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GENERALIZED DIFFERENCE SEQUENCE SPACES ON SEMINORMED SPACE DEFINED BY ORLICZ FUNCTIONS

Binod Chandra Tripathy* — Yavuz Altin** — Mikail Et**

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ABSTRACT. In this paper we define the sequence space $\ell_M(\Delta^m, p, q, s)$ on a seminormed complex linear space by using an Orlicz function. We study its different algebraic and topological properties like solidness, symmetricity, monotonicity, convergence free etc. We prove some inclusion relations involving $\ell_M(\Delta^m, p, q, s)$.

1. Introduction

Let ℓ_∞, c and c_0 be the linear spaces of *bounded*, *convergent* and *null* sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N}$, the set of positive integers. Throughout the paper $w(X)$, $\ell_\infty(X)$, $c(X)$ and $c_0(X)$ denote class of *all*, *bounded, convergent* and *null* X-valued sequences, where (X, q) is a seminormed space, seminormed by q. The zero sequence is denoted by $\bar{\theta} = (\theta, \theta, \dots)$, where θ is the zero element of X. These spaces are seminormed spaces seminormed by $g(x) = \sup_{k \in \mathbb{N}} q(x_k)$. For $X = \mathbb{C}$, the set of complex numbers, these represent the corresponding scalar valued sequence spaces.

The idea of difference sequence sets was introduced by Kizmaz [7] and this subject was generalized by Et and Çolak [4]. After then the difference sequence spaces have been studied by various mathematicians such as Et [3], Et and Nuray [5], Malkowsky and Parashar [13], Mursaleen [14], Tripathy [18], [19], Tripathy et. al. [20].

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The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Indeed, Lindberg [9] got interested in Orlicz spaces in connection with finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c_0 or ℓ_p $(1 \leq p < \infty)$. Subsequently Lindenstrauss and Tzafriri [10] investigated Orlicz sequence spaces in more detail and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(1 \leq p < \infty)$.

Parashar and Choudhary [16] have introduced and discussed some properties of the four sequence spaces defined by using an Orlicz function M, which generalized the sequence space ℓ_M and strongly summable sequence spaces $[C,1,p]$, $[C,1,p]_0$ and $[C,1,p]_\infty$. Later on different types of sequence spaces were introduced by using an Orlicz function by Bektas and Altin [1], Tripathy [19], Tripathy et al. [20], Tripathy and Mahanta [21], [22] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces introduced in [8]. Orlicz spaces find a number of useful applications in the theory of nonlinear integral equations. Whereas the Orlicz sequence spaces are the generalizations of ℓ_p-spaces, the L_p-spaces find themselves enveloped in Orlicz spaces.

The main purpose of this paper is to introduce and study the sequence space $\ell_M(\Delta^m, p, q, s)$ which arises from the notation of generalized difference operator Δ^m and the concept of an Orlicz function.

2. Definitions and background

In this section, using the generalized difference operator Δ^m and the concept of an Orlicz function, we generalize the sequence space $\ell_M(p)$ which was introduced by Parashar an Choudhary [16].

The difference sequence spaces, $Z(\Delta) = \big\{x = (x_k) : \ \Delta x \in Z\big\}$, where $Z = \ell_\infty$, c and c_0, were studied by Kizmaz [7]. The notion of difference sequence spaces was generalized by Et and Çolak [4] as follows:

$$Z(\Delta^m) = \big\{x = (x_k) : \ (\Delta^m x_k) \in Z\big\},$$

for $Z = \ell_\infty$, c and c_0, where $m \in \mathbb{N}$, $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ and so $\Delta^m x_k = \sum\limits_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$. These sequence spaces are BK-spaces with the norm $\|x\|_\Delta = \sum\limits_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty$.

It is trivial that the generalized difference operator Δ^m is a linear operator.

An Orlicz function is a function $M\colon [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\to\infty$, as $x\to\infty$ (for detail see Krasnoselskii and Rutickii [8]).

If the convexity of Orlicz function M is replaced by $M(x+y)\le M(x)+M(y)$ then this function is called modulus function, introduced by Nakano [15] and further investigated by Ruckle [17], Maddox [11], Bilgin [2] and others. Lindenstrauss and Tzafriri [10] defined the sequence space ℓ_M such as:

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm $\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$ becomes a Banach space which is called an Orlicz sequence space. For $M(t)=t^p$, $1\le p<\infty$, the spaces ℓ_M coincide with the classical sequence spaces ℓ_p.

Definition 1. Let $p=(p_k)$ be a sequence of strictly positive real numbers, X be a seminormed space with the seminorm q and M be an Orlicz function. We define the sequence space $\ell_M(\Delta^m,p,q,s)$ as follows:

$$\ell_M(\Delta^m,p,q,s) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right]^{p_k} < \infty,\ \ s\ge 0,\ \ \rho>0 \right\}.$$

We get the following sequence spaces from $\ell_M(\Delta^m,p,q,s)$ on giving particular values to p and s. Taking $p_k=1$ for all $k\in\mathbb{N}$ we have

$$\ell_M(\Delta^m,q,s) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right] < \infty,\ \ s\ge 0,\ \ \rho>0 \right\}.$$

If we take $s=0$, then we have

$$\ell_M(\Delta^m,p,q) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} \left[M\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right]^{p_k} < \infty,\ \ \rho>0 \right\}.$$

If we take $p_k=1$ for all $k\in\mathbb{N}$ and $s=0$, then we have

$$\ell_M(\Delta^m,q) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} \left[M\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right] < \infty,\ \ \rho>0 \right\}.$$

In addition to the above sequence spaces, we have $\ell_M(\Delta^m,p,q,s)=\ell_M(p)$ due to Parashar and Choudhary [16], on taking $m=0$, $s=0$, $q(x)=|x|$ and $X=\mathbb{C}$.

The sequence space $\ell_M(\Delta^m,p,q,s)$ contains some unbounded sequences for $m\ge 1$. This is clear from the following example.

Example 1. Let $X=\mathbb{C}$, $s=0$, $M(x)=x$ and $q(x)=|x|$ and $p_k=1$ for all $k\in\mathbb{N}$. Let $x_k=k^{m-1}$ for all $k\in\mathbb{N}$. Then $(x_k)\in\ell_M(\Delta^m,p,q,s)$ and $(x_k)\notin\ell_\infty$.

A sequence space E is said to be *symmetric* if $\left(x_{\pi(k)}\right) \in E$, whenever $(x_k) \in E$, where π is a permutation of $\mathbb{N}$.

A sequence space E is said to be *convergence free* if $(y_k) \in E$, whenever $(x_k) \in E$ and $y_k = \theta$ when $x_k = \theta$.

A sequence space E is said to be *solid* (or *normal*) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$, for all sequences (α_k) of scalars with $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$.

A sequence space E is said to be *monotone* if it contains the canonical pre-images of all its step-spaces (see Kamthan and Gupta [6, p. 48]).

Remark 1. It is well known that a sequence spaces E is normal implies that E is monotone (see for instance Kamthan and Gupta [6, p. 48]).

Remark 2. If M is a convex function and $M(0) = 0$, then $M(\lambda x) \le \lambda M(x)$ for all λ with $0 < \lambda < 1$.

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a sequence of strictly positive real numbers with $0 < p_k \le \sup_{k \in \mathbb{N}} p_k = G$, and let $D = \max\{1, 2^{G-1}\}$. Then for all $a_k, b_k \in \mathbb{C}$, we have

$$|a_k + b_k|^{p_k} \le D\left\{|a_k|^{p_k} + |b_k|^{p}\right\} \qquad ([12]). \tag{1}$$

3. Main results

In this section we will prove the results of this article involving the sequence space $\ell_M(\Delta^m, p, q, s)$.

Theorem 1. *The sequence space $\ell_M(\Delta^m, p, q, s)$ is linear space over $\mathbb{C}$.*

Proof. Let $x, y \in \ell_M(\Delta^m, p, q, s)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist some positive numbers ρ_1 and ρ_2 such that

$$\sum_{k=1}^{\infty} k^{-s} \left[M\left(q\left(\frac{\Delta^m x_k}{\rho_1}\right)\right)\right]^{p_k} < \infty$$

and

$$\sum_{k=1}^{\infty} k^{-s} \left[M\left(q\left(\frac{\Delta^m y_k}{\rho_2}\right)\right)\right]^{p_k} < \infty.$$

Define $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since M is non-decreasing convex function, q is a seminorm and Δ^m is linear, we have

$$\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\Delta^m(\alpha x_k+\beta y_k)}{\rho_3}\right)\right)\right]^{p_k}$$
$$\le D\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\Delta^m x_k}{\rho_1}\right)\right)\right]^{p_k} + D\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\Delta^m y_k}{\rho_2}\right)\right)\right]^{p_k} < \infty.$$

This proves that $\ell_M(\Delta^m, p, q, s)$ is linear space. □

Theorem 2. *Let $p = (p_k) \in \ell_\infty$ and $s > 1$. The sequence space $\ell_M(\Delta^m, p, q, s)$ is paranormed (not necessarily totally paranormed) space, paranormed by*

$$g_\Delta(x) = \sum_{k=1}^{m} q(x_k) + \inf\left\{\rho^{\frac{p_n}{H}} : M\left(q\left(\tfrac{\Delta^m x_k}{\rho}\right)\right) \le 1, \ n \in \mathbb{N}\right\},$$

where $H = \max\left\{1, \sup_{k\in\mathbb{N}} p_k\right\}$.

Proof. Clearly $g_\Delta(x) = g_\Delta(-x)$. Let $x = \bar{\theta} \in \ell_M(\Delta^m, p, q, s)$. Then there exist $\rho_1 > 0$, $\rho_2 > 0$ such that

$$M\left(q\left(\frac{\Delta^m x_k}{\rho_1}\right)\right) \le 1 \qquad \text{and} \qquad M\left(q\left(\frac{\Delta^m y_k}{\rho_2}\right)\right) \le 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\begin{aligned} M\left(q\left(\frac{\Delta^m(x_k+y_k)}{\rho}\right)\right) &= M\left(\frac{\rho_1}{\rho}q\left(\frac{\Delta^m x_k}{\rho_1}\right) + \frac{\rho_2}{\rho}q\left(\frac{\Delta^m y_k}{\rho_2}\right)\right) \\ &\le \frac{\rho_1}{\rho}M\left(q\left(\frac{\Delta^m x_k}{\rho_1}\right)\right) + \frac{\rho_2}{\rho}M\left(q\left(\frac{\Delta^m y_k}{\rho_2}\right)\right) \le 1. \end{aligned}$$

Hence

$$\begin{aligned} g_\Delta(x+y) &= \sum_{k=1}^{m} q(x_k+y_k) + \inf\left\{\rho^{\frac{p_n}{H}} : M\left(q\left(\tfrac{\Delta^m(x_k+y_k)}{\rho}\right)\right) \le 1, \ n \in \mathbb{N}\right\}, \\ &\le \sum_{k=1}^{m} q(x_k) + \sum_{k=1}^{m} q(y_k) + \inf\Big\{(\rho_1+\rho_2)^{\frac{p_n}{H}} : M\left(q\left(\tfrac{\Delta^m x_k}{\rho_1}\right)\right) \le 1, \\ &\qquad\qquad M\left(q\left(\tfrac{\Delta^m y_k}{\rho_2}\right)\right) \le 1, \ n \in \mathbb{N}\Big\}, \\ &\le g_\Delta(x) + g_\Delta(y) \qquad \rho = \rho_1 + \rho_2. \end{aligned}$$

Finally we prove that the scalar multiplication is continuous. Let λ be any number. From the linearity of Δ^m and the definition,

$$\begin{aligned} g_\Delta(\lambda x) &= \sum_{k=1}^{m} q(\lambda x_k) + \inf\left\{\rho^{\frac{p_n}{H}} : M\left(q\left(\tfrac{\lambda\Delta^m x_k}{\rho}\right)\right) \le 1,\ n\in\mathbb{N}\right\} \\ &= |\lambda|\sum_{k=1}^{m} q(x_k) + \inf\left\{(|\lambda| r)^{\frac{p_n}{H}} : M\left(q\left(\tfrac{\Delta^m x_k}{r}\right)\right) \le 1,\ n\in\mathbb{N}\right\}, \end{aligned}$$

where $r = \frac{\rho}{\lambda}$.

Now it can be easily verified that $\lambda \to 0$ and x fixed implies $g_\Delta(\lambda x) \to 0$; λ fixed and $x \to \bar{\theta}$ implies $g_\Delta(x) \to 0$ in $\ell_M(\Delta^m, p, q, s)$; $\lambda \to 0$ and $x \to \bar{\theta}$ implies $g_\Delta(x) \to 0$ in $\ell_M(\Delta^m, p, q, s)$. □

Remark 3. The space $\ell_M(\Delta^m, p, q, s)$ will be a totally paranormed sequence space if $m = 0$ and the seminorm q is replaced by a norm.

Theorem 3. *Let M, M_1, M_2 be Orlicz functions and s, s_1, s_2 be non-negative real numbers. Then we have*

(i) $\ell_{M_1}(\Delta^m, p, q, s) \bigcap \ell_{M_2}(\Delta^m, p, q, s) \subseteq \ell_{M_1+M_2}(\Delta^m, p, q, s)$.

(ii) *If $s_1 \le s_2$, then $\ell_M(\Delta^m, p, q, s_1) \subseteq \ell_M(\Delta^m, p, q, s_2)$.*

Proof.

(i) From (1) we have

$$\begin{aligned} &k^{-s}\left[(M_1+M_2)\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right]^{p_k} \\ &= k^{-s}\left[M_1\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right) + M_2\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right]^{p_k} \\ &\le Dk^{-s}\left[M_1\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right]^{p_k} + Dk^{-s}\left[M_2\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right)\right]^{p_k}. \end{aligned}$$

Let $x \in \ell_{M_1}(\Delta^m, p, q, s) \cap \ell_{M_2}(\Delta^m, p, q, s)$; when adding the above inequality from $k = 1$ to ∞, we get $x \in \ell_{M_1+M_2}(\Delta^m, p, q, s)$.

(ii) Let $s_1 \le s_2$ and $x \in \ell_M(\Delta^m, p, q, s_1)$. Since $k^{-s_2} \le k^{-s_1}$, we have $x \in \ell_M(\Delta^m, p, q, s_2)$.

This completes the proof of Theorem 3. □

Theorem 4. *Let $m \ge 1$, then the inclusion $\ell_M(\Delta^{m-1}, q, s) \subset \ell_M(\Delta^m, q, s)$ is strict. In general $\ell_M(\Delta^i, q, s) \subset \ell_M(\Delta^m, q, s)$ for all $i = 0, 1, 2, \dots, m-1$ and the inclusion is strict.*

Proof. Let $x \in \ell_M(\Delta^{m-1}, q, s)$. Then we have

$$\sum_{k=1}^{\infty} k^{-s} \left[M \left(q \left(\frac{\Delta^{m-1} x_k}{\rho} \right) \right) \right] < \infty \tag{2}$$

for some $\rho > 0$. Since M is non-decreasing convex function and q is a seminorm, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} k^{-s} \left[M \left(q \left(\frac{\Delta^{m} x_k}{2\rho} \right) \right) \right] \\
&= \sum_{k=1}^{\infty} k^{-s} \left[M \left(q \left(\frac{\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}}{2\rho} \right) \right) \right] \\
&\le \sum_{k=1}^{\infty} k^{-s} \left[\frac{1}{2} M \left(q \left(\frac{\Delta^{m-1} x_k}{\rho} \right) \right) \right] + \sum_{k=1}^{\infty} k^{-s} \left[\frac{1}{2} M \left(q \left(\frac{\Delta^{m-1} x_{k+1}}{\rho} \right) \right) \right] \\
&< \infty, \qquad \text{by (2).}
\end{aligned}$$

Thus $\ell_M(\Delta^{m-1}, q, s) \subset \ell_M(\Delta^m, q, s)$. Proceeding in this way one will have $\ell_M(\Delta^i, q, s) \subset \ell_M(\Delta^m, q, s)$ for all $i = 0, 1, 2, \dots, m-1$. □

To show that the inclusion is strict, consider the following example.

Example 2. Let $X = \mathbb{C}$, $M(x) = x^p$, $q(x) = |x|$, $s = 0$. Consider the sequence $(x_k) = \left(k^{m-1}\right)$. Then $(x_k) \in \ell_M(\Delta^m, q, s)$, but $(x_k) \notin \ell_M(\Delta^{m-1}, q, s)$ since $\Delta^m x_k = 0$ and $\Delta^{m-1} x_k = (-1)^{m-1}(m-1)!$ for all $k \in \mathbb{N}$.

Theorem 5. *The space $\ell_M(\Delta^m, p, q, s)$ is not convergence free.*

Proof. The result follows from the following example. □

Example 3. Let $X = \mathbb{C}$, $q(x) = |x|$, $M(x) = x$, $m = 1$, $s = 2$, $p_k = 2$ for all $k \in \mathbb{N}$. Consider the sequence (x_k) defined by $x_k = 1$ for $k = 2n-1$, $n \in \mathbb{N}$, and $x_k = 0$ otherwise. Then $(x_k) \in \ell(\Delta, 2, 2)$. Now consider the sequence (y_k) defined by $y_k = k$ for $k = 2n-1$, $n \in \mathbb{N}$, and $y_k = 0$ otherwise. Then $(y_k) \notin \ell(\Delta, 2, 2)$. Hence $\ell(\Delta, 2, 2)$ is not convergence free.

Theorem 6. *The space $\ell_M(\Delta^m, p, q, s)$ is not symmetric in general.*

Proof. The result follows from the following example. □

Example 4. Let $X = \mathbb{C}$, $q(x) = |x|$, $M(x) = x^p$ for $p \ge 1$, $m = 1$, $s = 2$, $p_k = 2$ for all $k \in \mathbb{N}$. Consider the sequence (x_k) defined by $x_k = k$ for all $k \in \mathbb{N}$. Then $(x_k) \in \ell(\Delta, 2, 2)$. Now consider the rearranged sequence (y_k) of (x_k) defined by

$$(y_k) = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \dots).$$

Then $(y_k) \notin \ell(\Delta, 2, 2)$. Hence $\ell(\Delta, 2, 2)$ is not symmetric.

THEOREM 7. *Let $0 < p_k \leq t_k < \infty$ for each $k \in \mathbb{N}$. Then $\ell_M(\Delta^m, p, q) \subseteq \ell_M(\Delta^m, t, q)$.*

Proof. Let $x \in \ell_M(\Delta^m, p, q)$. Then there exists some $\rho > 0$ such that

$$\sum_{k=1}^{\infty} \left[M \left(q \left(\frac{\Delta^m x_k}{\rho} \right) \right) \right]^{p_k} < \infty.$$

This implies that $M\left(q\left(\frac{\Delta^m x_k}{\rho}\right)\right) \leq 1$ for sufficiently large values of k, say $k \geq k_0$ for some fixed $k_0 \in \mathbb{N}$. Since $p_k \leq t_k$ for each $k \in \mathbb{N}$, we get

$$M \left(q \left(\frac{\Delta^m x_k}{\rho} \right) \right)^{t_k} \leq M \left(q \left(\frac{\Delta^m x_k}{\rho} \right) \right)^{p_k}$$

for all $k \geq k_0$ and therefore

$$\sum_{k \geq k_0} \left[M \left(q \left(\frac{\Delta^m x_k}{\rho} \right) \right) \right]^{t_k} \leq \sum_{k \geq k_0} \left[M \left(q \left(\frac{\Delta^m x_k}{\rho} \right) \right) \right]^{p_k} < \infty.$$

Hence $x \in \ell_M(\Delta^m, t, q)$. □

The following result is a consequence of Theorem 7.

COROLLARY 8.

(i) *If $0 < p_k \leq 1$ for each $k \in \mathbb{N}$, then $\ell_M(\Delta^m, p, q) \subseteq \ell_M(\Delta^m, q)$.*

(ii) *If $p_k \geq 1$ for all $k \in \mathbb{N}$, then $\ell_M(\Delta^m, q) \subseteq \ell_M(\Delta^m, p, q)$.*

PROPOSITION 9. *For any two sequences $p = (p_k)$ and $t = (t_k)$ of strictly positive real numbers and any two seminorms q_1 and q_2 we have $\ell_M(\Delta^m, p, q_1, r) \cap \ell_M(\Delta^n, t, q_2, s) \neq \emptyset$ for all $m, n \in \mathbb{N}$ and $r > 0$, $s > 0$.*

Remark 4. In general it becomes difficult to predict about the intersection relation in the above result. For this consider the following examples.

Example 5. Let $X = \mathbb{C}$, $M(x) = x^p$ for $p \geq 1$, $q_1(x) = |x| = q_2(x)$, $n = 0$, $m > 0$, $r > 2$, $s = 0$. Let $p_k = 1$ and $t_k = 2 + k^{-1}$ for all $k \in \mathbb{N}$. Then the sequence $(x_k) = (k^m)$ belongs to $\ell_M(\Delta^m, p, q_1, r)$, but does not belong to $\ell_M = \ell_M(\Delta^n, t, q_2, s)$.

Example 6. Let $X = c_0$, $m = n = 0$, $p_k = t_k = 1$ for all $k \in \mathbb{N}$, $r = s = 0$ and $M(x) = x$. Let the sequence $x = \left(x^{(k)}\right)$ where $x^{(k)} \in c_0$ for all $k \in \mathbb{N}$ be defined by $x^{(1)} = x^{(2)} = (1, 1, 0, 0, \dots)$ and for $k > 2$, let $x^{(k)} = \left(x_i^{(k)}\right) = \left(1, 1, 0, \dots, 0, k^{-2}, 0, 0, \dots\right)$ where the k^{-2} appears at the kth place. Consider the seminorms

$$q_1\left(\left(x_i^{(k)}\right)\right) = \sup_{i \in \mathbb{N}} |x_i^{(k)}| \qquad \text{and} \qquad q_2\left(\left(x_i^{(k)}\right)\right) = |x_1^{(k)} - x_2^{(k)}| + \sup_{i > 2} |x_i^{(k)}|.$$

Then $q_1\left(\left(x^{(k)}\right)\right)=1$ for all $k\in\mathbb{N}$ and $q_2\left(\left(x^{(k)}\right)\right)=k^{-2}$ for all $k\in\mathbb{N}$. Thus $\left(x^{(k)}\right)\in\ell_M(q_1)$, but $\left(x^{(k)}\right)\notin\ell_M(q_2)$.

Theorem 10. *The sequence space $\ell_M(p,q,s)$ is solid.*

Proof. Let $(x_k)\in\ell_M(p,q,s)$, i.e.

$$\sum_{k=1}^{\infty}k^{-s}\left[M\left(q\left(\frac{x_k}{\rho}\right)\right)\right]^{p_k}<\infty.$$

Let (α_k) be sequence of scalars such that $|\alpha_k|\le 1$ for all $k\in\mathbb{N}$. Then the result follows from the following inequality

$$\sum_{k=1}^{\infty}k^{-s}\left[M\left(q\left(\frac{\alpha_k x_k}{\rho}\right)\right)\right]^{p_k}\le\sum_{k=1}^{\infty}k^{-s}\left[M\left(q\left(\frac{x_k}{\rho}\right)\right)\right]^{p_k}.$$

□

Remark 5. In general it is difficult to predict about the solidity of $\ell_M(\Delta^m,p,q,s)$ when $m>0$. For this consider the following example.

Example 7. Let $m=1$ and $p_k=1$ for all $k\in\mathbb{N}$, $M(x)=x$ and $q(x)=|x|$ for all $x\in\mathbb{C}$ and $s=0$. Then $(x_k)=\left(k^{-1}\right)\in\ell_M(\Delta)$ but $(\alpha_k x_k)\notin\ell_M(\Delta)$ when $\alpha_k=(-1)^k$ for all $k\in\mathbb{N}$. Hence $\ell_M(\Delta)$ is not solid.

We have the following result in view of Remark 2 and Theorem 10.

Corollary 11. *The sequence space $\ell_M(p,q,s)$ is monotone.*

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* *Mathematical Sciences Division*
Institue of Advanced Study
in Science and Technology
Paschim Baragoan, Garchuk
Guwahati-781 035
INDIA
E-mail: tripathybc@yahoo.com
tripathybc@rediffmail.com

** *Department of Mathematics*
Firat University
23119, Elazig
TURKEY
E-mail: yaltin23@yahoo.com
mikailet@yahoo.com