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DEDUCTIVE SYSTEMS OF A CONE ALGEBRA – II: ISOMORPHISM THEOREM

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ABSTRACT. We prove that there is an isomorphism φ of the lattice of deductive systems of a cone algebra onto the lattice of convex ℓ-subgroups of a lattice ordered group (determined by the cone algebra) such that for any deductive system A of the cone algebra, A is respectively a prime, normal or polar if and only if $\varphi(A)$ is a prime convex ℓ-subgroup, ℓ-ideal or polar subgroup of the ℓ-group, thus generalizing and extending the result of Rachůnek that the lattice of ideals of a pseudo MV-algebra is isomorphic to the lattice of convex ℓ-subgroups of a unital lattice ordered group.

Introduction

Recently, J. Rachůnek has proved [5] that the lattice of ideals of a generalized MV-algebra (equivalently, a pseudo MV-algebra) is isomorphic to the lattice of convex ℓ-subgroups of a unital lattice ordered group, using a theorem of A. Dvurečenskij [3] that every pseudo MV-algebra is an interval $[0, u]$ of a unital ℓ-group with strong order unit u. Now by [7, Theorems 2.8, 3.13] (Part I), a subset A of a pseudo MV-algebra C is an *ideal* of C if and only if A is a *deductive system* of the brick equivalent to C. Hence Rachůnek's theorem ([5, Theorem 2]) can be interpreted as proving that lattice of deductive systems of a brick is isomorphic to the lattice of convex ℓ-subgroups of a unital ℓ-group. The main object of this paper is to generalize Rachůnek's theorem (in this form) to cone algebras (of which, bricks are a subclass). Stated briefly, we prove the following theorem (for various definitions involved, please see inside):

Main Theorem. *Let C be a cone algebra; then there exists a lattice ordered group $G = G(C)$ such that C is a subalgebra of the cone algebra of the ℓ-group cone G^+ and an isomorphism φ of the lattice of deductive systems of C onto the*

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lattice of convex ℓ-subgroups of G such that a deductive system A of C is, respectively, prime, normal or polar if and only if $\varphi(A)$ is a prime convex ℓ-subgroup, an ℓ-ideal or a polar subgroup of G.

The full version of the above theorem will be given in Section 6 (Theorem 6.8) and we will deduce the theorem of Rachůnek in the concluding part of this paper.

This part contains the Sections 4 through 6. In Section 4, we define the *enveloping ℓg-cone* of a cone algebra, which plays the same role for cone algebras in our work, as Dvurečenskij's theorem ([3]) has done for pseudo MV-algebras in the proof of Rachůnek's Theorem 2 ([5]).

Sections 5 and 6 are devoted to the proof of the main theorem, given in two steps.

4. Enveloping ℓg-cone of a cone algebra

In this section, we lay the foundation for the proof of the main theorem; and for this purpose, the following theorem due to Bosbach ([1, p. 64]) is very pivotal.

First Embedding Theorem (Bosbach)**.** *Necessary and sufficient condition for an algebra $(R;*,:)$ to admit an extension $(S;*,:)$, which is the residuation groupoid of some ℓ-group cone is that $(R;*,:)$ is a cone algebra.*

We recall from Bosbach [2] that a right residuation groupoid is a binary algebra $(A;\circ)$ satisfying the equations:

(1) $(a \circ a) \circ b = b$,

(2) $(a \circ b) \circ (a \circ c) = (b \circ a) \circ (b \circ c)$,

(3) $a \circ (b \circ b) = c \circ c$, and

(4) $a \circ b = c \circ c = b \circ a$ imply $a = b$.

Also, an algebra $(A;*,:)$ is called a residuation groupoid, if and only if both $(A;*)$ and $(A;:)$ are right residuation groupoids. For instance, the positive cone G^+ of a (not necessarily abelian) ℓ-group $(G;+,\leq)$ with operations $*$ and $:$ defined by

$$a * b = (-a + b) \vee 0 \qquad \text{and} \qquad a : b = (a - b) \vee 0$$

is a residuation groupoid called the residuation groupoid of the ℓ-group cone G^+, and $(G^+;*,:)$ is also called the cone algebra of the ℓ-group cone $(G^+;+,\leq)$.

Now let C be a cone algebra, then by the first embedding theorem of Bosbach [1], C is a subalgebra of the cone algebra of the positive cone G^+ of some ℓ-group $(G;+,\leq)$. If $\widehat{C}$ is the subsemigroup of $(G^+;+)$ generated by C, then $(\widehat{C};*,:)$ is an ℓ-group cone ([6, Lemma 4.2]), which contains C as a convex subalgebra ([6, Theorem 4.6]). We now prove:

Lemma 4.1. *Let $(C;*,:)$ be a cone algebra and let $(K;*,:)$ be the cone algebra of the positive cone of an ℓ-group. Also, let $\sigma\colon C \to K$ be a homomorphism of cone algebras; then σ can be extended to a homomorphism $\widehat{\sigma}\colon \widehat{C} \to K$ of cone algebras.*

Proof. Let $C_0 = C$ and $C_k = C_{k-1} + C_{k-1}$, for $k \geq 1$. Then it follows that each C_k is a cone subalgebra of the ℓ-group cone $\widehat{C}$, by induction and the following identities:

$$\left.\begin{array}{rl} & (a+b)*(c+d) = (b*(a*c)) + [((a*c)*b)*((c*a)*d)] \\ \text{and} & (d+c):(b+a) = [(d:(a:c)):(b:(c:a))] + ((c:a):b) \end{array}\right\} \qquad (\#)$$

which are valid in every ℓ-group cone.

Let $a, b, c, d \in C$ and suppose that $a+b = c+d$. Then, since $x+y=0 \implies x=y=0$ in an ℓ-group cone, we obtain from (#):

$$b*(a*c) = 0 \qquad \text{and} \qquad ((a*c)*b)*((c*a)*d) = 0$$

since $(a+b)*(c+d) = 0$. Since σ is a homomorphism of cone algebras, we obtain

$$\sigma(b)*(\sigma(a)*\sigma(c)) = 0 \qquad \text{and} \qquad ((\sigma(a)*\sigma(c))*\sigma(b))*((\sigma(c)*\sigma(a))*\sigma(d)) = 0$$

from which it follows that $(\sigma(a)+\sigma(b))*(\sigma(c)+\sigma(d)) = 0$. By symmetry, we also get $(\sigma(c)+\sigma(d))*(\sigma(a)+\sigma(b)) = 0$ and hence $\sigma(a)+\sigma(b) = \sigma(c)+\sigma(d)$ whenever $a+b = c+d$. Hence if we put $\sigma_1(a+b) = \sigma(a)+\sigma(b)$, then $\sigma_1\colon C_1 \to K$ is well defined; and it is routine to verify that σ_1 is a homomorphism of cone algebras (using the identities (#)) and that σ_1 extends σ.

Now

(1) $\{C_k\}$ is an increasing sequence of cone algebras, each contained in $\widehat{C}$, and
(2) for each $k \geq 0$, there is a homomorphism $\sigma_k\colon (C_k;*,:) \to (K;*,:)$ of cone algebras, where $\sigma_0 = \sigma$ and for $k \geq 1$, $\sigma_k(a+b) = \sigma_{k-1}(a) + \sigma_{k-1}(b)$ for all $a, b \in C_{k-1}$, so that σ_k extends σ_{k-1}.

Now $\widehat{C} = \bigcup\limits_{k\geq 0} C_k$ and put $\widehat{\sigma} = \bigcup\limits_{k\geq 0} \sigma_k$. Then since $(\widehat{C};+)$ is generated by C, $\widehat{\sigma}\colon \widehat{C} \to K$ is a homomorphism of cone algebras. $\square$

Now, if $a_1 + a_2 + \cdots + a_{2k} \in C_k$, then by induction

$$\begin{aligned} \widehat{\sigma}(a_1 + a_2 + \cdots + a_{2k}) &= \sigma_k(a_1 + a_2 + \cdots + a_{2k}) \\ &= \sigma_{k-1}(a_1 + a_2 + \cdots + a_k) + \sigma_{k-1}(a_{k+1} + \cdots + a_{2k}) \\ &= \sigma(a_1) + \sigma(a_2) + \cdots + \sigma(a_{2k}). \end{aligned}$$

Since $0 \in C$, it follows that

$$\widehat{\sigma}(a_1 + a_2 + \cdots + a_n) = \sigma(a_1) + \sigma(a_2) + \cdots + \sigma(a_n) \qquad \text{for all} \quad n \geq 1.$$

Lemma 4.2. *Let $\mu\colon G^+ \to K^+$ be a homomorphism of the cone algebras of the ℓ-group cones G^+ and K^+; then $\mu(a+b) = \mu(a) + \mu(b)$ for all $a, b \in G^+$.*

Proof. $\mu(a) * \mu(a+b) = \mu(a * (a+b)) = \mu(b) = \mu(a) * (\mu(a) + \mu(b))$ and $\mu(a+b) * \mu(a) = \mu((a+b) * a) = \mu(0) = (\mu(a) + \mu(b)) * \mu(a)$. Hence $\mu(a+b) = \mu(a) + \mu(b)$ by Lemma 3.6 (Part I). □

Lemma 4.3. *With the notations preceding the above* Lemma 4.2*, $\widehat{\sigma}\colon \widehat{C} \to K^+$ is the* unique *extension of σ.*

Proof. Let $\bar{\sigma}\colon \widehat{C} \to K^+$ be a homomorphism of cone algebras extending σ; then by Lemma 4.2,

$$\begin{aligned}\bar{\sigma}(a_1 + a_2 + \cdots + a_n) &= \bar{\sigma}(a_1) + \bar{\sigma}(a_2) + \cdots + \bar{\sigma}(a_n)\\ &= \sigma(a_1) + \sigma(a_2) + \cdots + \sigma(a_n)\\ &= \widehat{\sigma}(a_1 + a_2 + \cdots + a_n) \qquad \text{for all} \quad a_1, \dots, a_n \in C.\end{aligned}$$

Since $(\widehat{C}; +)$ is generated by C, we have $\bar{\sigma} = \widehat{\sigma}$. □

Summarizing, we have:

Theorem 4.4. *Let C be a cone algebra, then there exists an ℓ-group cone $\widehat{C}$ such that*

1. *$(C; *, :)$ is a convex subalgebra of the cone algebra $(\widehat{C}; *, :)$ of the ℓ-group cone $\widehat{C}$;*
2. *$(\widehat{C}; +)$ is generated by C; and*
3. *if K is an ℓ-group cone, then every homomorphism σ of C into the cone algebra of the ℓ-group cone K can be* uniquely *extended to a homomorphism $\widehat{\sigma}\colon \widehat{C} \to K$ of cone algebras; and $\widehat{\sigma}$ is then also a monoid homomorphism of $\widehat{C} \to K$.*

Remark 4.5. We have seen that if K is an ℓ-group cone, then $(K; *, :, +)$ is an *ℓg-cone* where $(K; *, :)$ is the cone algebra of K. Hence $\widehat{\sigma}\colon \widehat{C} \to K$ can be described as a *homomorphism of ℓg-cones*.

Corollary 4.6. *Given a cone algebra C, the ℓ-group cone $\widehat{C}$ satisfying the conditions* (1) *and* (2) *of the above* Theorem 4.4*, is uniquely determined up to isomorphism.*

Proof. Routine. □

We now formally introduce the following definition:

Definition 4.7. The ℓ-group cone $\widehat{C}$ described in the above Corollary 4.6, will be called *the enveloping* ℓ-group cone (or simply, ℓg-cone) of C.

The following theorem sharpens Theorem 4.4(3) above.

Theorem 4.8. *Let C be a convex subset of the positive cone of an ℓ-group $(G;+,\leq)$, which generates the semigroup $(G^+;+)$. Then*

(i) *C is a cone algebra with $(G^+;*,:)$ as its enveloping ℓg-cone $\widehat{C}$, and*

(ii) *every cone algebra homomorphism σ of C into the cone algebra of the positive cone of an ℓ-group $(K;+,\leq)$ can be uniquely extended to an ℓ-homomorphism $\widetilde{\sigma}\colon G \to K$ of ℓ-groups.*

Proof.

(i) Since $a * b \leq b$ and $a : b \leq a$, the convexity of C implies that $(C;*,:)$ is a cone algebra contained in G^+; and since C generates $(G^+;+)$, $(G^+;*,:)$ is an enveloping ℓg-cone of C.

(ii) By Theorem 4.4(3), $\sigma\colon C \to K^+$ can be uniquely extended to a homomorphism $\widehat{\sigma}\colon \widehat{C} \to K^+$ and $\widehat{\sigma}$ is also a monoid homomorphism of $\widehat{C} \to K^+$. Further, $\widehat{\sigma}(a\wedge b) = \widehat{\sigma}(a : (b * a)) = \widehat{\sigma}(a) : (\widehat{\sigma}(b) * \widehat{\sigma}(a)) = \widehat{\sigma}(a) \wedge \widehat{\sigma}(b)$. If we define $\widetilde{\sigma}(a) = \widehat{\sigma}(a^+) - \widehat{\sigma}(a^-)$ for each $a \in G$, then it is a routine verification to show that $\widetilde{\sigma}$ is an ℓ-homomorphism $G \to K$. Uniqueness of $\widetilde{\sigma}$ is clear. $\square$

Theorem 4.8 above is the cone algebra version of [4, Corollary 7.5] of Kühr where a similar result has been proved by Kühr for pseudo LBCK-algebras. Clearly, Theorem 4.8 establishes directly a categorical equivalence of the category of cone algebras and homomorphisms with the category $\mathcal{LG}$ whose objects are pairs (G,X) where $(G;+,\leq)$ is an ℓ-group and X is a convex subset of G^+, which generates the semigroup $(G^+;+)$ and whose morphisms $f\colon (G,X) \to (H,Y)$ are ℓ-group homomorphisms $f\colon G \to H$ such that $f(X) \subseteq Y$. We omit the details, which are similar to those presented by Kühr in [4].

Now Lemma 4.2 says that for ℓg-cones (or equivalently, ℓ-group cones — see Section 3, Part I), every homomorphism of their cone algebra reducts is also a homomorphism of the ℓg-cones. However, this is not true for semi-ℓg-cones as shown by the following example.

Example 4.9. Let $(C;*,:,\vee)$ be a Boolean cone (Remark 3.8, Part I) — which is a semi-ℓg-cone, in which every element is idempotent — and let $\widehat{C}$ be the enveloping ℓ-group cone of its reduct $(C;*,:)$. Then $(\widehat{C};*,:,+)$ is an ℓg-cone and hence a semi-ℓg-cone, and since $(C;*,:)$ is a subalgebra of $(\widehat{C};*,:)$, the inclusion mapping $j\colon C \to \widehat{C}$ is a monomorphism of cone algebras. Now let $a,b \in C$; then $a \vee b$ is a common upper bound of a and b in $\widehat{C}$ also, since the partial ordering in $\widehat{C}$ is an extension of the partial ordering in C. Hence, if u is least upper bound of a and b in $\widehat{C}$ (which exists, since every semi-ℓg-cone is a join semilattice), we have $u \leq a \vee b$; and since C is a convex subset of $\widehat{C}$ (Theorem 4.4(1)) and $a\vee b \in C$, we get $u \in C$ and hence $u = a\vee b$. But $u \neq a+b$ unless $a \wedge b = 0$ and hence j is not a homomorphism of monoids. So j is not a homomorphism of the ℓg-cones.

5. Proof of Main Theorem — First step

This section presents the first step in the proof of our main theorem. In the following, we let $(C; *, :)$ be a cone algebra and $(\widehat{C}; *, :, +)$ its enveloping ℓ-group cone. Also, if $A \subseteq C$, we write $\widehat{A}$ for the subsemigroup of $(\widehat{C}; +)$ generated by A.

Lemma 5.1 (Riesz Decomposition). *Let $(G^+; *, :, +)$ be an ℓ-group cone, $c, a_1, a_2, \ldots, a_n \in G^+$ and let*

$$c \le a_1 + a_2 + \cdots + a_n.$$

Then there exist $c_1, c_2, \ldots, c_n \in G^+$ such that $c_i \le a_i$ for $i = 1, 2, \ldots, n$ and

$$c = c_1 + c_2 + \cdots + c_n.$$

The proof of this well known lemma is omitted.

Lemma 5.2. *If A is a convex subset of C, then $\widehat{A}$ is a convex subset of $\widehat{C}$.*

Proof. Assume $a \in \widehat{A}$ and $c \le a$; and write $a = a_1 + a_2 + \cdots + a_n$ where $a_1, a_2, \ldots, a_n \in A$. By the above Lemma 5.1, we have $c = c_1 + c_2 + \cdots + c_n$ where each $c_i \le a_i$ for $i = 1, 2, \ldots, n$. Since A is convex, each $c_i \in A$ and hence $c = c_1 + \cdots + c_n \in \widehat{A}$. Hence $\widehat{A}$ is convex. □

Corollary 5.3. *If A is a nonempty convex subset of C, then $\widehat{A}$ is an ideal of $\widehat{C}$* (Definition 2.9(ii), Part I).

Proof. $\widehat{A}$ is a subsemigroup of $\widehat{C}$ by construction, and convex by Lemma 5.2. So $\widehat{A}$ is an ideal of $\widehat{C}$. □

Lemma 5.4. *Let A be a deductive system of C, $a_1, a_2, \ldots, a_n \in C$ and $a_1 + a_2 + \cdots + a_n \in C$; then $a_1 + a_2 + \cdots + a_n \in A$ if and only if $a_i \in A$ for $i = 1, 2, \ldots, n$.*

Proof. By Theorem 2.8 (Part I), every deductive system of a cone algebra C is a convex subset of C. Hence if $a_1 + a_2 + \cdots + a_n \in A$, then each $a_i \in A$ since $a_i \le a_1 + a_2 + \cdots + a_n$ for all i. Conversely, assume that $a_1, a_2, \ldots, a_n \in A$; since $a_1 + a_2 + \cdots + a_k \le a_1 + a_2 + \cdots + a_n \in C$ and C is convex subset of $\widehat{C}$ (Theorem 4.4(1)), $a_1 + a_2 + \cdots + a_k \in C$ for all $k \le n$. Now if $k < n$ and $a_1 + a_2 + \cdots + a_k \in A$, then $a_1 + a_2 + \cdots + a_{k+1} \in A$ since $(a_1 + a_2 + \cdots \cdots + a_k) * (a_1 + a_2 + \cdots + a_{k+1}) = a_{k+1} \in A$. For $n = 1$, $a_1 \in A$; hence by induction, $a_1 + a_2 + \cdots + a_n \in A$. □

Lemma 5.5. *If A is a deductive system of C, then $\widehat{A}$ is an ideal of $\widehat{C}$ and $\widehat{A} \cap C = A$.*

Proof. If A is a deductive system of C, then A is a nonempty convex subset of C and hence, by Corollary 5.3, $\widehat{A}$ is an ideal of $\widehat{C}$.

Now, let $g \in \widehat{A} \cap C$; then we can write $g = a_1 + a_2 + \cdots + a_n \in C$ where $a_1, a_2, \ldots, a_n \in A$. Then by Lemma 5.4, $g \in A$. Hence $\widehat{A} \cap C \subseteq A$ and obviously $A \subseteq \widehat{A} \cap C$. Hence $\widehat{A} \cap C = A$. □

Corollary 5.6. *If A and B are deductive systems of C, then $A \subseteq B \iff \widehat{A} \subseteq \widehat{B}$.*

Proof. $A \subseteq B \implies \widehat{A} \subseteq \widehat{B} \implies \widehat{A} \cap C \subseteq \widehat{B} \cap C \implies A \subseteq B$. $\square$

Lemma 5.7. *If J is an ideal of $\widehat{C}$, then $J \cap C$ is a deductive system of C and $\widehat{J \cap C} = J$.*

Proof. Clearly, $0 \in J \cap C$; and now let $a \in J \cap C, b \in C$ and $a * b \in J \cap C$. Then $a \in J$, $a * b \in J$ and $b \in \widehat{C}$ and since J is an ideal of $\widehat{C}$, we get $b \in J$ (by Theorem 2.8, Part I) and hence $b \in J \cap C$. Hence $J \cap C$ is a deductive system of C.

Now let $a \in J$; then, since $J \subseteq \widehat{C}$, we can write $a = c_1 + c_2 + \cdots + c_n$ where $c_1, c_2, \ldots, c_n \in C$. Now each $c_i \leq a$ and J is an ideal $\widehat{C}$ and hence $c_i \in J \cap C$ for $i = 1, 2, \ldots, n$. Hence $a \in \widehat{J \cap C}$ so that $J \subseteq \widehat{J \cap C}$. On the other hand, J, being an ideal of $\widehat{C}$, is a subsemigroup of $\widehat{C}$ and $J \cap C \subseteq J$. Hence $\widehat{J \cap C} \subseteq J$. Hence $\widehat{J \cap C} = J$. $\square$

We now combine Lemmas 5.5 and 5.7 with Corollary 5.6 to obtain the following preliminary theorem.

Theorem 5.8. *Let C be a cone algebra and $\widehat{C}$ its enveloping ℓ-group cone. Then the mapping $A \mapsto \widehat{A}$ is an isomorphism of the lattice of deductive systems of C onto the lattice of ideals of $\widehat{C}$. The inverse isomorphism is given by $J \mapsto J \cap C$.*

We will now show that under the isomorphism described above, the prime, normal and polar deductive systems of C correspond to the prime, normal and polar ideals of $\widehat{C}$, respectively. We continue with the notation used in Theorem 5.8.

Lemma 5.9. *A is a prime deductive system of C if and only if $\widehat{A}$ is a prime ideal of $\widehat{C}$.*

Proof. Recall that a deductive system D of C (an ideal D of $\widehat{C}$) is said to be *prime* if and only if

(i) $D \subsetneqq C$ ($D \subsetneqq \widehat{C}$) and
(ii) $a \wedge b \in D \implies a \in D$ or $b \in D$.

Observe that, by Theorem 5.8, A is proper if and only if $\widehat{A}$ is proper; and now assume that A is a prime deductive system of C. Let $a, b \in \widehat{C}$, $a \wedge b \in \widehat{A}$ and $b \notin \widehat{A}$. Write $a = a_1 + a_2 + \cdots + a_n$ and $b = b_1 + b_2 + \cdots + b_m$ where all the a_i's and b_j's are in C. Since $b \notin \widehat{A}$, some b_j, say b_1, is not in A. Then for each j, $a_j \wedge b_1 \in C$ and since $a_j \wedge b_1 \leq a \wedge b \in \widehat{A}$ and $\widehat{A}$ is convex (since A is convex) we have $a_j \wedge b_1 \in \widehat{A} \cap C = A$. Since A is prime and $b_1 \notin A$, we get $a_j \in A$ for all j. Hence $a = a_1 + a_2 + \cdots + a_n \in \widehat{A}$. Hence $\widehat{A}$ is a prime ideal of $\widehat{C}$.

Conversely, assume $\widehat{A}$ is a prime ideal of $\widehat{C}$; and let $a, b \in C$ and $a \wedge b \in A$. Then $a, b \in \widehat{C}$ and $a \wedge b \in \widehat{A}$; and hence $a \in \widehat{A}$ or $b \in \widehat{A}$. Hence $a \in \widehat{A} \cap C = A$ or $b \in \widehat{A} \cap C = A$. Hence A is a prime deductive system of C. □

Lemma 5.10. *If J is a normal ideal of $\widehat{C}$, then $J \cap C$ is a normal deductive system of C.*

Proof. We know that $J \cap C$ is a deductive system of C; and now let $a, b \in C$. Then $a*b$ and $b : a$ are in C; and hence $a*b \in J \cap C \iff a*b \in J \iff b : a \in J$ (since J is a normal ideal of $\widehat{C}$ and $a, b \in \widehat{C}$) $\iff b : a \in J \cap C$. Hence $J \cap C$ is a normal deductive system of C. □

Lemma 5.11. *In any ℓ-group cone, the following statements are equivalent:*

(i) $x = a + b$

(ii) $a * x = b$ *and* $x * a = 0$

(iii) $x : b = a$ *and* $b : x = 0$.

Proof. Routine. □

Lemma 5.12. *Let A be a normal deductive system of C, $a \in A$ and $c \in C$; then there exist $b, d \in \widehat{A}$ such that $c + a = b + c$ and $a + c = c + d$.*

Proof. From the identities (#) (Section 4) valid in every ℓ-group cone, we know

$$\begin{aligned} c * (a + c) &= (0 + c) * (a + c) \\ &= (c * (0 * a)) + [((0 * a) * c) * ((a * 0) * c)] \\ &= (c * a) + ((a * c) * c), \end{aligned}$$

and dually,

$$(c + a) : c = (c : (c : a)) + (a : c).$$

Since A is normal ([7, Theorem 2.13, Definition 2.14] (Part I)) $(a * c) * c \in A$ and $c : (c : a) \in A$; and $c * a \le a \in A$ and $a : c \le a \in A$ and A is convex, so that $c * a \in A$ and $a : c \in A$. Hence if we put $b = (c + a) : c$ and $d = c * (a + c)$, then $b, d \in \widehat{A}$ and, by Lemma 5.11, $c + a = b + c$ and $a + c = c + d$. □

Corollary 5.13. *Let A be a normal deductive system of C, $a \in \widehat{A}$ and $c \in C$; then there exist $b, d \in \widehat{A}$ such that $c + a = b + c$ and $a + c = c + d$.*

Proof. Write $a = a_1 + a_2 + \cdots + a_n$ where each $a_i \in A$; then, by the above Lemma 5.12, there exist $b_1, b_2, \ldots, b_n, d_1, d_2, \ldots, d_n \in \widehat{A}$ such that $c + a_i = b_i + c$ and $a_i + c = c + d_i$ for $i = 1, 2, \ldots, n$. Put $b = b_1 + b_2 + \cdots + b_n$ and $d = d_1 + d_2 + \cdots + d_n$; then $b, d \in \widehat{A}$, $c + a = b + c$ and $a + c = c + d$. □

Corollary 5.14. *If A is a normal deductive system of C, then $\widehat{A}$ is a normal ideal of $\widehat{C}$.*

Proof. If $c \in C$ and $a \in \widehat{A}$, then by Corollary 5.13, $c+a \in \widehat{A}+c$ and $a+c \in c+\widehat{A}$ and hence $c+\widehat{A} \subseteq \widehat{A}+c \subseteq c+\widehat{A}$. Hence $c+\widehat{A} = \widehat{A}+c$ for all $c \in C$. Now if $c \in \widehat{C}$, we can write $c = c_1 + c_2 + \cdots + c_n$ where each $c_i \in C$ and it follows that $c+\widehat{A} = \widehat{A}+c$ by the above. Hence $\widehat{A}$ is a normal ideal of $\widehat{C}$ (by [7, Theorem 2.13, Definition 2.14] (Part I)). □

Now we combine Lemma 5.10 with Corollary 5.14 to obtain the following lemma.

Lemma 5.15. *A is a normal deductive system of C if and only if $\widehat{A}$ is a normal ideal of $\widehat{C}$.*

Lemma 5.16. *If A is a subset of C, then*

$$\widehat{A^{\perp}} = (\widehat{A}\,)^{\perp}.$$

Proof. Note that $A^{\perp}$ is computed in C, while $(\widehat{A}\,)^{\perp}$ is computed in $\widehat{C}$.

Now suppose that $a, b_1, b_2 \in \widehat{C}$ and $a \wedge b_1 = a \wedge b_2 = 0$; then $b_1 * a = b_2 * a = a$ and hence $(b_1+b_2) * a = b_2 * (b_1 * a) = b_2 * a = a$ so that $(b_1+b_2) \wedge a = 0$. On the other hand, if $a \wedge (b_1+b_2) = 0$, then $a \wedge b_1 = a \wedge b_2 = 0$ since $b_1 \le b_1 + b_2$ and $b_2 \le b_1 + b_2$. Hence $a \wedge (b_1+b_2) = 0 \iff a \wedge b_1 = a \wedge b_2 = 0$. Hence by induction and symmetry, we get $(a_1 + a_2 + \cdots + a_m) \wedge (b_1 + b_2 + \cdots + b_n) = 0$ if and only if $a_i \wedge b_j = 0$ for all i, j where $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in \widehat{C}$. In particular, this is true if $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in C$. Now let $c_1, c_2, \ldots, c_m \in C$; then

$$c_1 + c_2 + \cdots + c_m \in (\widehat{A}\,)^{\perp} \iff (c_1 + c_2 + \cdots + c_m) \wedge (a_1 + a_2 + \cdots + a_n) = 0$$

for all $a_1, a_2, \ldots, a_n \in A \iff (c_1 + c_2 + \cdots + c_m) \wedge a = 0$ for all $a \in A \iff c_i \wedge a = 0$ for all $a \in A$ and $i = 1, 2, \ldots, m \iff c_i \in A^{\perp}$ for $i = 1, 2, \ldots, m \iff c_1 + c_2 + \cdots + c_m \in \widehat{A^{\perp}}$. Hence $\widehat{A^{\perp}} = (\widehat{A}\,)^{\perp}$. □

Lemma 5.17. *A is a polar deductive system of C if and only if $\widehat{A}$ is a polar ideal of $\widehat{C}$.*

Proof. Assume A is a polar deductive system of C; then $A = A^{\perp\perp}$. Hence $\widehat{A} = \widehat{(A^{\perp\perp})} = (\widehat{A}\,)^{\perp\perp}$ by applying Lemma 5.16 twice. Hence $\widehat{A}$ is a polar ideal of $\widehat{C}$.

Conversely, assume that $\widehat{A}$ is a polar ideal of $\widehat{C}$; then $A^{\perp} = \widehat{A^{\perp}} \cap C$ (since $A^{\perp}$ is a deductive system of C) $= (\widehat{A}\,)^{\perp} \cap C$ (by Lemma 5.16). Hence

$$\begin{aligned} A^{\perp\perp} &= (\widehat{A^{\perp}})^{\perp} \cap C && \text{(replacing } A \text{ by } A^{\perp} \text{ in the above)} \\ &= (\widehat{A}\,)^{\perp\perp} \cap C && \text{(by Lemma 5.16)} \\ &= (\widehat{A}\,) \cap C && \text{(since } \widehat{A} \text{ is a polar ideal of } \widehat{C}) \\ &= A. \end{aligned}$$

Hence A is a polar deductive system of C. □

In the next section, we will prove our main theorem.

6. Proof of Main Theorem — Final step

Let $(G; +, \leq)$ be an arbitrary lattice ordered group and A an ideal of the ℓg-cone $(G^{+}; *, :, +)$; then we know that A is a convex ℓ-submonoid of $(G^{+}; +, \leq)$. Then

Lemma 6.1. *If $g \in G$, then the following are equivalent:*

(α) *$|g| \in A$, where $|g| = g \vee -g$; and*

(β) *there exists $a \in A$ such that $g + a \in A$.*

Proof. Assume (α); then since A is convex and $0 \leq g \vee 0 \leq |g| \in A$ and $0 \leq -g \vee 0 \leq |g|$, both $g \vee 0$ and $-g \vee 0$ are in A. Also, $g + (-g \vee 0) = g \vee 0 \in A$. Hence ($\alpha$) $\implies$ (β).

Assume (β); and write $g + a = b \in A$. Then $g = b - a$ and $|g| = (b-a) \vee (a-b) = |a - b| \leq |a| + |b| + |a| \in A$ since A is a submonoid of $(G^{+}; +)$. Since A is also convex, it follows that $|g| \in A$. Hence (β) $\implies$ (α). □

Lemma 6.2. *Let A be an ideal of $(G^{+}; *, :, +)$ and write*

$$\varphi(A) = \{g \in G \ ; \ |g| \in A\}.$$

Then $\varphi(A)$ is the convex ℓ-subgroup of G generated by A.

Proof. If $g, h \in \varphi(A)$, then $|g - h| \leq |g| + |h| + |g| \in A$ and hence $|g - h| \in A$, since A is convex. Hence $g - h \in \varphi(A)$ so that $\varphi(A)$ is a subgroup of G. If $g \in \varphi(A)$ and $|h| \leq |g|$, then $|g| \in A$ and hence $|h| \in A$ since A is convex. Hence $h \in \varphi(A)$ so that $\varphi(A)$ is a convex subgroup of G. Also, if $g \in \varphi(A)$, then $0 \leq g \vee 0 \leq |g| \in A$ and so $g \vee 0 \in \varphi(A)$. Hence $\varphi(A)$ is also a sublattice of $(G; \leq)$. Thus $\varphi(A)$ is a convex ℓ-subgroup of G.

Now let K be a convex ℓ-subgroup of G containing A; and let $g \in \varphi(A)$. Then by Lemma 6.1, there exist $a, b \in A$ such that $g + a = b$ and hence $|g| = |a - b| \leq a + b + a \in A \subseteq K$; and since K is convex, $|g| \in K$. Hence $g \in K$ so that $\varphi(A) \subseteq K$. Hence $\varphi(A)$ is the convex ℓ-subgroup of G, generated by A. □

Lemma 6.3. *Let K be a convex ℓ-subgroup of G and let $A = K \cap G^+$; then A is an ideal of $(G^+; *, :, +)$ and $\varphi(A) = K$.*

Proof. If $A = K \cap G^+$, then A is clearly a convex submonoid of $(G^+; +, \leq)$ and hence is an ideal of $(G^+; *, :, +)$. Now, if $g \in G$, then $g \in \varphi(K \cap G^+) \iff |g| \in K \cap G^+ \iff |g| \in K \iff g \in K$ since K is a convex ℓ-subgroup of G. Hence $K = \varphi(A)$. □

Lemma 6.4. *If A is an ideal of $(G^+; *, :, +)$, then $\varphi(A) \cap G^+ = A$.*

Proof. $g \in \varphi(A) \cap G^+ \implies g = |g| \in A$ so that $\varphi(A) \cap G^+ = A$. □

Theorem 6.5. *Let $(G; +, \leq)$ be an arbitrary ℓ-group and let $(G^+; *, :, +)$ be its positive cone. Then the mapping $\varphi\colon A \mapsto \varphi(A)$ is an isomorphism of the lattice of ideals of $(G^+; *, :, +)$ onto the lattice of convex ℓ-subgroups of $(G; +, \leq)$. Further,*

- (α) *A is a prime ideal of $(G^+; *, :, +)$ if and only if $\varphi(A)$ is a prime convex ℓ-subgroup of $(G; +, \leq)$,*
- (β) *A is a normal ideal of $(G^+; *, :, +)$ if and only if $\varphi(A)$ is an ℓ-ideal of $(G; +, \leq)$, and*
- (γ) *A is a polar ideal of $(G^+; *, :, +)$ if and only if $\varphi(A)$ is a polar subgroup of $(G; +, \leq)$.*

Proof. We need only prove (α), (β) and (γ).

(α) Assume A is a prime ideal of $(G^+; *, :, +)$ and let $g, h \in G$ and $g \wedge h \in \varphi(A)$. Then by Lemma 6.1, there exists $a \in A$ such that $(g \wedge h) + a \in A$, i.e., $(g + a) \wedge (h + a) \in A$. Since $A \subseteq G^+$, $0 \leq (g + a) \wedge (h + a)$ and hence $g + a, h + a \in G^+$. Since A is prime ideal, $g + a \in A$ or $h + a \in A$. Hence by Lemma 6.1 again $g \in \varphi(A)$ or $h \in \varphi(A)$ so that $\varphi(A)$ is a prime ℓ-subgroup. The converse is clear.

(β) Assume that A is a normal ideal of $(G^+; *, :, +)$; then $c + A = A + c$ for all $c \in G^+$. Now, if $g \in G$, then $a = -g \vee 0 \in G^+$ and $g + a \in G^+$. Hence $(g + A) + a = g + (A + a) = g + (a + A) = (g + a) + A = A + (g + a) = (A + g) + a$ and hence $A + g = g + A$, since $(G^+; +, 0)$ is a cancellative monoid. Hence if $g \in G$, then $g + A - g \subseteq A$.

Now let $k \in \varphi(A)$ and $g \in G$; then $|g + k - g| = (g + k - g) \vee (g - k - g) = g + |k| - g \in A$ since $|k| \in A$. Hence $g + k - g \in \varphi(A)$; and hence $\varphi(A)$ is a normal subgroup of G and hence an ℓ-ideal of G.

Conversely, assume that $\varphi(A)$ is a normal subgroup of G, and let $k \in A$. Then $k \in G^+$ and hence for any $g \in G$, $g + k - g \in \varphi(A) \cap G^+ = A$ (since $A \subseteq \varphi(A)$). Hence $g + A = A + g$ for all $g \in G$, and in particular, $c + A = A + c$ for all $c \in G^+$. Hence A is a normal ideal of $(G^+; *, :, +)$.

(γ) We first prove that $\varphi(A^\perp) = (\varphi(A))^\perp$; note that $A^\perp$ is computed in G^+, while $(\varphi(A))^\perp$ is computed in G. Now $g \in \varphi(A^\perp) \iff |g| \in A^\perp \iff |g| \wedge a = 0$ for all $a \in A \iff |g| \wedge |k| = 0$ for all $k \in \varphi(A) \iff g \in \varphi(A)^\perp$.

Hence $\varphi(A^{\perp}) = (\varphi(A))^{\perp}$. Hence, if A is a polar ideal of $(G^{+}; *, :, +)$, then $\varphi(A) = \varphi(A^{\perp\perp}) = (\varphi(A^{\perp}))^{\perp} = (\varphi(A))^{\perp\perp}$ so that $\varphi(A)$ is a polar subgroup of G. Conversely, if $\varphi(A)$ is a polar subgroup of G, then $\varphi(A) = (\varphi(A))^{\perp\perp} = \varphi(A^{\perp\perp})$ and hence $A = \varphi(A) \cap G^{+} = \varphi(A^{\perp\perp}) \cap G^{+} = A^{\perp\perp}$ (by Lemma 6.4) and hence A is a polar ideal of $(G^{+}; *, :, +)$. □

We now assume that $(C; *, :)$ is a cone algebra, $(\widehat{C}; *, :, +)$ its enveloping ℓ-group cone and $(\widehat{C}; +, \leq)$ is the positive cone of the lattice ordered group $(G; +, \leq)$ so that $\widehat{C} = G^{+}$. Further, let A be a nonempty convex subset of C; then, with the above notation,

Lemma 6.6. *$\varphi(\widehat{A})$ is the subgroup of G, generated by A.*

Proof. By Corollary 5.3, $\widehat{A}$ is an ideal of $\widehat{C}$ and so, by Lemma 6.2, $\varphi(\widehat{A})$ is the convex ℓ-subgroup of G, generated by $\widehat{A}$. Now let H be a subgroup of G containing A, then $\widehat{A} \subseteq H$. If $g \in \varphi(\widehat{A})$, then by Lemma 6.1, there exist $a, b \in \widehat{A}$ such that $g + a = b$ so that $g = b - a \in H$. Hence $\varphi(\widehat{A}) \subseteq H$. Thus $\varphi(\widehat{A})$ is the subgroup of G, generated by A. □

Corollary 6.7. *With the same notation as above, the subgroup of G generated by a convex subset A of C — in particular, a deductive system of C — is just the convex ℓ-subgroup of G generated by A.*

Hence we obtain our main theorem.

Theorem 6.8. *Let C be a cone algebra and suppose that its enveloping ℓ-group cone $\widehat{C}$ is the positive cone of the lattice ordered group G. Also, if A is a subset of C, then write $[A]$ for the subgroup of G generated by A. Then*

(a) *if A is a deductive system of C, then $[A]$ is a convex ℓ-subgroup of G; and*

(b) *the mapping $A \mapsto [A]$ is an isomorphism of the lattice of deductive systems of C* onto *the lattice of convex ℓ-subgroups of G.*

Further,

(i) *A is a prime deductive system of C if and only if $[A]$ is a prime convex ℓ-subgroup of G;*

(ii) *A is a normal deductive system of C if and only if $[A]$ is an ℓ-ideal of G; and*

(iii) *A is a polar deductive system of C if and only if $[A]$ is a polar convex ℓ-subgroup of G.*

Remark 6.9. Kühr has proved ([4, Theorem 7.7]) the above Theorem 6.8 for pseudo LBCK-algebras except (i) and (iii); also observe that a normal deductive system has been called a *compatible* deductive system in [4].

Remark 6.10. By [7, Theorem 3.13], A is an ideal of a pseudo MV-algebra $(C;\oplus,^-,^\sim,0,1)$ if and only if A is an ideal of the equivalent semi-ℓg-cone $(C;*,:,\oplus)$ if and only if A is deductive system of the *brick* $(C;*,:)$. If C in Theorem 6.8 is a brick, then $\widehat{C}$ is the positive cone of a unital ℓ-group with strong order unit u and $C=[0,u]$ (see [6, Lemma 4.4] and also [3]). Hence Theorem 6.8 yields:

Theorem (Rachůnek)**.** ([5, Theorem 2, p. 157]) *The lattice of ideals of a pseudo MV-algebra is isomorphic to the lattice of convex ℓ-subgroups of a unital lattice ordered group.*

Remark 6.11. Let $(C;*,:)$ be a precone algebra and A a normal deductive system of C; then the relation Θ_A defined by

$$(a,b)\in\Theta_A \iff a*b\in A \qquad \text{and} \qquad b*a\in A$$

is a congruence relation on C with $A=\Theta_A[0]$; and if Θ is a congruence relation on C, then $\Theta[0]$ is a normal deductive system of C such that $\Theta=\Theta_{\Theta[0]}$ (see Kühr [4, Section 7], for a discussion on commutative pseudo BCK-algebras, which we know are equivalent to precone algebras). Thus, the lattice of normal deductive systems of a precone algebra C is isomorphic to the lattice of congruences of C. Hence, if C is a cone algebra, then by Theorem 6.8, the congruence lattice of C is isomorphic to the lattice of ℓ-ideals of an ℓ-group G, which is, as is well known, isomorphic to the congruence lattice of G. Since the congruence lattice of an ℓ-group is distributive, so is the congruence lattice of a cone algebra. Now, ℓ-groups are congruence permutable, but we show below that cone algebras are not. (By the way, the example shows a well-known fact that commutative BCK-algebras are not congruence permutable.)

Example 6.12. Let $(C;*,:)$ be the cone algebra given in [7, Example 1.8] (Part I). Then $(C;*,:)$ is a symmetric cone algebra and hence all deductive systems are normal. If Θ and Φ are the congruences respectively determined by the deductive systems $(a]$ and $(b]$, then $(a,b)\in\Theta\circ\Phi$ and $(a,b)\notin\Phi\circ\Theta$. Hence $\Theta\circ\Phi\neq\Phi\circ\Theta$.

However, we have:

Theorem 6.13. *If C is a cone algebra directed above, then the congruence lattice of C is permutable.*

Proof. Since a commutative pseudo BCK-algebra is equivalent to a precone algebra, it follows by [4, Lemma 3.8], that a directed cone algebra (recall that a directed precone algebra is a cone algebra) is a lattice and hence, by [4, Proposition 4.4], is permutable. □

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