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NOTE ON A SUBGROUP OF LEVY'S GROUP

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ABSTRACT. In the paper, there are proved some properties of the asymptotic density using the permutations.

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Let $\mathbb{N}$ be the set of natural numbers. For any subset $A \subseteq \mathbb{N}$ and $x > 0$, let $A(x)$ be the cardinality of $A \cap [0, x)$. The value $\limsup\limits_{x\to\infty} x^{-1}A(x) := \overline{d}(A)$ is called *the upper asymptotic density* of A, the value $\liminf\limits_{x\to\infty} x^{-1}A(x) := \overline{d}(A)$ is called *the lower asymptotic density* of A. If $\overline{d}(A) = \underline{d}(A)$ then we say that A has an *asymptotic density* and the value $\overline{d}(A) = \underline{d}(A)$ is called the *asymptotic density of the set* A. It is easy to see that this holds if and only if the limit $\lim\limits_{x\to\infty} x^{-1}A(x) := d(A)$ $(= \overline{d}(A) = \underline{d}(A))$ exists. For more details on the asymptotic density we refer to the paper [5].

By $\mathscr{D}$ we denote the set of all subsets of $\mathbb{N}$ which have an asymptotic density. Suppose that for a permutation $g\colon \mathbb{N} \to \mathbb{N}$ the following properties hold:

i) $\forall A \subseteq \mathbb{N};\ \ A \in \mathscr{D} \Longrightarrow g(A) \in \mathscr{D}$,

ii) $\forall A \in \mathscr{D};\ \ d\big(g(A)\big) = d(A)$.

Then we say that g *preserves the asymptotic density* (as usually $g(A) = \big\{g(a) : a \in A\big\}$). Denote by G the set of all permutations $g\colon \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{N\to\infty} \frac{1}{N}\big|\big\{j \le N :\ g(j) > N\big\}\big| = 0.$$

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This set is a group with respect to the composition law and it is called Lévy's group, introduced in [4]. It can be proved easily that the permutations from G preserve the asymptotic density. The set $\mathscr{D}$ has the following properties:

1) If $A, B \in \mathscr{D}$ and $A \cap B = \emptyset$, then $A \cup B \in \mathscr{D}$.
2) If $A, B \in \mathscr{D}$ and $A \subseteq B$, then $B \setminus A \in \mathscr{D}$.

In the paper [2] the following two theorems are proved:

iii) Let γ be a finitely additive probability measure on $\mathscr{D}$ such that for every $g \in G$ and every $A \in \mathscr{D}$ it holds that $\gamma(g(A)) = \gamma(A)$. Then $\gamma = d$.

iv) Let $A \in \mathscr{D}$. If for every $g \in G$ we have $d\big(A \ominus g(A)\big) = 0$ ($\ominus$ is the symmetric difference), then $d(A) = 1$ or $d(A) = 0$.

In this paper we prove these results for a proper subgroup of G.

Denote by G_0 the set of all permutations $g\colon \mathbb{N} \to \mathbb{N}$ which fulfill the condition

$$\lim_{n\to\infty} \frac{g(n)}{n} = 1. \tag{1}$$

Proposition 1.

a) *G_0 is a subgroup of G.*

b) *The set $G \setminus G_0$ has the cardinality of the continuum.*

Proof.

a) It is easy to prove that G_0 is a group. Thus it suffices to prove $G_0 \subset G$. Suppose that a permutation $g\colon \mathbb{N} \to \mathbb{N}$ does not belong to G. Then there exists $\varepsilon > 0$ and an infinite sequence $\{n_k\}$ of positive integers such that the number $j \in \mathbb{N}$, $j \le n_k$ verifying $g(j) > n_k$ exceeds the εn_k. Denote j_k an element from the set $\{1, \dots, n_k\}$ such that $g(j_k) = \max\big\{g(j) :\ 1 \le j \le n_k\big\}$. Then, between $g(j_k)$ and n_k, there are at least $\varepsilon n_k - 1$ numbers and so we have $g(j_k) \ge n_k + \varepsilon n_k = (1+\varepsilon)n_k \ge (1+\varepsilon)j_k$. Thus $\frac{g(j_k)}{j_k} \ge 1 + \varepsilon$. Clearly $j_k \to \infty$ and so $g \notin G_0$.

b) Let $A = \{a_1 < a_2 < \dots\}$ be an infinite subset of $\mathbb{N}$ such that $\lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k} \to \infty$ and $d(A) = 0$. Put $A_1 = \{a_{2k-1} :\ k = 1, 2, \dots\}$ and $A_2 = \{a_{2k} :\ k = 1, 2, \dots\}$. Denote H the set of all permutations $g\colon \mathbb{N} \to \mathbb{N}$ such that $g(x) = x$ for $x \notin A$ and $g(A_1) = A_2$, $g(A_2) = A_1$. It can be easily proved that $H \subset G$. But for $g \in H$ and $a_k \in A$ we have $\frac{g(a_k)}{a_k} \ge \frac{a_{k+1}}{a_k}$ or $\frac{g(a_k)}{a_k} \le \frac{a_{k-1}}{a_k}$ thus $g \notin G_0$. Therefore $H \subset G \setminus G_0$. It is easy to see that H is uncountable. $\square$

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Now we use the following well known characterization of asymptotic density of the infinite sets. (cf. [3] or [6])

v) Let $A \subset \mathbb{N}$ be an infinite set and $\{a_n\}$ be the sequence of all its elements ordered with respect to the magnitude. Then $A \in \mathscr{D}$ if and only if there exists limit $\lim\limits_{n\to\infty} \frac{n}{a_n} = \gamma$ and in this case it holds that $d(A) = \gamma$.

vi) For any set $A \subset \mathbb{N}$ it holds that

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{n}{a_n} \qquad \text{and} \qquad \overline{d}(A) = \limsup_{n\to\infty} \frac{n}{a_n}.$$

Analogously as in [2] this lemma will be crucial.

Lemma 1. *Let $A, B \in \mathscr{D}$ be such that $A \cap B = \emptyset$, $d(A) = d(B) = \alpha \in (0,1)$. Then there exists such $g \in G_0$ that $g(A) = B$.*

P r o o f. Since $\alpha > 0$, the sets A, B are infinite. Denote

$$A = \{a_1 < a_2 < \dots\}, \qquad B = \{b_1 < b_2 < \dots\},$$
$$\mathbb{N} \setminus A = \{a'_1 < a'_2 < \dots\}, \qquad \mathbb{N} \setminus B = \{b'_1 < b'_2 < \dots\}.$$

Define the permutation $g\colon \mathbb{N} \to \mathbb{N}$ as follows: $g(a_i) = b_i$, $g(a'_i) = b'_i$. Then $\frac{g(n)}{n} = \frac{a_i}{b_i}$ or $\frac{g(n)}{n} = \frac{a'_i}{b'_i}$, thus v) yields that $g \in G_0$. □

The following statement is proved in [2], and it can be easily derived from v) (cf. for instance [3]).

Lemma 2. *Let $A \in \mathscr{D}$. For every $\alpha \in [0, d(A)]$ there exists a subset $B \subset A$ that $B \in \mathscr{D}$ and $d(B) = \alpha$.*

Remark that Lemma 2 is called "Darboux property of the asymptotic density".

Theorem.

c) *Let γ be a finitely additive probability measure on $\mathscr{D}$ such that for every $g \in G_0$ and every $A \in \mathscr{D}$ it holds that $\gamma(g(A)) = \gamma(A)$. Then $\gamma = d$.*

d) *Let $A \in \mathscr{D}$. If for every $g \in G_0$ we have*

$$d\big(A \ominus g(A)\big) = 0 \tag{2}$$

($\ominus$ is the symmetric difference), then $d(A) = 1$ or $d(A) = 0$.

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P r o o f. The proof is a straightforward transcription of the proof from [2].

c) Suppose that $A \in \mathscr{D}$ and $d(A) = \frac{1}{q}$ where $q \in \mathbb{N}$, $q > 1$. Due to Lemma 2 the set $\mathbb{N}$ can be written $\mathbb{N} = A \cup A_1 \cup \dots \cup A_{q-1}$ as a disjoint decomposition where the subsets A_j are elements of $\mathscr{D}$ with the density $\frac{1}{q}$, $j = 1, \dots, q-1$. From Lemma 1 we have that $A_j = g_j(A)$ for suitable permutations $g_j \in G_0$, $j = 1, \dots, q-1$; and so from the assumption of c) we deduce that $q \cdot \gamma(A) = 1$ thus $\gamma(A) = \frac{1}{q} = d(A)$. If now $B \in \mathscr{D}$ and $d(B)$ is a non-zero rational number then due to Lemma 2 the set B has a disjoint decomposition $B = B_1 \cup \dots \cup B_p$ where $d(B_i) = \frac{1}{q}$ thus also in this case we have $\gamma(B) = d(B)$. If $d(B) = 0$ then $d(\mathbb{N} \setminus B) = 1$ and so $\gamma(\mathbb{N} \setminus B) = 1$ thus $\gamma(B) = 0$. We have proved that $d(B) = \gamma(B)$ if $d(B)$ is rational. Suppose that $B \in \mathscr{D}$ and $d(B)$ is irrational. From Lemma 2, for any rational number $r > d(B)$ there exists a set B_r in $\mathscr{D}$ such that $B \subset B_r$ and $d(B_r) = r$. But in this case $\gamma(B_r) = r$, and so $\gamma(B) < r$; consequently $d(B) \geq \gamma(B)$. The same inequality for $\mathbb{N} \setminus B$ leads to $d(B) \leq \gamma(B)$, hence $d(B) = \gamma(B)$.

d) Suppose that for the set A from d) it holds that $0 < d(A) < 1$. Since the condition (2) holds also for $\mathbb{N} \setminus A$ we can suppose that $0 < d(A) \leq \frac{1}{2}$.

Lemma 2 implies that there exists a set $A_1 \subset \mathbb{N} \setminus A$ with $d(A) = d(A_1)$. Therefore from Lemma 1 we have that there exists a $g \in G_0$ such that $g(A) = A_1$. We have a contradiction with (2) because the sets A, $g(A) = A_1$ are disjoint. $\square$

Let us conclude with following remarks.

Proposition 2. *Let $g \colon \mathbb{N} \to \mathbb{N}$ be an injective mapping and $A_\alpha = \left\{n : \frac{g(n)}{n} \leq \alpha\right\}$, for $\alpha \geq 0$. Then $\overline{d}(A_\alpha) \leq \alpha$ for $\alpha \geq 0$.*

P r o o f. The assertion is trivial when the set A_α is a finite one. Let $A_\alpha = \{n_1 < n_2 < \dots < n_k < \dots\}$. Then $g(n_k) \leq \alpha n_k$, $k = 1, 2, \dots$. Let $\beta < \overline{d}(A_\alpha)$. Then from vi) we have that there exists a $k' \in \mathbb{N}$ such that $\frac{k'}{n_{k'}} \geq \beta$. The set $\{g(n_1), \dots, g(n_{k'})\}$ has k' elements which does not exceed $\alpha n_{k'}$, thus $k' \leq \alpha n_{k'}$. This yields $k' \leq k' \frac{\alpha}{\beta}$ and so $\beta \leq \alpha$ and the assertion follows. $\square$

From Proposition 2 we can immediately derive that for an injective mapping $g \colon \mathbb{N} \to \mathbb{N}$ the inequality

$$\limsup_{n \to \infty} \frac{g(n)}{n} \geq 1.$$

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holds. Therefore the group G_0 consists of the permutations g with the lowest possible value of $\limsup\limits_{n\to\infty} \frac{g(n)}{n}$.

Proposition 3. *Let $A, B \subset \mathbb{N}$ be two infinite sets $A = \{a_1 < a_2 < \dots\}$, $B = \{b_1 < b_2 < \dots\}$ such that*

$$\liminf_{n\to\infty} \frac{b_n}{a_n} > 1 \qquad \textit{or} \qquad \limsup_{n\to\infty} \frac{b_n}{a_n} < 1.$$

Then there does not exist a permutation $g \in G_0$ such that $g(A) = B$.

P r o o f. Let $g\colon \mathbb{N} \to \mathbb{N}$ be a permutation such that $g(A) = B$. Then $b_n = g(a_{k_n})$, $n = 1, 2, \dots$, and $\{k_1, k_2, \dots\} = \mathbb{N}$. It is easy to see that there exists an infinite subsequence $\{k_{n_s}\}$ such that $k_{n_s} \le n_s$. This yields

$$\frac{g(a_{k_{n_s}})}{a_{k_{n_s}}} = \frac{b_{n_s}}{a_{k_{n_s}}} \ge \frac{b_{n_s}}{a_{n_s}}$$

and so $\limsup\limits_{k\to\infty} \frac{g(k)}{k} > 1$ in the first case, thus $g \notin G_0$. The second case follows analogously. $\square$

Thus for instance there is not a permutation from G_0 between the sets $\{n^2 : n \in \mathbb{N}\}$, $\{n^3 : n \in \mathbb{N}\}$.

Question. *How should be the minimal set of permutations of the natural numbers which preserve the asymptotic density F, such that iii) and iv) hold when G is replaced by F?*

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