

I AND I^* -CONVERGENCE OF DOUBLE SEQUENCES

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ABSTRACT. The idea of I -convergence was introduced by Kostyrko et al (2001) and also independently by Nuray and Ruckle (2000) (who called it generalized statistical convergence) as a generalization of statistical convergence (Fast (1951), Schoenberg(1959)). For the last few years, study of these convergences of sequences has become one of the most active areas of research in classical Analysis. In 2003 Muresaleen and Edely introduced the concept of statistical convergence of double sequences. In this paper we consider the notions of I and I^* -convergence of double sequences in real line as well as in general metric spaces. We primarily study the inter-relationship between these two types of convergence and then investigate the category and porosity position of bounded I and I^* -convergent double sequences.

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1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [5] and Schoenberg [21]. A lot of developments have been made in this area after the works of Šalát [18] and Fridy [6] (see [7], [20] where other references can be found).

The concept of I -convergence of real sequences was introduced by Kostyrko et al [10] as a generalization of statistical convergence which is based on the structure of the ideal I of subsets of the set of natural numbers. In [17], Nuray and Ruckle independently introduced the same with another name 'generalized statistical convergence'. Several works have been done on I -convergence in

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the last five years (see [1], [11], [12], [13]). In [10], the concept of I^* -convergence was also introduced and a detailed study was made to explore its relation with I -convergence.

The notion of statistical convergence was introduced for double sequences by Muresaleen and Edely [15] (also by Móricz [14] who introduced it for multiple sequences). Very recently some works on I -convergence of double sequences have also been done (see [2], [4], [22]) and in particular in [4] the idea of I -convergence of a double sequence was introduced. It seems therefore reasonable to investigate the concepts of I and I^* -convergence for the double sequences. In this paper we do the same and primarily investigate their inter-relationship. We also study the category and porosity positions of the classes of bounded I and I^* -convergent sequences which extends the results of [15].

2. Definitions and notations

Throughout the paper \mathbb{N} denotes the set of all positive integers, χ_A -the characteristic function of $A \subset \mathbb{N}$, \mathbb{R} the set of all real numbers.

Recall that a subset A of \mathbb{N} is said to have asymptotic density $d(A)$ if

$$d(A) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \chi_A(k).$$

DEFINITION 1. ([5]) A sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers is said to be *statistically convergent* to $\xi \in \mathbb{R}$ if for any $\epsilon > 0$, we have $d(A(\epsilon)) = 0$, where $A(\epsilon) = \{n \in \mathbb{N} : |x_n - \xi| \geq \epsilon\}$.

By the convergence of a double sequence we mean the convergence in Pringsheim's sense (see [8], [18]):

A double sequence $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$ of real numbers is said to be convergent to $\xi \in \mathbb{R}$ if for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $|x_{mn} - \xi| < \epsilon$ whenever $m, n \geq N_\epsilon$. In this case we write $\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$.

A double sequence $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$ for all $m, n \in \mathbb{N}$. That is $\|x\|_{(\infty,2)} = \sup_{m,n} |x_{mn}| < \infty$.

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let $K(n, m)$ be the number of $(j, k) \in K$ such that $j \leq n, k \leq m$. If the sequence $\left\{ \frac{K(n, m)}{n \cdot m} \right\}_{n, m \in \mathbb{N}}$ has a limit in Pringsheim's sense then we say that K has double natural density and is denoted by

$$d_2(K) = \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \frac{K(n, m)}{n \cdot m}.$$

DEFINITION 2. ([15]) A double sequence $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$ of real numbers is said to be *statistically convergent* to $\xi \in \mathbb{R}$ if for any $\epsilon > 0$, we have $d_2(A(\epsilon)) = 0$, where $A(\epsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - \xi| \geq \epsilon\}$.

A statistically convergent double sequence of elements of a metric space (X, ρ) is defined essentially in the same way ($\rho(x_{mn}, \xi) \geq \epsilon$ instead of $|x_{mn} - \xi| \geq \epsilon$).

Next we recall the following definitions, where X represents an arbitrary set.

DEFINITION 3. Let $X \neq \emptyset$. A class I of subsets of X is said to be an ideal in X provided:

- (i) $\emptyset \in I$.
- (ii) $A, B \in I$ implies $A \cup B \in I$.
- (iii) $A \in I, B \subset A$ implies $B \in I$.

I is called a *nontrivial ideal* if $X \notin I$.

DEFINITION 4. Let $X \neq \emptyset$. A non empty class F of subsets of X is said to be a *filter* in X provided:

- (i) $\emptyset \notin F$.
- (ii) $A, B \in F$ implies $A \cap B \in F$.
- (iii) $A \in F, A \subset B$ implies $B \in F$.

If I is a nontrivial ideal in $X, X \neq \emptyset$, then the class

$$F(I) = \{M \subset X : (\exists A \in I)(M = X \setminus A)\}$$

is a filter on X , called the *filter associated with I* .

DEFINITION 5. A nontrivial ideal I in X is called *admissible* if $\{x\} \in I$ for each $x \in X$.

Throughout the paper we take I as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

DEFINITION 6. A nontrivial ideal I of $\mathbb{N} \times \mathbb{N}$ is called *strongly admissible* if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to I for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Let $I_0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \implies (i, j) \notin A)\}$.

Then I_0 is a nontrivial strongly admissible ideal and clearly an ideal I is strongly admissible if and only if $I_0 \subset I$.

We now introduce the following definitions.

DEFINITION 7. (see also [4]) A double sequence $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$ of real numbers is said to *converge to $\xi \in \mathbb{R}$ with respect to the ideal I* , if for every $\epsilon > 0$ the set $A(\epsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - \xi| \geq \epsilon\} \in I$. In this case we say that x is I -convergent and we write $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$.

Remark 1. Note that if I is the ideal I_0 then I -convergence coincides with the usual convergence and if we take $I_d = \{A \subset \mathbb{N} \times \mathbb{N} : d_2(A) = 0\}$ then I_d -convergence becomes statistical convergence.

I -convergent double sequences may be unbounded, for example, let I be the ideal I_0 of $\mathbb{N} \times \mathbb{N}$. If we define $\{x_{mn}\}_{m,n \in \mathbb{N}}$ by

$$\begin{aligned} x_{mn} &= n & \text{if } m = 1, \\ &= 2 & \text{if } m \neq 1. \end{aligned}$$

Then $\{x_{mn}\}_{m,n \in \mathbb{N}}$ is unbounded but I -convergent.

DEFINITION 8. A double sequence $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$ of real numbers is said to be I^* -convergent to $\xi \in \mathbb{R}$ if and only if there exists a set $M \in F(I)$ (i.e. $\mathbb{N} \times \mathbb{N} \setminus M \in I$) such that $\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty \\ (m,n) \in M}} x_{mn} = \xi$ and we write $I^* - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$.

Remark 2. In principle I -convergence of single sequences and double sequences are the same. In fact any bijection between $\mathbb{N} \times \mathbb{N}$ and \mathbb{N} transforms double sequences into single sequences and also ideals of $\mathbb{N} \times \mathbb{N}$ to ideals of \mathbb{N} and by this transformation I -convergence and condition (AP) (defined later) are both preserved. This implies that we can sometimes use known facts of I -convergence of single sequences to prove results of double sequences. However in this paper we desist from that and give proofs in terms of double sequences only. Also it should be noted that some concepts like I^* -convergence is not preserved by the above mentioned transformation.

3. I and I^* -convergence in metric spaces

In this section we consider the I and I^* -convergence of double sequences in the more general structure of a metric space (X, ρ) (say). Unless otherwise mentioned we shall denote the metric space (X, ρ) by X only.

DEFINITION 9. A double sequence $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$ of elements of X is said to be I -convergent to $\xi \in X$ if and only if for every $\epsilon > 0$ we have $A(\epsilon) \in I$, where $A(\epsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, \xi) \geq \epsilon\}$ and we write $I - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$.

DEFINITION 10. A double sequence $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$ of elements of X is said to be I^* -convergent to $\xi \in X$ if and only if there exists a set $M \in F(I)$ (i.e. $\mathbb{N} \times \mathbb{N} \setminus M \in I$) such that $\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty \\ (m,n) \in M}} x_{mn} = \xi$ and we write $I^* - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$.

THEOREM 1. Let I be a strongly admissible ideal. If $I^* - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$ then $I - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$.

P r o o f. Since $I^* - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$, so there exists a set $M \in F(I)$ (i.e. $\mathbb{N} \times \mathbb{N} \setminus M = H \in I$) such that

$$\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty \\ (m,n) \in M}} x_{mn} = \xi. \quad (1)$$

Let $\epsilon > 0$. Then by (1) there exists $k_0 \in \mathbb{N}$ such that $\rho(x_{mn}, \xi) < \epsilon$ for all m, n such that $(m, n) \in M$ and $m, n \geq k_0$. Then

$$\begin{aligned} A(\epsilon) &= \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, \xi) \geq \epsilon\} \\ &\subset H \cup (M \cap ((\{1, 2, \dots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (k_0 - 1)\}))). \end{aligned}$$

Now since $H \cup (M \cap ((\{1, 2, \dots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (k_0 - 1)\}))) \in I$, so $A(\epsilon) \in I$. Therefore $I - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$. \square

The converse of the above theorem depends essentially on the structure of the metric space (X, ρ) .

THEOREM 2. *Let (X, ρ) be a metric space.*

- (i) *If X has no accumulation point, then I and I^* -convergence coincide for each strongly admissible ideal I .*
- (ii) *If X has an accumulation point ξ , then there exists a strongly admissible ideal I and a double sequence $\{y_{mn}\}_{m,n \in \mathbb{N}}$ for which $I - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} y_{mn} = \xi$ but*

$$I^* - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} y_{mn} \text{ does not exist.}$$

P r o o f.

(i) Let $\xi \in X$ and $I - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$. Then by virtue of Theorem 1, it is sufficient to prove that $I^* - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$. Since X has no accumulation point, so there exists $\delta > 0$ such that

$$B(\xi, \delta) = \{x \in X : \rho(x, \xi) < \delta\} = \{\xi\}.$$

Since $I - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$, so $\{(m, n) : \rho(x_{mn}, \xi) \geq \delta\} \in I$. This gives

$$\{(m, n) : \rho(x_{mn}, \xi) < \delta\} = \{(m, n) : x_{mn} = \xi\} \in F(I).$$

So $I^* - \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$.

(ii) Since ξ is an accumulation point of X , so there exists a sequence $\{z_j\}_{j \in \mathbb{N}}$ of distinct points all different from ξ in X which is convergent to ξ such that the sequence $\{\rho(z_j, \xi)\}_{j \in \mathbb{N}}$ is decreasing to 0. Let $\{E_j\}_{j \in \mathbb{N}}$ be a decomposition of \mathbb{N} onto infinite sets and put $\Delta_j = \{(m, n) : \min\{m, n\} \in E_j\}$. Then $\{\Delta_j\}_{j \in \mathbb{N}}$ is a decomposition of $\mathbb{N} \times \mathbb{N}$ and the ideal $I = \{A \subset \mathbb{N} \times \mathbb{N} : A \text{ is included in a finite union of } \Delta_j\text{'s}\}$ is a strongly admissible ideal. Put

$x_{mn} = z_j$ if and only if $(m, n) \in \Delta_j$. Put $\epsilon_n = \rho(z_n, \xi)$ for $n \in \mathbb{N}$. Let $\eta > 0$ be given. Choose $\gamma \in \mathbb{N}$ such that $\epsilon_\gamma < \eta$. Then $A(\eta) = \{(m, n) : \rho(x_{mn}, \xi) \geq \eta\} \subset \Delta_1 \cup \Delta_2 \cdots \cup \Delta_\gamma$. Hence $A(\eta) \in I$ and $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$.

Now suppose that $I^*\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$. Then there exists $H \in I$ such that for $M = \mathbb{N} \times \mathbb{N} \setminus H$ we have $\lim_{\substack{m \rightarrow \infty \\ (m, n) \in M}} x_{mn} = \xi$. By definition of the ideal I , there exists $l \in \mathbb{N}$ such that $H \subset \Delta_1 \cup \Delta_2 \cdots \cup \Delta_l$. But then $\Delta_{l+1} \subset \mathbb{N} \setminus H = M$. Then from the construction of Δ_{l+1} it follows that for any $n_0 \in \mathbb{N}$, $\rho(x_{mn}, \xi) = \epsilon_{l+1} > 0$ hold for infinitely many (m, n) 's with $(m, n) \in M$ and $m, n \geq n_0$. This contradicts that $\lim_{\substack{m \rightarrow \infty \\ (m, n) \in M}} x_{mn} = \xi$. Also the assumption $I^*\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = p$ for $p \neq \xi$ leads to the contradiction. \square

4. Condition (AP2)

In [10] it was proved that I and I^* -convergence of ordinary sequences of real numbers (so also of elements of a metric space possessing at least one accumulation point) are equivalent if and only if the ideal $I \subset 2^{\mathbb{N}}$ satisfies the following condition (AP):

(AP) An admissible ideal $I \subset 2^{\mathbb{N}}$ satisfies the condition (AP) if for every countable family of mutually disjoint sets $\{A_n\}_{n \in \mathbb{N}}$ belonging to I , there exists a countable family of sets $\{B_n\}_{n \in \mathbb{N}}$ such that $A_n \Delta B_n$ is a finite set for $n \in \mathbb{N}$ and $B = \bigcup_{n=1}^{\infty} B_n \in I$.

If $I \subset 2^{\mathbb{N} \times \mathbb{N}}$ is an admissible ideal fulfilling the condition (AP) (the definition of (AP) for ideals of subsets of $\mathbb{N} \times \mathbb{N}$ is in practice the same as above) then as in [10, Theorem 3.2] we can easily prove that for any double sequence $\{x_{mn}\}_{m, n \in \mathbb{N}}$ in X , $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$ implies $I^*\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$. However unlike single sequences, the condition (AP) is not necessary for the equivalence of I and I^* -convergence of double sequences. For example consider the ideal I_0 (which corresponds to the Pringsheim's convergence). Obviously for the ideal I_0 , I_0 and I_0^* -convergence are equivalent. But note that the sets $B_i = \{i\} \times \mathbb{N}$ belong to I_0 and they form a decomposition of $\mathbb{N} \times \mathbb{N}$. If we omit from $\mathbb{N} \times \mathbb{N}$ only finitely many elements of each B_i (or some B_i 's), the resulting set does not belong to I_0 . This shows that the ideal I_0 does not have the property (AP) (we shall prove the same for the ideal I_d also, later in this section).

From above we can come to the conclusion that the situation is different for double sequences and we now introduce the following condition:

(AP2) We say that an admissible ideal $I \subset 2^{\mathbb{N} \times \mathbb{N}}$ satisfies the condition (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, \dots\}$ belonging to I , there exists a countable family of sets $\{B_1, B_2, \dots\}$ such that $A_j \Delta B_j \in I_0$ i.e. $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in I$ (hence $B_j \in I$ for each $j \in \mathbb{N}$).

THEOREM 3. *If I is an admissible ideal of $\mathbb{N} \times \mathbb{N}$ having the property (AP2) and (X, ρ) is an arbitrary metric space, then for an arbitrary double sequence $\{x_{mn}\}_{m,n \in \mathbb{N}}$ of elements of X , $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$ implies $I^*\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$.*

Proof. Let I satisfy (AP2). Let $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$. Then for any $\epsilon > 0$, $A(\epsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, \xi) \geq \epsilon\} \in I$.

Put $A_1 = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, \xi) \geq 1\}$ and $A_k = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{k} \leq \rho(x_{mn}, \xi) < \frac{1}{k-1}\}$ for $k \geq 2$. Obviously $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_i \in I$ for each $i \in \mathbb{N}$. By virtue of (AP2) there exists a sequence $\{B_k\}_{k \in \mathbb{N}}$ of sets such that $A_j \Delta B_j$ is included in finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each j and $B = \bigcup_{j=1}^{\infty} B_j \in I$.

We shall prove that for $M = \mathbb{N} \times \mathbb{N} \setminus B$ we have $\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty \\ (m,n) \in M}} x_{mn} = \xi$.

Let $\eta > 0$ be given. Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \eta$. Then $\{(m, n) : \rho(x_{mn}, \xi) \geq \eta\} \subset \bigcup_{j=1}^k A_j$. Since $A_j \Delta B_j$, $j = 1, 2, \dots, k$, are included in finite union of rows and columns, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \left(\bigcup_{j=1}^k B_j \right) \cap \{(m, n) : m \geq n_0 \wedge n \geq n_0\} \\ = \left(\bigcup_{j=1}^k A_j \right) \cap \{(m, n) : m \geq n_0 \wedge n \geq n_0\}. \end{aligned}$$

If $m, n \geq n_0$ and $(m, n) \notin B$ then $(m, n) \notin \bigcup_{j=1}^k B_j$ and so $(m, n) \notin \bigcup_{j=1}^k A_j$. This implies that $\rho(x_{mn}, \xi) < \frac{1}{k} < \eta$. This completes the proof of the theorem. \square

For the converse we have the following theorem.

THEOREM 4. *If (X, ρ) has at least one accumulation point and for any arbitrary double sequence $\{x_{mn}\}_{m,n \in \mathbb{N}}$ of elements of X and for each $\xi \in X$, $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$ implies $I^*\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$, then I has the property (AP2).*

Proof. Suppose $\xi \in X$ is an accumulation point of X . There exists a sequence $\{z_k\}_{k \in \mathbb{N}}$ of distinct elements of X such that $z_k \neq \xi$ for any k , $\xi = \lim_{k \rightarrow \infty} z_k$ and the sequence $\{\rho(z_k, \xi)\}_{k \in \mathbb{N}}$ is a decreasing sequence converging to zero. Put $\epsilon_k = \rho(z_k, \xi)$ for $k \in \mathbb{N}$. Let $\{A_j\}_{j \in \mathbb{N}}$ be a disjoint family of nonempty sets from I . Define a sequence $\{x_{mn}\}_{m,n \in \mathbb{N}}$ in the following way: $x_{mn} = z_j$ if $(m, n) \in A_j$ and $x_{mn} = \xi$ if $(m, n) \notin A_j$ for any j .

Let $\eta > 0$. Choose $k \in \mathbb{N}$ such that $\epsilon_k < \eta$. Then $A(\eta) = \{(m, n) : \rho(x_{mn}, \xi) \geq \eta\} \subset A_1 \cup A_2 \cdots \cup A_k$. Hence $A(\eta) \in I$ and so $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$. By virtue of our assumption, we then have $I^*\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$. So there exists a set $B \in I$ such that $M = \mathbb{N} \times \mathbb{N} \setminus B \in F(I)$ and

$$\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty \\ (m,n) \in M}} x_{mn} = \xi. \quad (2)$$

Put $B_j = A_j \cap B$ for $j \in \mathbb{N}$. Then $B_j \in I$ for each $j \in \mathbb{N}$. Moreover $\bigcup_{j=1}^{\infty} B_j = B \cap \bigcup_{j=1}^{\infty} A_j \subset B$ and so $\bigcup_{j=1}^{\infty} B_j \in I$. Fix $j \in \mathbb{N}$. If $A_j \cap M$ is not included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$, then M must contain an infinite sequence of elements $\{(m_k, n_k)\}$ where both $m_k, n_k \rightarrow \infty$ and $x_{m_k n_k} = z_j \neq \xi$ for all $k \in \mathbb{N}$ which contradicts (2). Hence $A_j \cap M$ must be contained in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$. Hence $A_j \Delta B_j = A_j \setminus B_j = A_j \setminus B = A_j \cap M$ is also included in the finite union of rows and columns. This proves that the ideal I has the property (AP2). \square

The next question which comes naturally is what is the relation between the conditions (AP) and (AP2). Obviously (AP) implies (AP2). But the converse is not true as we have already seen that the ideal I_0 does not satisfy (AP) though it has the property (AP2) (in view of Theorem 4). Thus (AP) is essentially stronger than (AP2) when considered for double sequences. Below we take another very important ideal $I_d = \{K \subset \mathbb{N} \times \mathbb{N} : d_2(K) = 0\}$ and show that I_d fulfills (AP2) but not (AP).

By virtue of Theorem 3 and 4 it is sufficient to prove that I_d -convergence implies I_d^* -convergence. The method of proof is similar to that of [6], but we present it for the sake of completeness (the equivalence of I_d -convergence and I_d^* -convergence for multiple sequences was also proved in [14]).

THEOREM 5. *I_d -convergence of a double sequence implies I_d^* -convergence.*

Proof. Since the proof is practically the same for double sequences of real numbers and double sequences of elements of a metric space, we shall deal with real numbers.

Let $\{x_{mn}\}_{m,n \in \mathbb{N}}$ be a double sequence I_d -convergent to $\xi \in \mathbb{R}$. Put $A_1 = \{(m, n) : |x_{mn} - \xi| \geq 1\}$ and $A_k = \{(m, n) : \frac{1}{k} \leq |x_{mn} - \xi| < \frac{1}{k-1}\}$. From the assumption it follows that $d_2(A_k) = 0$ for each $k \in \mathbb{N}$.

Observe that also $d_2\left(\bigcup_{k=1}^p A_k\right) = 0$ for $p \in \mathbb{N}$. For $p \in \mathbb{N}$, let T_p be a natural number such that

$$\frac{1}{m \cdot n} \text{card}\left\{(j, k) : j \leq m \wedge k \leq n \wedge (j, k) \in \bigcup_{i=1}^p A_i\right\} < \frac{1}{p}$$

for $n \geq T_p$ and $m \geq T_p$. We can obviously assume that the sequence $\{T_p\}_{p \in \mathbb{N}}$ is increasing.

Let $C_p = \{(j, k) : T_p \leq \min\{j, k\} < T_{p+1}\}$, $D_p = C_p \cap \bigcup_{i=1}^p A_i$ for $p \in \mathbb{N}$ and $D = \bigcup_{p=1}^{\infty} D_p$. We shall show that $d_2(D) = 0$. Indeed, if $\eta > 0$ and $p \in \mathbb{N}$ is such that $\frac{1}{p} < \eta$, then for $(m, n) \in C_p$ we have

$$(\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}) \cap D \subset (\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}) \cap \bigcup_{i=1}^p A_i,$$

so $\frac{1}{m \cdot n} \text{card}\{(j, k) : j \leq m \wedge k \leq n \wedge (j, k) \in D\} < \frac{1}{p}$ for such n and m . Hence $d_2(D) = 0$.

Simultaneously for $n \geq T_p$, $m \geq T_p$, $(m, n) \notin D$ we have $|x_{mn} - \xi| < \frac{1}{p}$, so $\{x_{mn}\}_{m,n \in \mathbb{N}}$ I_d^* -converges to ξ . Hence the ideal I_d has the property (AP2).

Now we shall show that I_d does not fulfill (AP).

First, let $\{E_p\}_{p \in \mathbb{N}}$ be a sequence of subsets of \mathbb{N} of density zero such that $\bigcup_{p=1}^{\infty} E_p = \mathbb{N}$.

Put $A_p = E_p \times \mathbb{N}$ for $p \in \mathbb{N}$. Then it is easy to see that $d_2(A_p) = 0$ for $p \in \mathbb{N}$. Let $\{B_p\}_{p \in \mathbb{N}}$ be an arbitrary sequence of subsets of $\mathbb{N} \times \mathbb{N}$ such that $\text{card}(A_p \Delta B_p) < \aleph_0$. Then there exists a sequence of finite sets $\{F_p\}_{p \in \mathbb{N}}$ such that $B_p \supset A_p \setminus F_p$. We shall show that $d_2\left(\bigcup_{p=1}^{\infty} B_p\right) \neq 0$ (actually $\overline{d_2}\left(\bigcup_{p=1}^{\infty} (A_p \setminus F_p)\right) = \overline{d_2}\left(\bigcup_{p=1}^{\infty} B_p\right) = 1$).

Let m be an arbitrary natural number. We shall show that for each $\eta > 0$ there exists $n \in \mathbb{N}$ such that $n \geq m$ and $\frac{1}{m \cdot n} \text{card} \left\{ (j, k) : j \leq m \wedge k \leq n \wedge (j, k) \in \bigcup_{p=1}^{\infty} (A_p \setminus F_p) \right\} > 1 - \eta$.

For this we first choose $p_0 \in \mathbb{N}$ such that $\bigcup_{i=1}^{p_0} E_i \supset \{1, 2, \dots, m\}$, since $\bigcup_{p=1}^{\infty} E_i = \mathbb{N}$. So $\bigcup_{i=1}^{p_0} A_i \supset \{1, 2, \dots, m\} \times \mathbb{N}$.

Hence $\bigcup_{i=1}^{p_0} (A_i \setminus F_i) \supset (\{1, 2, \dots, m\} \times \mathbb{N}) \setminus F$, where F is a finite set. So for each $n \in \mathbb{N}$ we have

$$\begin{aligned} & (\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}) \cap \bigcup_{i=1}^{\infty} (A_i \setminus F_i) \\ & \supset (\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}) \cap \bigcup_{i=1}^{p_0} (A_i \setminus F_i) \\ & \supset (\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}) \setminus F \end{aligned}$$

(where F does not depend on n) and for sufficiently big $n \in \mathbb{N}$ (when $n \geq m$ also) we have the inequality

$$\frac{1}{m \cdot n} \text{card} \left((\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}) \cap \bigcup_{i=1}^{\infty} (A_i \setminus F_i) \right) > 1 - \eta.$$

This shows that $\overline{d_2} \left(\bigcup_{p=1}^{\infty} B_p \right) = 1$ and so $\bigcup_{p=1}^{\infty} B_p \notin I_d$ which implies that I_d does not fulfill (AP). \square

5. Category and porosity studies

Let I be an admissible ideal of $\mathbb{N} \times \mathbb{N}$ and m_{0_2} be the set of real I -convergent double sequences. Then $m_{0_2} \subset w_2$, where w_2 is the set of all real double sequences.

We now give some properties of m_{0_2} .

THEOREM 6.

- (a) Let I be a strongly admissible ideal. If $\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$, then $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$;
- (b) Let I be an admissible ideal.

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- (i) If $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$, $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} y_{mn} = \eta$, then $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} (x_{mn} + y_{mn}) = \xi + \eta$;
- (ii) If $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = \xi$, $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} y_{mn} = \eta$, then $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} (x_{mn} \cdot y_{mn}) = \xi \cdot \eta$.

P r o o f. The proof is straightforward and so is omitted. \square

Let Z_2 be the class of all admissible ideals in $\mathbb{N} \times \mathbb{N}$. The class Z_2 is partially ordered by inclusion. If $Z_{0_2} \subset Z_2$ is a nonvoid linearly ordered subset of Z_2 , then $\bigcup Z_{0_2}$ is an admissible ideal in $\mathbb{N} \times \mathbb{N}$ which is an upper bound of Z_{0_2} . So by Zorn's Lemma Z_2 has a maximal element i.e. a maximal admissible ideal. The next lemma is a characterization of maximal admissible ideal.

LEMMA 1. *Let I_0 be an ideal in $\mathbb{N} \times \mathbb{N}$ which contains all singletons. Then I_0 is a maximal admissible ideal if and only if*

$$A \in I_0 \quad \text{or} \quad (\mathbb{N} \times \mathbb{N} \setminus A) \in I_0$$

holds for each $A \subset \mathbb{N} \times \mathbb{N}$.

P r o o f. The proof is parallel to the proof of [11, Lemma 2.1] and so is omitted. \square

We now consider bounded double sequences of real numbers and investigate some structural properties of the sets of bounded I and I^* -convergent double sequences in the normed linear space of all bounded real double sequences.

Let m_2 denote the normed linear space of all bounded double sequences of real numbers (with norm, $\|x\| = \sup_{m,n} |x_{mn}|$, where $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$) and let m_{I_2} denote the set all bounded I -convergent double sequences of real numbers.

Now we prove the following theorem.

THEOREM 7. *Let I be an admissible ideal in $\mathbb{N} \times \mathbb{N}$. Then $m_{I_2} = m_2$ if and only if I is a maximal admissible ideal of $\mathbb{N} \times \mathbb{N}$.*

P r o o f. If I is a maximal ideal of $\mathbb{N} \times \mathbb{N}$ then we can prove $m_{I_2} = m_2$ as in [11, Theorem 2.2].

For the converse part let us assume I to be not maximal. Then by Lemma 1 there exists a set $M \subset \mathbb{N} \times \mathbb{N}$ such that $M \notin I$ and $(\mathbb{N} \times \mathbb{N} \setminus M) \notin I$. Now we define a sequence $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$ by

$$\begin{aligned} x_{mn} &= 2 & \text{if } (m, n) \in M, \\ &= 0 & \text{if } (m, n) \notin M. \end{aligned}$$

Then $x \in m_2$ and $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn}$ does not exist. Indeed for every $\xi \in \mathbb{R}$ we can find a sufficiently small $\epsilon > 0$ such that the set $\{(m, n) : |x_{mn} - \xi| \geq \epsilon\}$ is equal to either M or $(\mathbb{N} \times \mathbb{N} \setminus M)$ or to $(\mathbb{N} \times \mathbb{N})$ and neither of them belong to I . \square

Remark 3. The above theorem can not be extended for unbounded sequences. For this we consider the following example.

Example 1. Let I be an admissible ideal. We define a sequence $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$ as follows,

$$x_{mn} = \max\{m, n\} \quad \text{for } (m, n) \in \mathbb{N} \times \mathbb{N}.$$

Then x is an unbounded sequence which is not I -convergent.

We now study some topological properties of m_{I_2} and $m_{I_2^*}$ in m_2 .

THEOREM 8. *The set m_{I_2} is a closed linear subspace of the normed linear space m_2 , when I is a nontrivial admissible ideal of $\mathbb{N} \times \mathbb{N}$.*

Proof. From Theorem 5.(b)(i) and (ii) it is easy to see that m_{I_2} is a linear subspace of the normed linear space m_2 . So we only show that m_{I_2} is closed in m_2 .

Let $x^{(p)} \in m_{I_2}$ ($p = 1, 2, \dots$) and $x^{(p)} \rightarrow x \in m_2$. We have to show that $x \in m_{I_2}$.

Since $x^{(p)} \in m_{I_2}$, for each p there exists a real number a_p such that $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn}^{(p)} = a_p$ ($p = 1, 2, \dots$), where $x^{(p)} = \{x_{mn}^{(p)}\}_{m,n \in \mathbb{N}}$. We shall now prove that

- (a) the sequence $\{a_p\}_{p \in \mathbb{N}}$ converges to a real number a (say), and
- (b) $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = a$, where $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$.

The result will then follow from (a) and (b).

Proof of (a):

Since $x^{(p)} \rightarrow x \in m_2$, for each $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that for each $q \geq r \geq n_\epsilon$ we have,

$$\|x^{(q)} - x^{(r)}\| < \frac{\epsilon}{3}.$$

Now since $x^{(q)}, x^{(r)} \in m_{I_2}$, so $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn}^{(q)} = a_q$ and $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn}^{(r)} = a_r$. Therefore

$A_r = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn}^{(r)} - a_r| < \frac{\epsilon}{3}\} \in F(I)$ and $A_q = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn}^{(q)} - a_q| < \frac{\epsilon}{3}\} \in F(I)$. Then $A_q \cap A_r \in F(I)$. Since I is nontrivial and admissible so $A_r \cap A_q$ must be infinite. Choose $(m_0, n_0) \in A_r \cap A_q$. Then $|x_{m_0 n_0}^{(q)} - a_q| < \frac{\epsilon}{3}$ and $|x_{m_0 n_0}^{(r)} - a_r| < \frac{\epsilon}{3}$. Hence for each $q \geq r \geq n_\epsilon$ we have,

$$|a_q - a_r| \leq |a_q - x_{m_0 n_0}^{(q)}| + |x_{m_0 n_0}^{(q)} - x_{m_0 n_0}^{(r)}| + |x_{m_0 n_0}^{(r)} - a_r| < \epsilon.$$

So $\{a_p\}_{p \in \mathbb{N}}$ is a Cauchy sequence in real numbers and so it must converge to a real number say, a . Hence $\lim_{p \rightarrow \infty} a_p = a$.

Proof of (b):

Let $\eta > 0$. Since $x^{(p)} \rightarrow x$, there exists $q \in \mathbb{N}$ such that

$$\|x^{(q)} - x\| < \frac{\eta}{3}. \quad (3)$$

The number q can be chosen in such a way that together with (3) the inequality $|a_q - a| < \frac{\eta}{3}$ also holds. Since $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn}^{(q)} = a_q$, so $A_q = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn}^{(q)} - a_q| < \frac{\eta}{3}\} \in F(I)$. Now for each $(m, n) \in A_q$ we have,

$$|x_{mn} - a| \leq |x_{mn} - x_{mn}^{(q)}| + |x_{mn}^{(q)} - a_q| + |a_q - a| < \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta.$$

Therefore $A_q \subset \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - a| < \eta\}$, which implies that $\{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - a| \geq \eta\} \in I$. Hence $I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = a$. This completes the proof of the theorem. \square

Remark 4. It is well known that every closed linear subspace of an arbitrary normed linear space E , different from E is a nowhere dense set in E (cf. [9, p. 37, Example 4], [16]). Hence on account of Theorem 5, it is obvious that if for the ideal I , $m_{I_2} \neq m_2$, then m_{I_2} is nowhere dense in m_2 .

We now introduce the following notation. Let $m_{I_2^*}$ denote the set of all bounded I^* -convergent double sequences of real numbers. It is easy to see that $m_{I_2^*}$ is a linear subspace of m_2 . Now let I be a strongly admissible ideal, then we have $m_{I_2^*} \subset m_{I_2}$, and the equality $m_{I_2^*} = m_{I_2}$ holds if I has the property (AP2). Further $m_{I_2} = m_2$ if and only if I is a maximal ideal. Thus if I is strongly admissible and not maximal, then

$$m_{I_2^*} \subset m_{I_2} \subsetneq m_2.$$

We further show that $m_{I_2^*}$ is dense in m_{I_2} when I is a strongly admissible ideal.

THEOREM 9. *Let I be a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$. Then*

$$\overline{m_{I_2^*}} = m_{I_2},$$

where $\overline{m_{I_2^*}}$ is the closure of $m_{I_2^*}$ in m_2 .

Proof. We have $m_{I_2^*} \subset m_{I_2}$. Since m_{I_2} is closed in m_2 , so $\overline{m_{I_2^*}} \subset m_{I_2}$ (Theorem 1). To prove the theorem we need to show only $m_{I_2} \subset \overline{m_{I_2^*}}$.

Put $B(z, \delta) = \{x \in m_2 : \|x - z\| < \delta\}$ for $z \in m_2$, $\delta > 0$ (ball in m_2). It is sufficient to prove that for each $y \in m_{I_2}$ and $\delta > 0$ we have

$$B(y, \delta) \cap m_{I_2^*} \neq \emptyset. \quad (4)$$

Let $l = I\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} y_{mn}$. Choose an arbitrary $\epsilon \in (0, \delta)$. Then

$$A(\epsilon) = \{(m, n) : |y_{mn} - l| \geq \epsilon\} \in I_2.$$

We define a sequence $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$ as follows

$$\begin{aligned} x_{mn} &= y_{mn} & \text{if } (m, n) \in A(\epsilon), \\ &= l & \text{otherwise.} \end{aligned}$$

Then $x \in m_2$, $I^*\text{-}\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} x_{mn} = l$ and $x \in B(y, \epsilon)$. So (4) holds and the proof is completed. \square

It is a well known fact that a closed linear subspace E of a normed linear space X , different from X is nowhere dense in X . This result encourages one to study the porosity of E . We now observe that as in [11] certain observations can be made regarding porosity of m_{I_2} and $m_{I_2^*}$.

We first recall some basic definitions of porosity in a metric space.

Let (X, ρ) be a metric space, $M \subset X$. If $x \in X$, and $r > 0$, we denote, as before by $B(x, r) = \{y \in X : \rho(x, y) < r\}$ the ball with center x and radius r . Let

$$\gamma(x, r, M) = \sup\{t > 0 : (\exists z \in B(x, r)) (B(z, t) \subset B(x, r) \setminus M)\}.$$

If there is no such $t > 0$, we put $\gamma(x, r, M) = 0$.

The numbers $\underline{p}(x, M) = \liminf_{r \rightarrow 0} \frac{\gamma(x, r, M)}{r}$, $\overline{p}(x, M) = \limsup_{r \rightarrow 0} \frac{\gamma(x, r, M)}{r}$ are called the *lower* and *upper porosity* of M at x .

If $\overline{p}(x, M) > 0$, then M is said to be *porous* at x . If $\overline{p}(x, M) > 0$ for all $x \in X$, then M is said to be *porous* in X . Obviously any set porous in X is nowhere dense in X .

If $\overline{p}(x, M) \geq c > 0$, then M is said to be *c-porous* at x . If $\overline{p}(x, M) \geq c > 0$ for all $x \in X$, then M is said to be *c-porous* in X .

If $\underline{p}(x, M) > 0$, then M is said to be *very porous* at x . If $\underline{p}(x, M) > 0$ for all $x \in X$, then M is said to be *very porous* in X .

If $\underline{p}(x, M) = \overline{p}(x, M)$ ($= p(x, M)$), then the number $p(x, M)$ is called the *porosity* of M at x .

We now recall the [11, Theorem 2.5].

THEOREM A. ([11]) *Suppose that X is a normed linear space and W is its closed linear subspace, $W \neq X$. Then W is veryporous set in X , in more detail*

- (a) *If $x \in X \setminus W$, then $p(x, W) = 1$,*
- (b) *If $x \in W$, then $p(x, W) = \frac{1}{2}$.*

We can easily apply Theorem A to the study the porosity position of m_{I_2} and $m_{I_2^*}$ in m_2 .

The following theorem is a direct application of Theorem A.

THEOREM 10. *Let I be an admissible ideal in $\mathbb{N} \times \mathbb{N}$ which is not maximal. Then the following holds:*

- (i) *If $x \in m_2 \setminus m_{I_2}$, then $p(x, m_{I_2}) = 1$.*
- (ii) *If $x \in m_{I_2}$, then $p(x, m_{I_2}) = \frac{1}{2}$.*

Since $m_{I_2^*} \subset m_{I_2} = \overline{m_{I_2^*}}$ (see Theorem 8) holds for each strongly admissible ideal, we get the following corollary.

COROLLARY 1. *Let I be a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$ which is not maximal. Then the following holds:*

- (i) *If $x \in m_2 \setminus m_{I_2^*}$, then $p(x, m_{I_2^*}) = 1$.*
- (ii) *If $x \in m_{I_2^*}$, then $p(x, m_{I_2^*}) = \frac{1}{2}$.*

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