

SEQUENCE SPACES OF FUZZY REAL NUMBERS DEFINED BY ORLICZ FUNCTIONS

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(Communicated by Pavel Kostyrko)

ABSTRACT. In this article we study different properties of convergent, null and bounded sequence spaces of fuzzy real numbers defined by an Orlicz function, like completeness, solidness, symmetricity, convergence free etc. We prove some inclusion results, too.

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1. Introduction

An *Orlicz function* M is mapping $M: [0, \infty) \rightarrow [0, \infty)$ such that it is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$, as $x \rightarrow \infty$. If the convexity of Orlicz function M is replaced by

$$M(x + y) \leq M(x) + M(y),$$

then this function is called the *modulus function*, introduced by Nakano [6].

Remark 1. It is well known if M is an Orlicz function, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct the sequence space,

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

2000 Mathematics Subject Classification: Primary 40A05, 40D25, 46A45, 46E30.

Keywords: fuzzy real number, Orlicz function, solid space, symmetric space.

The work of the first author was carried under University Grants Commission of India project No.-F. No. 30-240/2004 (RS).

The space ℓ_M becomes a Banach space, with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left(\frac{|x_k|}{\rho} \right) \leq 1 \right\},$$

which is called an Orlicz sequence space.

The space ℓ_M is closely related to the space ℓ^p , which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

Later on Orlicz sequence spaces were investigated by Parashar and Choudhary [8], Tripathy et.al. [13], Savas and Rhoades [9] and many others.

Let D denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on \mathbb{R} , the real line. For $X, Y \in D$ we define

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|),$$

where $X = [a_1, a_2]$ and $Y = [b_1, b_2]$. It is known that (D, d) is a complete metric space.

A *fuzzy real number* X is a fuzzy set on \mathbb{R} , i.e. a mapping $X : \mathbb{R} \rightarrow I (= [0, 1])$ associating each real number t with its grade of membership $X(t)$.

A fuzzy real number X is called *convex* if

$$X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r)),$$

where $s < t < r$.

If there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *upper-semi continuous* if, for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon])$, for all $a \in I$ is open in the usual topology of \mathbb{R} .

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $R(I)$ and throughout the article, by a fuzzy real number we mean that the number belongs to $R(I)$.

The α -level set $[X]^\alpha$ of the fuzzy real number X , for $0 < \alpha \leq 1$, is defined as $[X]^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\}$. If $\alpha = 0$, then it is the closure of the strong 0-cut.

The set \mathbb{R} of all real numbers can be embedded in $R(I)$. For $r \in \mathbb{R}$, $\bar{r} \in R(I)$ is defined by

$$\bar{r}(t) = \begin{cases} 1, & \text{for } t = r, \\ 0, & \text{for } t \neq r. \end{cases}$$

The arithmetic operations for α -level sets are defined as follows:

Let $X, Y \in R(I)$ and the α -level sets be $[X]^\alpha = [a_1^\alpha, b_1^\alpha]$, $[Y]^\alpha = [a_2^\alpha, b_2^\alpha]$, $\alpha \in [0, 1]$. Then

$$\begin{aligned} [X \oplus Y]^\alpha &= [a_1^\alpha + a_2^\alpha, b_1^\alpha + b_2^\alpha], \\ [X - Y]^\alpha &= [a_1^\alpha - b_2^\alpha, b_1^\alpha - a_2^\alpha], \end{aligned}$$

$$[X \otimes Y]^\alpha = \left[\min_{i,j \in \{1,2\}} a_i^\alpha b_j^\alpha, \max_{i,j \in \{1,2\}} a_i^\alpha b_j^\alpha \right]$$

and

$$[Y^{-1}]^\alpha = \left[\frac{1}{b_2^\alpha}, \frac{1}{a_2^\alpha} \right], \quad \text{if } 0 \notin Y.$$

The *absolute value*, $|X|$ of $X \in R(I)$ is defined by (see for instance Kaleva and Seikkala [2])

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

A fuzzy real number X is called *non-negative* if $X(t) = 0$, for all $t < 0$. The set of all non-negative fuzzy real numbers is denoted by $R^*(I)$.

Let $\bar{d}: R(I) \times R(I) \rightarrow \mathbb{R}$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d([X]^\alpha, [Y]^\alpha).$$

Then \bar{d} defines a metric on $R(I)$.

The additive identity and multiplicative identity in $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

2. Definitions and preliminaries

After the introduction of $R(I)$, different classes of sequences were introduced and studied by Tripathy and Nanda [12], Savas [9], Das and Choudhury [1], Subrahmanyam [11] and many others.

A sequence (X_k) of fuzzy real numbers is said to be *convergent* to the fuzzy real number X_0 if, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\bar{d}(X_k, X_0) < \varepsilon$, for all $k \geq n_0$.

A fuzzy real number sequence (X_k) is said to be *bounded* if $\sup_k |X_k| \leq \mu$, for some $\mu \in R^*(I)$.

For $r \in \mathbb{R}$ and $X \in R(I)$ we define

$$rX(t) = \begin{cases} X(r^{-1}t), & \text{if } r \neq 0, \\ \bar{0}, & \text{if } r = 0. \end{cases}$$

Throughout the article w^F , ℓ_∞^F , c^F and c_0^F denote the classes of *all*, *bounded*, *convergent* and *null* fuzzy real number sequences respectively.

A sequence space E^F is said to be *normal* (or *solid*) if $(Y_k) \in E^F$, whenever $|Y_k| \leq |X_k|$, for all $k \in \mathbb{N}$, for some $(X_k) \in E^F$.

Let $K = \{k_1 < k_2 < k_3 < \dots\} \subseteq \mathbb{N}$ and E^F be a sequence space. A K -step space of E^F is a sequence space $\lambda_K^{E^F} = \{(X_{k_n}) \in w^F : (X_n) \in E^F\}$.

A canonical pre-image of a sequence $(X_{k_n}) \in \lambda_K^{E^F}$ is a sequence $(Y_n) \in w^F$ defined as follows:

$$Y_n = \begin{cases} X_n, & \text{if } n \in K, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

A canonical pre-image of a step space $\lambda_K^{E^F}$ is a set of canonical pre-images of all elements in $\lambda_K^{E^F}$, i.e., Y is in canonical pre-image $\lambda_K^{E^F}$ if and only if Y is canonical pre-image of some $X \in \lambda_K^{E^F}$.

A sequence space E^F is said to be *monotone* if E^F contains the canonical pre-images of all its step spaces.

From the above definitions we have the following remark.

Remark 2. A sequence space E^F is solid $\implies E^F$ is monotone.

A sequence space E^F is said to be *symmetric* if $(X_{\pi(n)}) \in E^F$, whenever $(X_k) \in E^F$, where π is a permutation of \mathbb{N} .

A sequence space E^F is said to be *convergence free* if $(Y_k) \in E^F$, whenever $(X_k) \in E^F$ and $X_k = \bar{0}$ implies $Y_k = \bar{0}$.

In this article we introduce the following sequence spaces of fuzzy real numbers defined by Orlicz function:

$$\begin{aligned} (\ell_\infty)_F(M) &= \left\{ (X_k) : \sup_k M \left(\frac{\bar{d}(X_k, \bar{0})}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}. \\ c_F(M) &= \left\{ (X_k) : \lim_{k \rightarrow \infty} M \left(\frac{\bar{d}(X_k, L)}{\rho} \right) = 0, \text{ for some } \rho > 0 \text{ and } L \in R(I) \right\}. \\ (c_0)_F(M) &= \left\{ (X_k) : \lim_{k \rightarrow \infty} M \left(\frac{\bar{d}(X_k, \bar{0})}{\rho} \right) = 0, \text{ for some } \rho > 0 \right\}. \end{aligned}$$

3. Main results

THEOREM 3.1. *The spaces $(\ell_\infty)_F(M)$, $c_F(M)$ and $(c_0)_F(M)$ are complete metric spaces under the metric*

$$f(X, Y) = \inf \left\{ \rho > 0 : \sup_k M \left(\frac{\bar{d}(X_k, Y_k)}{\rho} \right) \leq 1 \right\}.$$

P r o o f. Consider the sequence space $(\ell_\infty)_F(M)$. Let (X^i) be a Cauchy sequence in $(\ell_\infty)_F(M)$. Let $\varepsilon > 0$ be given. For a fixed $x_0 > 0$, choose $r > 0$ such that $M\left(\frac{rx_0}{2}\right) \geq 1$. Now there exists $m_0 \in \mathbb{N}$ such that

$$f(X^i, X^j) < \frac{\varepsilon}{rx_0}, \quad \text{for all } i, j \geq m_0.$$

By the definition of f we have,

$$\sup_k M\left(\frac{\bar{d}(X_k^i, X_k^j)}{\rho}\right) \leq 1 \quad \text{for all } i, j \geq m_0 \tag{1}$$

$$\implies M\left(\frac{\bar{d}(X_k^i, X_k^j)}{f(X^i, X^j)}\right) \leq 1 \leq M\left(\frac{rx_0}{2}\right) \text{ for all } i, j \geq m_0 \text{ and for each } k \in \mathbb{N}.$$

$$\implies \bar{d}(X_k^i, X_k^j) < \frac{\varepsilon}{2} \text{ for all } i, j \geq m_0 \text{ and for each } k \in \mathbb{N}.$$

Thus (X_k^i) is a Cauchy sequence of fuzzy real numbers. So there exists a fuzzy real number X_k such that $\lim_{j \rightarrow \infty} X_k^j = X_k$ for each $k \in \mathbb{N}$.

Since M is continuous, so taking $j \rightarrow \infty$ in (1) we get

$$\sup_k M\left(\frac{\bar{d}(X_k^i, X_k)}{\rho}\right) \leq 1.$$

Taking infimum of such ρ 's we get

$$f(X^i, X) < \varepsilon, \quad \text{for all } i \geq m_0.$$

Let $i \geq m_0$, then using the triangle inequality, $f(X, \bar{0}) \leq f(X, X^i) + f(X^i, \bar{0})$, we have $X \in (\ell_\infty)_F(M)$.

Hence $(\ell_\infty)_F(M)$ is complete. Similarly it can be shown that the other spaces are also complete. □

THEOREM 3.2. *The sequence spaces $(\ell_\infty)_F(M)$ and $(c_0)_F(M)$ are solid and hence monotone.*

P r o o f. Consider the sequence space $(\ell_\infty)_F(M)$. Let $(X_k) \in (\ell_\infty)_F(M)$ and (Y_k) be such that $\bar{d}(Y_k, \bar{0}) \leq \bar{d}(X_k, \bar{0})$ for each $k \in \mathbb{N}$. Since M is increasing, we have

$$\sup_k M\left(\frac{\bar{d}(Y_k, \bar{0})}{\rho}\right) \leq \sup_k M\left(\frac{\bar{d}(X_k, \bar{0})}{\rho}\right).$$

Hence the space $(\ell_\infty)_F(M)$ is solid. Similarly $(c_0)_F(M)$ is also solid. □

PROPERTY 3.3. *The sequence space $c_F(M)$ is not monotone and hence not solid.*

The result follows from the following example:

Example 3.1. Let $J = \{k : k \text{ is even}\} \subseteq \mathbb{N}$. Let the sequence (X_k) be defined as: For all $k \in \mathbb{N}$,

$$X_k(t) = \begin{cases} t + 1, & \text{if } -1 \leq t \leq 0, \\ -t + 1, & \text{if } 0 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X_k) \in c_F(M)$. Let (Y_k) be the canonical pre-image of $(X_k)_J$ for the subsequence J of \mathbb{N} . Then

$$Y_k = \begin{cases} X_k, & \text{for } k \in J, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Then $(Y_k) \notin c_F(M)$. Hence $c_F(M)$ is not monotone.

THEOREM 3.4. *The sequence spaces $(\ell_\infty)_F(M)$, $c_F(M)$ and $(c_0)_F(M)$ are symmetric.*

Proof. It can be proved following the technique used for establishing the crisp set cases. □

The proof of the following two results are obvious.

PROPERTY 3.5. $Z(M_1) \cap Z(M_2) \subseteq Z(M_1 + M_2)$, for $Z = \ell_\infty^F(M)$, $c_F(M)$, $(c_0)_F(M)$.

PROPERTY 3.6. *If $M_1(x) \leq M_2(x)$ for all $x \in [0, \infty)$, then $Z(M_2) \subseteq Z(M_1)$ for $Z = (\ell_\infty)_F$, c_F and $(c_0)_F$.*

THEOREM 3.7. *Let M and M_1 be two Orlicz functions, then $Z(M_1) \subseteq Z(M \circ M_1)$, for $Z = (\ell_\infty)_F$, c_F and $(c_0)_F$.*

Proof. We prove it for the case $Z = (c_0)_F$, the other cases can be proved following similar technique. Let $\varepsilon > 0$ be given. Since M is continuous, so there exists $\eta > 0$ such that $M(\eta) = \varepsilon$. Let $\langle X_k \rangle \in (c_0)_F(M)$. Then there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} M_1 \left(\frac{\bar{d}(X_k, \bar{0})}{\rho} \right) < \eta & \quad \text{for all } k \geq k_0. \\ \implies M \circ M_1 \left(\frac{\bar{d}(X_k, \bar{0})}{\rho} \right) < \varepsilon & \quad \text{for all } k \geq k_0. \end{aligned}$$

$$\implies (X_k) \in (c_0)_F(M \circ M_1).$$

Hence $Z(M_1) \subseteq Z(M \circ M_1)$, for $Z = (\ell_\infty)_F$, c_F and $(c_0)_F$. □

PROPERTY 3.8. $Z(M) \subseteq (\ell_\infty)_F(M)$, for $Z = c_F$ and $(c_0)_F$. *The inclusions are strict.*

Proof. The result $Z(M) \subseteq (\ell_\infty)_F(M)$, for $Z = c_F$ and $(c_0)_F$ is obvious. To show that the inclusions are proper, consider the following example. \square

Example 3.2. Let the sequence (X_k) be defined by:
For k even,

$$X_k(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1, \\ -t + 2, & \text{if } 1 \leq t \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

For k odd,

$$X_k(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1, \\ -t + 2, & \text{if } 1 \leq t \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X_k) \in (\ell_\infty)_F(M)$, but (X_k) neither belongs to $c_F(M)$ nor to $(c_0)_F(M)$.

THEOREM 3.9. *The sequence spaces $(\ell_\infty)_F(M)$, $c_F(M)$ and $(c_0)_F(M)$ are not convergence free.*

The result follows from the following example.

Example 3.3. Consider the sequence space $c_F(M)$. Let the sequence (X_k) be defined by:

For $k = 1$, $X_k = \bar{0}$.

For $k \neq 1$,

$$X_k(t) = \begin{cases} t + 1, & \text{if } 0 \leq t \leq 1, \\ -k(k + 1)^{-1}t + 1, & \text{if } 1 \leq t \leq 1 + k^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Let the sequence (Y_k) be defined by:

For $k = 1$, $Y_k = \bar{0}$.

For $k \neq 1$,

$$Y_k(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1, \\ -(2k - 1)^{-1}t + 2k(2k - 1)^{-1}, & \text{if } 1 \leq t \leq 2k, \\ 0, & \text{otherwise.} \end{cases}$$

Here $(X_k) \in c_F(M)$ but $(Y_k) \notin c_F(M)$. Hence $c_F(M)$ is not convergent free.

Acknowledgement. The authors thank the referee for the careful reading of the paper and for the comments.

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Received 28. 9. 2006

Revised 13. 6. 2007

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