 3.2. *Let C be a nonempty boundedly compact closed and starshaped subset with respect to some point u in a separable Banach space X and $T\colon \Omega \times C \to X$ be an asymptotically nonexpansive random operator with $T(\omega, C) \subseteq C$ and $I - T(\omega, \cdot)$ be demiclosed for each $\omega \in \Omega$.*

If $\limsup\limits_{\substack{\|x\|\to\infty \\ x\in C}} \frac{G(\omega, T^n(\omega,x)-u,x)}{\|x\|^2} < 1$, then T has a random fixed point.

Now, we prove the following existence theorem under a condition different from that given in Theorem 3.1.

Theorem 3.3. *Let C be a nonempty unbounded closed starshaped subset with respect to u in a separable reflexive Banach space X and $T\colon \Omega \times C \to X$ be an asymptotically nonexpansive random operator satisfying condition* (A) *with $T(\omega, C) \subseteq C$ for each $\omega \in \Omega$. If $I - T(\omega, \cdot)$ is demiclosed for each ω in Ω, then T has a random fixed point.*

Proof. For each n, define the mapping $T_n\colon \Omega \times C \to X$ as $T_n(\omega, x) = \alpha_n(\omega)T^n(\omega, x) + (1 - \alpha_n(\omega))u$, where $\alpha_n(\omega) = \frac{\lambda_n(\omega)}{r_n(\omega)}$, and $\lambda_n\colon \Omega \to (0, 1)$ is a sequence of mappings with $\lim\limits_{n\to\infty} \lambda_n(\omega) = 1$, for each $\omega \in \Omega$. Since $T(\omega, C) \subseteq C$ for each ω in Ω, starshapedness of C with respect to u implies that $T_n(\omega, C) \subseteq C$ for each n. Now

$$\begin{aligned} \|T_n(\omega, x) - T_n(\omega, y)\| &= \alpha_n(\omega) \|T^n(\omega, x) - T^n(\omega, y))\| \\ &\leq \lambda_n(\omega) \|x - y\| . \end{aligned}$$

Following an argument similar to that in Theorem 3.1, we obtain a measurable mapping $\xi_n\colon \Omega \to X$ with $\xi_n(\omega) = T(\omega, \xi_n(\omega))$ for each positive integer n and $\omega \in \Omega$. Now we show that $\{\xi_n(\omega)\}$ is a bounded sequence for each $\omega \in \Omega$. If this is not the case, we may assume $\|\xi_n(\omega)\| \to \infty$ for some $\omega \in \Omega$. Let $\alpha \in (0, 1)$ and $\beta > 0$ be such that $\|T^n(\omega, x) - T(\omega, u)\| \leq \alpha \|x - u\|$, for each $\omega \in \Omega$ and $x \in C$ with $\|x\| \geq \beta$. For n large enough, consider

$$\begin{aligned} \|\xi_n(\omega)\| &= \|\alpha_n(\omega)T^n(\omega, \xi_n(\omega)) + (1 - \alpha_n(\omega))u\| \\ &\leq \alpha_n(\omega) (\|T^n(\omega, \xi_n(\omega)) - T(\omega, u)\| + \|T(\omega, u)\|) + (1 - \alpha_n(\omega)) \|u\| \\ &\leq \alpha_n(\omega) (\|T^n(\omega, \xi_n(\omega)) - T(\omega, u)\| + \|T(\omega, u)\|) + (1 - \alpha_n(\omega)) \|u\| \\ &\leq \alpha_n(\omega)\alpha \|\xi_n(\omega) - u\| + (1 - \alpha_n(\omega)) \|u\| . \end{aligned}$$

Dividing by $\|\xi_n(\omega)\|$ and taking limit $n \to \infty$, we arrive at the conclusion $1 \leq \alpha$, a contradiction. Thus $\{\xi_n(\omega)\}$ is a bounded sequence for each ω in Ω. Following similar arguments as those used in the proof of Theorem 3.1, we obtain a measurable mapping $\xi\colon \Omega \to C$ satisfying the random operator equation $T(\omega, x) = x$ for each ω in Ω. □

The following corollary is a generalization of [4, Theorem 3.2].

Corollary 3.4. *Let C be a nonempty unbounded closed starshaped subset with respect to some u in a separable reflexive Banach space X and $T\colon \Omega\times C \to X$ be a nonexpansive asymptotically contractive random operator with $T(\omega, C) \subseteq C$, for each ω in Ω. Then T has a random fixed point.*

Remark 3.5. If we take a uniformly convex Banach space in Theorems 3.1 and 3.2, the conclusion of these theorems remain valid even if we drop the condition of demiclosedness of $I - T(\omega, \cdot)$.

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