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ON AUTOMORPHISMS OF FINITE ABELIAN p-GROUPS

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ABSTRACT. Let A be a finitely generated abelian group. We describe the automorphism group $\mathrm{Aut}(A)$ using the rank of A and its torsion part p-part A_p.

For a finite abelian p-group A of type $(k_1, \ldots, k_n)$, simple necessary and sufficient conditions for an $n \times n$-matrix over integers to be associated with an automorphism of A are presented. Then, the automorphism group $\mathrm{Aut}(A)$ for a finite p-group A of type (k_1, k_2) is analyzed.

Introduction

Certainly, automorphisms of any finite abelian group A are determined by automorphisms of its p-primary components for any prime p. Furthermore, the automorphism group $\mathrm{Aut}(A)$ of any finite abelian p-group A of type $(k_1, \ldots, k_n)$ have been described in [5, Satz 112–114] up to some extension. Namely, a normal series of $\mathrm{Aut}(A)$ such that its quotient groups are completely determined in terms of well known groups has been constructed.

The purpose of this note is to complete those results and present (Theorem 1) simple necessary and sufficient conditions for an $n \times n$-matrix over integers to be associated with an automorphism of A. Then, the automorphism group $\mathrm{Aut}(A)$ is analyzed for finite p-group A of type (k_1, k_2). We point out that automorphisms of split metacyclic groups have been fully described in [4].

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At the end, given a finitely generated abelian group A, the group $\mathrm{Aut}(A)$ is determined by means of automorphisms of its free and torsion parts.

Main result

Let A be a finite multiplicative abelian group and $A = \bigoplus_p A_p$ its primary decomposition. Given an endomorphism $\varphi\colon A \to A$, we get $\varphi(A_p) \subseteq A_p$ for any prime p. Henceforth, unless explicitly stated otherwise, in the rest of the paper we assume that A is a finite abelian p-group for some prime p. Write e for its unit element, $o(a)$ for the order of $a \in A$ and let

$$A^0 = \big\{a \in A : \ o(a) = p \ \text{or} \ o(a) = 1\big\}.$$

We start with the following useful lemma:

Lemma 1. *Let $\varphi\colon A \to A$ be an endomorphism of A. Then φ is an automorphism if and only if $\varphi\big|_{A^0}\colon A^0 \to A^0$ is an automorphism of A^0.*

Proof. Certainly, $\varphi\big|_{A^0}$ is an automorphism of A^0, provided that φ is an automorphism of A.

Conversely, suppose that $\varphi\big|_{A^0}\colon A^0 \to A^0$ is an automorphism of A^0 and let $\varphi(a) = e$ for some $a \in A$ with $a \neq e$. Then, $a^{p^{n-1}} \in A^0$ and $a^{p^{n-1}} \neq e$, where $p^n = o(a)$. On the other hand, $\varphi(a^{p^{n-1}}) = e$ and so $a^{p^{n-1}} = e$. This contradiction shows that φ is an automorphism of A. $\square$

Fix an isomorphism

$$A \cong \mathbb{Z}/p^{k_1} \oplus \dots \oplus \mathbb{Z}/p^{k_n}$$

with $1 \leq k_1 \leq \dots \leq k_n$ and let $\mathbb{Z}/p^{k_i} = \big\langle a_i \big\rangle$ for $i = 1, \dots, n$, where $\mathbb{Z}/k$ denotes the cyclic group of order k. We also write $\mathbb{Z}/k$ for the ring of integers (mod k) and $(\mathbb{Z}/k)^\star$ for its group of invertible elements.

Given an endomorphism $\varphi\colon A \to A$ we get (up to the isomorphism above) $\varphi(a_j) = a_1^{\alpha_{1j}} \cdots a_n^{\alpha_{nj}}$ for some integers $0 \leq \alpha_{ij} < p^{k_i}$ with $i, j = 1, \dots, n$. Observe that the relation $a_j^{p^{k_j}} = e$ yields $\alpha_{ij} \equiv 0 \pmod{p^{k_i - k_j}}$ for $j \leq i$.

Let now

$$A \cong \bigoplus_{i=1}^{t_1} \mathbb{Z}/p^{l_1} \oplus \dots \oplus \bigoplus_{i=1}^{t_m} \mathbb{Z}/p^{l_m}$$

with some positive integers $1 \leq l_1 < \cdots < l_m$. Then, any endomorphism $\varphi\colon A \to A$ (up to that isomorphism) is represented by a square matrix

$$\begin{pmatrix} M_1 & * & * & \cdots & * \\ * & M_2 & * & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & M_m \end{pmatrix},$$

where M_s is a square $t_s \times t_s$-matrix over integers for $s = 1, \ldots, m$.

Note that

$$A^0 \cong \bigoplus_{i=1}^{t_1} \langle a_1^{p^{l_1-1}} \rangle \oplus \cdots \oplus \bigoplus_{i=1}^{t_m} \langle a_m^{p^{l_m-1}} \rangle.$$

Hence, the endomorphism $\varphi\big|_{A^0}\colon A^0 \to A^0$ is represented by the matrix

$$\begin{pmatrix} \overline{M}_1 & 0 & 0 & \cdots & 0 \\ \bar{*} & \overline{M}_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \bar{*} & \bar{*} & \bar{*} & \cdots & \overline{M}_m \end{pmatrix},$$

where a "bar" signifies that entries are considered $(\mathrm{mod}\ p)$ for $s = 1, \ldots, m$. The discussion above and Lemma 1 lead to the following result.

Theorem 2. *Let A be a finite abelian p-group.*

(1) *If A is of type $(k_1, \ldots, k_n)$ with $1 \leq k_1 \leq \cdots \leq k_n$ then the square matrix $[\alpha_{ij}]$ over integers represents an endomorphism of A if and only if $0 \leq \alpha_{ij} < p^{k_i}$ and $\alpha_{ij} \equiv 0 \pmod{p^{k_i - k_j}}$ for $j \leq i$.*

(2) *If A is of type $(\underbrace{l_1, \ldots, l_1}_{t_1}, \ldots, \underbrace{l_m, \ldots, l_m}_{t_m})$ with some positive integers $1 \leq l_1 < \cdots < l_m$ then the matrix*

$$[\alpha_{ij}] = \begin{pmatrix} M_1 & * & * & \cdots & * \\ * & M_2 & * & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & M_m \end{pmatrix},$$

where M_s is a square $t_s \times t_s$-matrix over integers for $s = 1, \ldots, m$ represents an automorphism of A if and only if:

(i) $0 \leq \alpha_{ij} < p^{k_i}$ *and* $\alpha_{ij} \equiv 0 \pmod{p^{k_i - k_j}}$ *for* $j \leq i$*;*

(ii) $\det \overline{M}_1 \cdots \det \overline{M}_m \neq 0$.

We apply the above to the group $A \cong \mathbb{Z}/p^m \times \mathbb{Z}/p^n$. If $n = m$ then certainly $\mathrm{Aut}(A) = GL_2(p^m)$, the group of invertible 2×2-matrices over the ring $\mathbb{Z}/p^m$

of integers (mod p^m). Then, the epimorphism $GL_2(p^m) \to GL_2(p)$ given by the assignment $M \mapsto \overline{M}$ for $M \in GL_2(p^m)$ leads to the short exact sequence

$$e \to GL_2'(p^m) \to GL_2(p^m) \to GL_2(p) \to e.$$

Note that $GL_2'(p^m) = \{E_2 + pM : \ M \in M_2(p^m)\}$, where E_2 is the unit 2×2-matrix and $M_2(p^m)$ the ring of all 2×2-matrices over $\mathbb{Z}/m$. Furthermore, by [2, Chapter 7], the group $GL_2(p)$ has the presentation

$$\begin{aligned} GL_2(p) = \langle Q, S, T;\ & Q^{p-1} = e,\ Q^{-1}SQ = S^n,\ Q^{-1}TSQ = TS^{-1/\alpha}TS^{-1/\alpha}T, \\ & S^p = e,\ T^2 = (ST)^3 = (S^4TS^{(p+1)/2}T)^2 \rangle, \end{aligned}$$

where α is a cyclic generator of the group $(\mathbb{Z}/p)^\star$.

For $m < n$, we derive:

Corollary 3. *Let $A \cong \mathbb{Z}/p^m \times \mathbb{Z}/p^n$ with $m < n$. Then, the matrix $\begin{pmatrix} i & r \\ j & s \end{pmatrix}$ represents:*

(1) *an endomorphism of A if and only if: $i \in \mathbb{Z}/p^m$, $j \equiv 0 \pmod{p^{n-m}}$, $r \in \mathbb{Z}/p^n$ and $s \in \mathbb{Z}/p^n$;*

(2) *an automorphism of A if and only if: $i \in (\mathbb{Z}/p^m)^\star$, $j \equiv 0 \pmod{p^{n-m}}$, $r \in \mathbb{Z}/p^n$ and $s \in (\mathbb{Z}/p^n)^\star$.*

Note that elements of a matrix associated to an endomorphism of the group $A \cong \mathbb{Z}/p^m \times \mathbb{Z}/p^n$ are in different rings. Nevertheless, the product of two such matrices is well defined as result of the congruence $j \equiv 0 \pmod{p^{n-m}}$. The composition of two endomorphisms of the group $A \cong \mathbb{Z}/p^m \times \mathbb{Z}/p^n$ is sent to the product of associated matrices.

Furthermore, for an odd prime p or $t \leq 2$ by means of [8, Chapter IV] it holds that $(\mathbb{Z}/p^t)^\star \cong \mathbb{Z}/p-1 \times \mathbb{Z}/p^{t-1}$. Let now $i = i'u$ for $i' \in \mathbb{Z}/p^{m-1}$, $u \in \mathbb{Z}/p-1$ and $s = s'v$ for $s' \in \mathbb{Z}/p^{n-1}$, $v \in \mathbb{Z}/p-1$. Then, given a matrix $\begin{pmatrix} i & r \\ j & s \end{pmatrix}$ associated to an automorphism of $A \cong \mathbb{Z}/p^m \times \mathbb{Z}/p^n$, we deduce that $\begin{pmatrix} i & r \\ j & s \end{pmatrix} = \begin{pmatrix} i' & r' \\ j' & s' \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ for $j' = u^{-1}j$ and $r' = v^{-1}r$. Observe that automorphisms corresponding to the matrices $\begin{pmatrix} i' & r' \\ j' & s' \end{pmatrix}$ and $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ form subgroups S_p and N of $\mathrm{Aut}(A)$ with:

(a) $S_p \cap N = E$, the trivial subgroup of $\mathrm{Aut}(A)$;

(b) $S_pN = \mathrm{Aut}(A)$;

(c) $S_p \lhd \mathrm{Aut}(A)$, i.e., S_p is a normal subgroup of $\mathrm{Aut}(A)$.

The properties (a) and (b) are clear. Because the order $O(\mathrm{Aut}(A)) = (p-1)^2 p^{2(m+n-1)}$ we see that $S_p \lhd \mathrm{Aut}(A)$ as the unique Sylow p-subgroup of $\mathrm{Aut}(A)$ and (c) follows. Hence, $\mathrm{Aut}(A) \cong S_p \rtimes N$, the semi-direct product of S_p and N, where the action of N on S_p is given by conjugation. More precisely, given a matrix $\begin{pmatrix} i' & r' \\ j' & s' \end{pmatrix}$ in S_p with $i' = (1+p)^k$, $j' = bp^{n-m}$, $r' \in \mathbb{Z}/p^n$ and $s' = (1+p)^l$ for some $0 \le k \le p^{m-1}$, $0 \le b \le p^m$ and $0 \le l \le p^{n-1}$, by conjugation with the matrix $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ in N, we obtain $\begin{pmatrix} (1+p)^k & uv^{-1}r' \\ u^{-1}vbp^{n-m} & (1+p)^l \end{pmatrix}$ for any $u, v \in \mathbb{Z}/p-1$. Furthermore, there is an isomorphism of the groups $N \cong \mathbb{Z}/p-1 \times \mathbb{Z}/p-1$.

Let now move to $p = 2$. Then, again by [8, Chapter IV], we get $(\mathbb{Z}/2^t)^\star \cong \mathbb{Z}/2 \times \mathbb{Z}/2^{t-2}$ provided $t \ge 3$. Let $i = i'u$ for $i' \in \mathbb{Z}/2^{m-2}$, $u \in \mathbb{Z}/2$ and $s = s'v$ for $s' \in \mathbb{Z}/2^{n-1}$, $v \in \mathbb{Z}/2$. Hence, given the matrix $\begin{pmatrix} i & r \\ j & s \end{pmatrix}$ associated to an automorphism of $\mathrm{Aut}(A)$, we deduce that $\begin{pmatrix} i & r \\ j & s \end{pmatrix} = \begin{pmatrix} i' & r' \\ j' & s' \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ for $j' = u^{-1}j$ and $r' = v^{-1}r$. One can easily check that automorphisms corresponding to matrices $\begin{pmatrix} i' & r' \\ j' & s' \end{pmatrix}$ and $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ form subgroups S_2 and N of $\mathrm{Aut}(A)$ with:

(a) $S_2 \cap N = E$;
(b) $S_2 N = \mathrm{Aut}(A)$;
(c) $S_2 \lhd \mathrm{Aut}(A)$.

Hence, $\mathrm{Aut}(A) \cong S_2 \rtimes N$. Furthermore, the order $O(\mathrm{Aut}(A)) = 2^{2(m+n-1)}$ and $N \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ provided $m \ge 3$. However, $N \cong \mathbb{Z}/2$ for $m = 2$ and $N = E$ for $m = 1$ and $n = 2$.

Now, we aim to describe the structure of the subgroup $S_p \lhd \mathrm{Aut}(A)$. Let $\langle \alpha \rangle = \mathbb{Z}/p^{m-1} \hookrightarrow (\mathbb{Z}/p^m)^\star$ and $\langle \beta \rangle = \mathbb{Z}/p^{n-1} \hookrightarrow (\mathbb{Z}/p^n)^\star$ for $p > 2$. Further, $\langle \alpha \rangle = \mathbb{Z}/2^{m-2} \hookrightarrow (\mathbb{Z}/2^m)^\star$ and $\langle \beta \rangle = \mathbb{Z}/2^{n-2} \hookrightarrow (\mathbb{Z}/2^n)^\star$ provided $2 \le m < n$. Then, consider the following cyclic subgroups of S_p for any prime p:
$H_p^1 = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$, $H_p^2 = \left\langle \begin{pmatrix} 1 & 0 \\ p^{n-m} & 1 \end{pmatrix} \right\rangle$, $H_p^3 = \left\langle \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$, $H_p^4 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \right\rangle$.

It is easy to see that $T_p^1 = \langle H_p^2, H_p^3 \rangle \cong H_p^2 \rtimes H_p^3$, where the action of H_p^3 on H_p^2 is determined by the assignment $\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ p^{n-m} & 1 \end{pmatrix} \right) \mapsto \begin{pmatrix} 1 & 0 \\ \alpha^{-1}p^{n-m} & 1 \end{pmatrix}$.

Further, the subgroup $T_p^2 = \langle H_p^1, H_p^4 \rangle \cong H_p^1 \rtimes H_p^4$. The action of H_p^4 on H_p^1 is determined by the assignment $\left(\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \mapsto \begin{pmatrix} 1 & \beta^{-1} \\ 0 & 1 \end{pmatrix}$.

Now, by the order argument, we derive $S_p = T_p^1 T_p^2 = T_p^2 T_p^1$.

Note that any matrix from H_p^3 and H_p^4, conjugated by the matrix $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$, is sent to itself for any $u, v \in \mathbb{Z}/p-1$. The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, by conjugation with $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$) is sent into $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & v^{-1} \\ 0 & 1 \end{pmatrix}$); the matrix $\begin{pmatrix} 1 & 0 \\ p^{n-m} & 1 \end{pmatrix}$ by conjugation with $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$) is sent into $\begin{pmatrix} 1 & 0 \\ u^{-1}p^{n-m} & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 0 \\ vp^{n-m} & 1 \end{pmatrix}$) for any $u, v \in \mathbb{Z}/p-1$.

But $H_p^1 \cong \mathbb{Z}/p^n$, $H_p^2 \cong \mathbb{Z}/p^m$ for any prime p and $H_p^3 \cong \mathbb{Z}/p^{n-1}$, $H_p^4 \cong \mathbb{Z}/p^{m-1}$ for $p > 2$. Whereas, $H_2^3 \cong \mathbb{Z}/2^{n-2}$ and $H_2^4 \cong \mathbb{Z}/2^{m-2}$. Therefore, we can summarize the discussion above as follows:

Proposition 4. *If $A \cong \mathbb{Z}/p^m \oplus \mathbb{Z}/p^n$ with $m < n$ then there is an isomorphism:*

(1) $\operatorname{Aut}(A) \cong S_p \rtimes (\mathbb{Z}/p-1 \oplus \mathbb{Z}/p-1)$, *where S_p is the unique Sylow p-subgroup of $\operatorname{Aut}(\mathbb{Z}/p^m \oplus \mathbb{Z}/p^n)$ provided p is an odd prime. Further, $S_p = T_p^1 T_p^2 = T_p^2 T_p^1$ is the product of two subgroups, where $T_p^1 \cong \mathbb{Z}/p^m \rtimes \mathbb{Z}/p^{n-1}$ and $T_p^2 \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/p^{m-1}$.*

The action of $\mathbb{Z}/p^{n-1}$ on $\mathbb{Z}/p^m$ is given by the composite of the injection $\mathbb{Z}/p^{n-1} \hookrightarrow (\mathbb{Z}/p^n)^\star \cong \operatorname{Aut}(\mathbb{Z}/p^n)$ and then taking the restriction to $\mathbb{Z}/p^m \hookrightarrow \mathbb{Z}/p^n$. Whereas the action of $\mathbb{Z}/p^{m-1}$ on $\mathbb{Z}/p^n$ is given by the composite of the injections $\mathbb{Z}/p^{m-1} \hookrightarrow (\mathbb{Z}/p^m)^\star \hookrightarrow (\mathbb{Z}/p^n)^\star \cong \operatorname{Aut}(\mathbb{Z}/p^n)$ with the automorphism of $\mathbb{Z}/p^n$ which sends an element to its inverse.

In particular, $\operatorname{Aut}(\mathbb{Z}/p^m \oplus \mathbb{Z}/p) \cong (\mathbb{Z}/p^{m-1} \oplus \mathbb{Z}/p \oplus \mathbb{Z}/p) \rtimes (\mathbb{Z}/p-1 \oplus \mathbb{Z}/p-1)$;

(2) $\operatorname{Aut}(A) \cong S_2 \rtimes (\mathbb{Z}/2 \oplus \mathbb{Z}/2)$. *Further, $S_2 = T_2^1 T_2^2 = T_2^2 T_2^1$ is the product of two subgroups, where $T_2^1 \cong \mathbb{Z}/2^m \rtimes \mathbb{Z}/2^{n-2}$ and $T_2^2 \cong \mathbb{Z}/2^n \rtimes \mathbb{Z}/2^{m-2}$.*

The action of $\mathbb{Z}/2^{n-2}$ on $\mathbb{Z}/2^m$ is given by the composite of the injection $\mathbb{Z}/2^{n-2} \hookrightarrow (\mathbb{Z}/2^n)^\star \cong \operatorname{Aut}(\mathbb{Z}/2^n)$ and then taking the restriction to $\mathbb{Z}/2^m \hookrightarrow \mathbb{Z}/2^n$. Whereas the action of $\mathbb{Z}/2^{m-2}$ on $\mathbb{Z}/2^n$ is given by the composite of the injections $\mathbb{Z}/2^{m-2} \hookrightarrow (\mathbb{Z}/2^m)^\star \hookrightarrow (\mathbb{Z}/2^n)^\star \cong \operatorname{Aut}(\mathbb{Z}/2^n)$ with the automorphism of $\mathbb{Z}/2^n$ which sends an element to its inverse.

In particular, $\operatorname{Aut}(\mathbb{Z}/2^m \oplus \mathbb{Z}/2) \cong (\mathbb{Z}/2^{m-2} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2) \rtimes (\mathbb{Z}/2)$ *for* $m \geq 2$.

To get the next conclusion, we recall that for an extension $e \to G_1 \to G \to G_2 \to e$ of groups, Wells [7] and then Buckley [1] construct an exact sequence with the group $\operatorname{Aut}(G, G_1)$ of automorphisms G which map G_1 into itself. We have interpreted in [3] that result for semi-direct products of two groups as follows.

Given an action $\tau\colon G_2 \to \operatorname{Aut}(G_1)$ write $\operatorname{Der}_\tau(G_2, G_1)$ for the set of crossed homomorphisms. For an extension $e \to G_1 \to G \to G_2 \to e$ there is an obvious G_2-action $\bar{\tau}\colon G_2 \to \operatorname{Aut}(\mathscr{Z}(G_1))$ on the center $\mathscr{Z}(G_1)$ of the subgroup G_1 and the map $\eta_G\colon \operatorname{Aut}(G, G_1) \to \operatorname{Aut}(G_1) \times \operatorname{Aut}(G_2)$.

Proposition 5. *For any G_2-action $\tau\colon G_2 \to \operatorname{Aut}(G_1)$ on an abelian group G_1 there is a short split exact sequence*

$$0 \to \operatorname{Der}_\tau(G_2, G_1) \longrightarrow \operatorname{Aut}(G_1 \rtimes_\tau G_2, G_1) \longrightarrow \operatorname{Im} \eta_{G_1 \rtimes_\tau G_2} \to 0.$$

Let now A be a finitely generated abelian group and $T(A)$ its torsion part. Then, $T(A)$ is a finite abelian group, invariant with respect to any automorphism of A and $A \cong T(A) \oplus F(A)$, where $F(A)$ is an abelian free group of finite rank $\operatorname{rk}(F(A))$. Hence, by means of Proposition 5 we obtain a short split exact sequence

$$0 \to \operatorname{Hom}(F(A), T(A)) \to \operatorname{Aut}(A) \to \operatorname{Aut}(T(A)) \times \operatorname{Aut}(F(A)) \to 0.$$

We close this paper with:

Corollary 6. *Let A be a finitely generated abelian group and $F(A)$, $T(A)$ its free and torsion parts, respectively. Then, there is an isomorphism*

$$\operatorname{Aut}(A) \cong T(A)^{\operatorname{rk}(F(A))} \rtimes (\operatorname{Aut}(T(A)) \times GL_{\operatorname{rk}(F(A))}(\mathbb{Z})),$$

where the semi-direct product is determined by the action $(\varphi, \psi) * \theta = \varphi \circ \theta \circ \psi$ for $(\varphi, \psi) \in \operatorname{Aut}(T(A)) \times GL_{\operatorname{rk}(F(A))}(\mathbb{Z})$ and $\theta \in T(A)^{\operatorname{rk}(F(A))} \cong \operatorname{Hom}(F(A), T(A))$.

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