VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-008-0074-3
Math. Slovaca **58** (2008), No. 3, 289–300

SEQUENTIAL CONVERGENCES ON MV-ALGEBRAS WITHOUT URYSOHN'S AXIOM

Ján Jakubík

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. In a previous author's paper, sequential convergences on an MV-algebra $\mathscr{A}$ have been studied; the Urysohn's axiom was assumed to be valid. The system of all such convergences was denoted by $\operatorname{Conv}\mathscr{A}$. In the present paper we investigate analogous questions without supposing the validity of the Urysohn's axiom; the corresponding system of convergences is denoted by $\operatorname{conv}\mathscr{A}$. Both $\operatorname{Conv}\mathscr{A}$ and $\operatorname{conv}\mathscr{A}$ are partially ordered by the set-theoretical inclusion. We deal with the properties of $\operatorname{conv}\mathscr{A}$ and the relations between $\operatorname{conv}\mathscr{A}$ and $\operatorname{Conv}\mathscr{A}$. We prove that each interval of $\operatorname{conv}\mathscr{A}$ is a distributive lattice. The system $\operatorname{conv}\mathscr{A}$ has the least element, but it does not possess any atom. Hence it is either a singleton set or it is infinite. We consider also the relations between $\operatorname{conv}\mathscr{A}$ and $\operatorname{conv}G$, where (G,u) is a unital lattice-ordered group with $\mathscr{A}=\Gamma(G,u)$.

1. Introduction

The present paper can be considered as a continuation of the article [7]. In [7], we studied sequential convergences on an MV-algebra $\mathscr{A}$ under the assumption that the Urysohn's axiom for the convergences under consideration is valid. The collection of such convergences was denoted by $\operatorname{Conv}\mathscr{A}$.

Now we will deal with sequential convergences on $\mathscr{A}$ without the Urysohn's axiom. The corresponding system of convergences is denoted by $\operatorname{conv}\mathscr{A}$. Both $\operatorname{Conv}\mathscr{A}$ and $\operatorname{conv}\mathscr{A}$ are partially ordered by the set-theoretical inclusion.

2000 Mathematics Subject Classification: Primary 06D35, 06F20.
Keywords: MV-algebra, sequential convergence, Urysohn's axiom, unital lattice-ordered group.
Supported by VEGA agency grant 1/2002/05.
This work has been partially supported by the Slovak Academy of Sciences via the project Center of Excellence – Physics of Information, Grant I/2/2005.

The well-known notion of o-convergence was studied for several types of lattice-ordered structures (cf., e.g. [1]). We remark that the o-convergence on an MV-algebra $\mathscr{A}$ belongs to $\operatorname{conv}\mathscr{A}$, but need not belong, in general, to the system $\operatorname{Conv}\mathscr{A}$.

For a lattice-ordered group G, the definitions of $\operatorname{Conv} G$ and of $\operatorname{conv} G$ are analogous to those of $\operatorname{Conv}\mathscr{A}$ and $\operatorname{conv}\mathscr{A}$. The system $\operatorname{Conv} G$ was studied in several papers; cf. e.g., [3], [4], [5]. The system $\operatorname{conv} G$ was dealt with in [6].

It is well-known (cf. e.g., [2]) that for each MV-algebra $\mathscr{A}$ there exists an abelian lattice-ordered group G with a strong unit u such that $\mathscr{A}$ is the interval $[0,u]$ of G; in this situation we write $\mathscr{A} = \Gamma(G,u)$. (For a more detailed formulation of this result cf. Section 2 below.)

We will investigate the relations between $\operatorname{Conv}\mathscr{A}$ and $\operatorname{conv}\mathscr{A}$, and the relations between $\operatorname{conv}\mathscr{A}$ and $\operatorname{conv} G$, where $\mathscr{A}$ and G are as above.

We prove that each interval of $\operatorname{conv}\mathscr{A}$ is a Brouwerian lattice. The system $\operatorname{conv}\mathscr{A}$ has the least element, but it does not possess any atom. Hence it is either a one-element set or it is infinite.

Though $\operatorname{Conv}\mathscr{A}$ and $\operatorname{conv}\mathscr{A}$ can be strongly different (e.g., it can happen that $\operatorname{Conv}\mathscr{A}$ is finite and $\operatorname{conv}\mathscr{A}$ is infinite), several methods used in [7] by studying the relations between $\operatorname{Conv}\mathscr{A}$ and $\operatorname{Conv} G$ can be applied without change or with minor modifications by the investigation of the relations between $\operatorname{conv}\mathscr{A}$ and $\operatorname{conv} G$.

Our investigation of systems of sequential convergences on a fixed MV-algebra is to a certain degree analogous to the investigation of systems of topologies on a given set which was performed in several papers. Let us remark that Section 54A10 of 2000 Mathematics Subject Classification is entitled "Several topologies on one set".

2. Preliminaries

An MV-algebra $\mathscr{A} = (A; \oplus, *, \neg, 0, 1)$ is defined to be an algebraic structure of type $(2,2,1,0,0)$ with the underlying set A such that the conditions (m_1)–(m_9) from [7] are satisfied.

For each x and y from A we set

$$x \vee y = (x * \neg y) \oplus y, \quad x \wedge y = \neg(\neg x \vee \neg y).$$

Then $\mathscr{L}(\mathscr{A}) = (A; \vee, \wedge)$ is a distributive lattice with the least element 0 and the greatest element 1. The corresponding partial order on A will be denoted by $\leqq$.

For lattice-ordered groups we apply the notation and the terminology as in [1].

An element u of a lattice-ordered group G is a *strong unit* of G if for each $g \in G$ there is $n \in \mathbb{N}$ with $g \leqq nu$. A lattice-ordered group with a fixed strong unit is called unital.

Let (G, u) be a unital abelian lattice-ordered group and let A be the interval $[0, u]$ of G. For each $x, y \in A$ we set

$$x \oplus y = (x + y) \wedge u, \quad \neg x = u - x, \quad 1 = u,$$
$$x * y = \neg(\neg x \oplus \neg y).$$

Then $\mathscr{A} = (A; \oplus, *, \neg, 0, 1)$ is an MV-algebra. We write $\mathscr{A} = \Gamma(G, u)$.

Conversely, for each MV-algebra $\mathscr{A}$ there exists a unital abelian lattice-ordered group (G, u) such that $\mathscr{A} = \Gamma(G, u)$. (Cf. [2].)

Let $\mathscr{A}$ be an MV-algebra. The notions of a sequence (x_n) in $\mathscr{A}$ and of a constant sequence $\operatorname{const} x$ have the usual meaning. If K is a subset of $A^{\mathbb{N}} \times A$ and $((x_n), x) \in K$, then we write $x_n \to_K x$.

For the sake of completeness, we recall the following conditions concerning a subset $K \subseteq A^{\mathbb{N}} \times A$ which will be applied by dealing with sequential convergences (cf. [7]):

(s(i)) If $x_n \to_K x$ and (y_n) is a subsequence of (x_n), then $y_n \to_K x$.

(s(ii)) If $(x_n) \in A^{\mathbb{N}}$ and if for each subsequence (y_n) of (x_n) there is a subsequence (z_n) of (y_n) such that $z_n \to_K x$, then $x_n \to_K x$.

(s(ii')) Let $(x_n), (y_n) \in A^{\mathbb{N}}$. Assume that there is $m \in \mathbb{N}$ such that $x_n = y_n$ for each $n \geqq m$. If $x_n \to_K x$, then $y_n \to_K x$.

(s(iii)) If $(x_n) \in A^{\mathbb{N}}$, $x \in A$, $(x_n) = \operatorname{const} x$, then $x_n \to_K x$.

(s(iv)) If $x_n \to_K x$ and $x_n \to_K y$, then $x = y$.

(s(v)) If $x_n \to_K x$ and $y_n \to_K y$, then $x_n \oplus y_n \to_K x \oplus y$, $x_n * y_n \to_K x * y$ and $\neg x_n \to \neg x$.

(s(vi)) If $x_n \leqq y_n \leqq z_n$ for each $n \in \mathbb{N}$ and $x_n \to_K x$, $z_n \to_K x$, then $y_n \to_K x$.

The condition (s(ii)) is the well-known Urysohn's axiom.

We denote by $\operatorname{Conv} \mathscr{A}$ the system of all $K \subseteq A^{\mathbb{N}} \times A$ satisfying the conditions (s(j)) for j = i–vi. Further, let $\operatorname{conv} \mathscr{A}$ be the system of all $K \subseteq A^{\mathbb{N}} \times A$ which satisfy the conditions (s(i)), (s(ii')) and (s(iii))–(s(vi)). Both $\operatorname{Conv} \mathscr{A}$ and $\operatorname{conv} \mathscr{A}$ are partially ordered by the set-theoretical inclusion. The elements of $\operatorname{conv} \mathscr{A}$ are called sequential convergences (or simply convergences) in $\mathscr{A}$. We obviously have $\operatorname{Conv} \mathscr{A} \subseteq \operatorname{conv} \mathscr{A}$.

Let $K(0)$ be the set consisting of all elements $((x_n), x)$ of $A^{\mathbb{N}} \times A$ such that there is $m \in \mathbb{N}$ with $x_n = x$ for each $n \geqq m$. Then $K(0)$ is the least element of $\operatorname{Conv} \mathscr{A}$. We say that $K(0)$ is the discrete convergence on $\mathscr{A}$.

We recall the notion of $\operatorname{Conv} G$ and $\operatorname{conv} G$ for a lattice ordered group G (cf. [6]). All lattice-ordered groups dealt with in the present paper are assumed to be abelian.

Let (g_n) and (h_n) be elements of $G^{\mathbb{N}}$. We set $(g_n) \sim (h_n)$ if there is $m \in \mathbb{N}$ such that $g_n = h_n$ for each $n \in \mathbb{N}$ with $n \geqq m$.

The positive cone $(G^{\mathbb{N}})^+$ of the lattice-ordered group $G^{\mathbb{N}}$ is a lattice ordered semigroup. We consider the following conditions for a subset α of $(G^{\mathbb{N}})^+$:

(I) If $(g_n) \in \alpha$, then each subsequence of (g_n) belongs to α.

(II) Let $(g_n) \in (G^{\mathbb{N}})^+$. If each subsequence of (g_n) has a subsequence belonging to α, then $(g_n) \in \alpha$.

(II') Let $(g_n) \in \alpha$ and $(h_n) \in (G^{\mathbb{N}})^+$. If $(h_n) \sim (g_n)$, then $(h_n) \in \alpha$.

(III) Let $g \in G$. Then $\operatorname{const} g$ belongs to α if and only if $g = 0$.

The system of all convex subsemigroups of $(G^{\mathbb{N}})^+$ which satisfy the conditions (I), (II) and (III) (or (I), (II') and (III)) will be denoted by $\operatorname{Conv} G$ (or by $\operatorname{conv} G$, respectively). Both $\operatorname{Conv} G$ and $\operatorname{conv} G$ are partially ordered by the set-theoretical inclusion. We have $\operatorname{Conv} G \subseteq \operatorname{conv} G$. Let $\alpha(0)$ be the set of all $(x_n) \in (G^{\mathbb{N}})^+$ such that $(x_n) \sim \operatorname{const} 0$. Then $\alpha(0)$ is the least element of both $\operatorname{Conv} G$ and $\operatorname{conv} G$; it is called the discrete convergence on G.

For $(g_n) \in G^{\mathbb{N}}$, $g \in G$ and $\alpha \in \operatorname{conv} G$ we put $g_n \to_\alpha g$ iff $(|g_n - g|) \in \alpha$.

We denote by $\alpha(o)$ the set of all sequences (g_n) in G having the property that there exists $(h_n) \in (G^{\mathbb{N}})^+$ such that

(i) $h_{n+1} \geqq h_n$ for each $n \in \mathbb{N}$;

(ii) $\bigwedge\limits_{n \in \mathbb{N}} h_n = 0$;

(iii) there is $m \in \mathbb{N}$ such that $h_n \geqq g_n$ for each $n \geqq m$.

Then $\alpha(o) \in \operatorname{conv} G$; on the other hand, $\alpha(o)$ need not belong to $\operatorname{Conv} G$ (cf. [6]). (From this it can be easily deduced that an analogous result holds for $\operatorname{Conv} \mathscr{A}$ and $\operatorname{conv} \mathscr{A}$.) We say that $\alpha(o)$ is the o-convergence in G.

3. The system $\operatorname{conv}_0 \mathscr{A}$

In this section we prove some auxiliary results which will be applied in the subsequent sections. As above, let $\mathscr{A}$ be an MV-algebra. If (g_n) and (h_n) belong to $A^{\mathbb{N}}$, then we define the meaning of $(f_n) \sim (h_n)$ analogously as we did for elements of $G^{\mathbb{N}}$ in Section 2.

For each $K \in \operatorname{conv} \mathscr{A}$ we set

$$K^0 = \{(x_n) \in A^{\mathbb{N}} : \ x_n \to_K 0\},$$
$$\operatorname{conv}_0 \mathscr{A} = \{K^0 : \ K \in \operatorname{conv} \mathscr{A}\}.$$

The direct power $\mathscr{A}^{\mathbb{N}}$ is defined in the usual way; its underlying set is $A^{\mathbb{N}}$.

From the definition of $\operatorname{conv} \mathscr{A}$ we obtain:

Lemma 3.1. *Let $K^1 \in \operatorname{conv}_0 \mathscr{A}$. Then the following conditions are satisfied:*

(i_1) *If $(g_n) \in K^1$, then each subsequence of (g_n) belongs to K^1.*

(ii_1) *Let $(g_n) \in K^1$ and $(h_n) \in A^{\mathbb{N}}$. If $(g_n) \sim (h_n)$, then $(h_n) \in K^1$.*

(iii_1) *Let $g \in A$. Then* $\operatorname{const} g \in K^1$ *if and only if $f = 0$.*

(iv_1) *K^1 is a convex subset of $A^{\mathbb{N}}$.*

(v_1) *K^1 is closed with respect to the operation $\oplus$.*

Let $\emptyset \neq K(1) \subseteq A^{\mathbb{N}}$. Consider the following conditions for $((x_n), x) \in A^{\mathbb{N}} \times A$:

($*$) There exist $(u_n), (v_n) \in A^{\mathbb{N}}$ such that
 (i) $u_n \leqq x_n$, $u_n \leqq x$ for each $n \in \mathbb{N}$ and $(\neg(u_n \oplus \neg x)) \in K(1)$;
 (ii) $v_n \geqq x_n$, $v_n \geqq x$ for each $n \in \mathbb{N}$ and $(\neg(x \oplus \neg v_n)) \in K(1)$.

We denote by $K(2)$ the set of all $((x_n), x) \in A^{\mathbb{N}} \times A$ such that the condition ($*$) is valid. If $((x_n), x) \in K(2)$, then we write $x_n \to_{K(2)} x$.

Lemma 3.2. *Let $K(1)$ and $K(2)$ be as above. Assume that $K(1)$ satisfies the condition* (ii_1)*. Then $K(2)$ satisfies the condition* (s(ii')) *from the definition of* $\operatorname{conv} \mathscr{A}$*.*

Proof. Let $((x_n), x) \in K(2)$ and $(y_n) \in A^{\mathbb{N}}$ such that $(y_n) \sim (x_n)$. Assume that (u_n) and (v_n) are as in ($*$). Put $u'_n = u_n \wedge y_n$ and $v'_n = v_n \vee y_n$ for each $n \in \mathbb{N}$. Hence we have

$$u'_n \leqq x, \quad u'_n \leqq y_n, \quad v'_n \geqq x, \quad v'_n \geqq y_n$$

for each $n \in \mathbb{N}$. Further,

$$(\neg(u'_n \oplus \neg x)) \sim (\neg(u_n \oplus \neg x)), \tag{1}$$
$$(\neg(x \oplus \neg v'_n)) \sim (\neg(x \oplus \neg v_n)), \tag{2}$$

thus both $(\neg(u'_n \oplus \neg x))$ and $(\neg(x \oplus \neg v'_n))$ belong to $K(1)$. Therefore $((y_n), x) \in K(2)$. $\square$

Lemma 3.3. *Assume that $K(1)$ satisfies the conditions from* 3.1*. Let $K(2)$ be as above. Then $K(2) \in \operatorname{conv} \mathscr{A}$.*

P r o o f. We have to verify that $K(2)$ satisfies the conditions (s(i)), (s(ii')), (s(iii))–(s(vi)) from the definition of $\operatorname{conv} \mathscr{A}$.

In view of 3.2, the condition (s(ii')) is satisfied. For the verification of the remaining conditions it suffices to apply the same method as in the proof of [7, 2.5]. □

Lemma 3.4. *Let $K(1)$ and $K(2)$ be as above. Then $(K(2))^0 = K(1)$.*

P r o o f. It suffices to use the same steps as in the proof of [7, Lemma 2.6]. □

From 3.1, 3.2 and 3.3 we obtain:

Lemma 3.5. *Let $K \in A^{\mathbb{N}}$. Then K belongs to $\operatorname{conv}_0 \mathscr{A}$ if and only if it satisfies the conditions from* 3.1*.*

Let $K \in \operatorname{conv} \mathscr{A}$ and $K(1) \in \operatorname{conv}_0 \mathscr{A}$. We put

$$f_1(K) = K^0, \qquad f_2(K(1)) = K(2),$$

where $K(2)$ is as above.

By the same method as in the proof of [7, Lemma 2.8] we obtain the relation $f_2(K^0) = K$ for each $K \in \operatorname{conv} \mathscr{A}$. Then according to 3.1–3.4 and in view of the fact that f_1 and f_2 are isotone we get:

Theorem 3.6. *The mapping f_2 is an isomorphism of the partially ordered set $\operatorname{conv}_0 \mathscr{A}$ onto $\operatorname{conv} \mathscr{A}$ and $f_1 = f_2^{-1}$.*

For the analogous result concerning $\operatorname{Conv}_0 \mathscr{A}$ and $\operatorname{Conv} \mathscr{A}$ cf. [7, Theorem 2.9].

Let $X \subseteq A^{\mathbb{N}}$. We denote by X^* the set of all sequences $(a_n) \in A^{\mathbb{N}}$ such that each subsequence of (a_n) has a subsequence belonging to X.

For an injective monotone mapping $\varphi\colon \mathbb{N} \to \mathbb{N}$ and a sequence (a_n) in A put $b_n = a_{\varphi(n)}$. Then (b_n) is a subsequence of (a_n); we say that (b_n) is generated by the mapping φ.

For $K_1 \in \operatorname{Conv} \mathscr{A}$ we set

$$K_1^0 = \bigl\{(x_n) \in A^{\mathbb{N}} :\ x_n \to_{K_1} 0\bigr\},$$
$$\operatorname{Conv}_0 \mathscr{A} = \bigl\{K_1^0 :\ K_1 \in \operatorname{Conv} \mathscr{A}\bigr\}.$$

According to [7, Corollary 2.7] we have:

Lemma 3.7. *The collection* $\operatorname{Conv}_0 \mathscr{A}$ *is the system of all subsets* K_1 *of* $A^{\mathbb{N}}$ *which satisfy the following conditions:*

(i) *If* $(g_n) \in K_1$*, then each subsequence of* (g_n) *belongs to* K_1*.*

(ii) K_1 *fulfils the Urysohn's condition.*

(iii) *Let* $g \in A$*. Then* $\operatorname{const} g \in K_1$ *iff* $g = 0$*.*

(iv) K_1 *is a convex subset of the lattice* $(A^{\mathbb{N}}, \vee, \wedge)$*.*

(v) K_1 *is closed with respect to the operation* $\oplus$*.*

Proposition 3.8. *Let* $K \in \operatorname{conv}_0 \mathscr{A}$*. Then* $K^* \in \operatorname{Conv}_0 \mathscr{A}$*. If* $K_1 \in \operatorname{Conv}_0 \mathscr{A}$ *and* $K_1 \geqq K$*, then* $K_1 \geqq K^*$*.*

Proof. For proving the relation $K^* \in \operatorname{Conv}_0 \mathscr{A}$ we have to verify that K^* satisfies the conditions (i)–(v) from 3.7. The validity of (i) and (ii) is a consequence of the properties of K. In view of the definition of K^*, (iii) is valid.

Assume that $(x_n) \in K^*$, $(y_n) \in A^{\mathbb{N}}$ and $y_n \leqq x_n$ for each $n \in \mathbb{N}$. Let (z_n) be a subsequence of (y_n); hence there exists an injective monotone mapping $\varphi\colon \mathbb{N} \to \mathbb{N}$ such that $z_n = y_{\varphi(n)}$ for each $n \in \mathbb{N}$. Put $z'_n = x_{\varphi(n)}$ for each $n \in \mathbb{N}$. Then (z'_n) is a subsequence of (x_n) and $z_n \leqq z'_n$ for each $n \in \mathbb{N}$. Thus $(z_n) \in K$. This yields that (y_n) belongs to K^*. Hence condition (iv) is satisfied.

The validity of (v) can be verified by applying the same idea we have just used for the condition (iv).

Assume that $K_1 \in \operatorname{Conv}_0 \mathscr{A}$ and $K_1 \supseteq K$. Since K_1 satisfies the Urysohn's axiom, we conclude that $K^* \subseteq K_1$. $\square$

A subset X of $A^{\mathbb{N}}$ is said to be regular with respect to $\operatorname{conv}_0 \mathscr{A}$ if there exists $K \in \operatorname{conv}_0 \mathscr{A}$ with $X \subseteq K$. The regularity with respect to $\operatorname{Conv}_0 \mathscr{A}$ is defined analogously.

Lemma 3.9. *Let* $X \in A^{\mathbb{N}}$*. Then* X *is regular with respect to* $\operatorname{conv}_0 \mathscr{A}$ *iff it is regular with respect to* $\operatorname{Conv}_0 \mathscr{A}$*.*

Proof. Let X be regular with respect to $\operatorname{conv}_0 \mathscr{A}$. Then in view of 3.8, it is regular with respect to $\operatorname{Conv}_0 \mathscr{A}$, since $K \subseteq K^*$ for each $K \in \operatorname{conv}_0 \mathscr{A}$. The converse assertion follows from the relation $\operatorname{Conv}_0 \mathscr{A} \subseteq \operatorname{conv}_0 \mathscr{A}$. $\square$

In view of 3.9 we say simply "regular" instead of "regular with respect to $\operatorname{conv}_0 \mathscr{A}$".

4. The relation between $\operatorname{conv}\mathscr{A}$ and $\operatorname{conv} G$

As above, let $\mathscr{A}$ be an MV-algebra and let (G, u) be a unital lattice-ordered group such that $\mathscr{A} = \Gamma(G, u)$.

The relation between $\operatorname{Conv} G$ and $\operatorname{Conv}\mathscr{A}$ was investigated in [7]. In the present section we deal with the relation between $\operatorname{conv} G$ and $\operatorname{conv}\mathscr{A}$. As we have already remarked in Section 1, we can apply some methods from [7].

Our present notation is slightly different from that used in [7]. Namely, instead of $\operatorname{Conv} G$ (in the sense defined in Section 2 above), the symbol $\operatorname{Conv}_0 G$ was applied in [7]. Our notation is in accordance with that of [6].

For $K \in \operatorname{conv} G$ we set

$$g_1(K) = A^{\mathbb{N}} \cap K.$$

If (x_n) and (y_n) are elements of K, then $(x_n \vee y_n)$, $(x_n \wedge y_n)$ and $(x_n + y_n)$ also belong to K. Then from the properties of $\operatorname{conv} G$ and from 3.7 we obtain:

Lemma 4.1. *For each $K \in \operatorname{conv} G$, $g_1(K) \in \operatorname{conv}_0 \mathscr{A}$.*

According to 3.9 and by applying the same method as in [7] (cf. [7, Lemma 3.8]) we get:

Lemma 4.2. *Let $X \in \operatorname{conv}_0 \mathscr{A}$. Then the set X is regular.*

For $X \in \operatorname{conv}_0 \mathscr{A}$ let $\overline{Y} = \{Y_i\}_{i\in I}$ be the system of all elements Y_i of $\operatorname{conv} G$ such that $X \subseteq Y_i$. In view of 4.2, $\overline{Y} \neq \emptyset$. Put

$$g_2(X) = \bigcap_{i\in I} Y_i.$$

From the definition of $\operatorname{conv} G$ we conclude that $\bigcap\limits_{i\in I} Y_i$ belongs to $\operatorname{conv} G$. Hence g_2 is a mapping of $\operatorname{conv}_0 \mathscr{A}$ into $\operatorname{conv} G$.

A more constructive description of $g_2(X)$ can be given as follows.

For each $\emptyset \neq Z \subseteq (G^{\mathbb{N}})^+$ we denote by

δZ — the set of all $(g_n) \in (G^{\mathbb{N}})^+$ such that (g_n) is a subsequence of some sequence belonging to Z;

$\langle Z\rangle$ — the set of all $(g_n) \in (G^{\mathbb{N}})^+$ such that there exist $k \in \mathbb{N}$ and $(g_n^1), \dots, (g_n^k) \in Z$ such that $g_n \leqq g_n^1 + \dots + g_n^k$ for each $n \in \mathbb{N}$;

Z^0 — the set of all $(g_n) \in (G^{\mathbb{N}})^+$ such that there exists $(h_n) \in Z$ with $(g_n) \sim (h_n)$.

From 4.2 and from [6, Proposition 2.3] we conclude:

Lemma 4.3. *Let $X \in \operatorname{conv}_0 \mathscr{A}$. Then $g_2(X) = \langle \delta X\rangle^0$.*

We remark that in [6, Proposition 2.3] the symbol $[Z]$ was also used; it is easy to verify that this symbol can be omitted.

Lemma 4.4. *Let $X_1, X_2 \in \operatorname{conv}_0 \mathscr{A}$. If $X_1 \subseteq X_2$, then $g_2(X_1) \subseteq g_2(X_2)$. If $X_1 \nsubseteq X_2$, then $g_2(X_1) \nsubseteq g_2(X_2)$.*

Proof. The first assertion is obvious. Let $X_1 \nsubseteq X_2$. By way of contradiction, assume that $g_2(X_1) \subseteq g_2(X_2)$. There exists $(x_n) \in X_1 \setminus X_2$. Since $X_1 \subseteq g_2(X_1)$, we get $(x_n) \in g_2(X_2)$. Hence according to 4.3, $(x_n) \in \langle \delta X_2 \rangle^0$.

From $X_2 \in \operatorname{conv}_0 \mathscr{A}$ we infer that $\delta X_2 = X_2$. Thus $(x_n) \in \langle X_2 \rangle^0$. Then there exists $(z_n) \in \langle X_2 \rangle$ such that $(z_n) \sim (x_n)$.

Further, there are $k \in \mathbb{N}$ and $(t_n^1), \ldots, (t_n^k) \in X_2$ such that

$$z_n \leqq t_n^1 + \cdots + t_n^k \quad \text{for each } n \in \mathbb{N}.$$

In view of $z_n \in A$ we get

$$z_n \leqq (t_n^1 + \cdots + t_n^k) \wedge u = t_n^1 \oplus \cdots \oplus t_n^k.$$

From this and from the relation $X_2 \in \operatorname{conv}_0 \mathscr{A}$ we obtain $(z_n) \in X_2$; finally, from $(z_n) \sim (x_n)$ we conclude that $(x_n) \in X_2$. We arrived at a contradiction. □

A sequence (x_n) of elements of G is *bounded* if there are $v_1, v_2 \in G$ with $v_1 \leqq v_2$ such that $v_1 \leqq x_n \leqq v_2$ for each $n \in \mathbb{N}$. It is clear that (x_n) is bounded iff there is $m \in \mathbb{N}$ such that $-mu \leqq x_n \leqq mu$ for each $m \in \mathbb{N}$.

For $K \in \operatorname{conv} G$ let K^b be the set of all $(x_n) \in K$ such that (x_n) is a bounded sequence in G. We put

$$\operatorname{conv}^b G = \{K \in \operatorname{conv} G : \ K = K^b\}.$$

If $K \in \operatorname{conv}^b G$ then we say that K is bounded. In view of the definition of $\operatorname{conv} G$, we have $K^b \in \operatorname{conv} G$ for each $K \in \operatorname{conv} G$. Hence $\operatorname{conv}^b G \subseteq \operatorname{conv} G$. Further,

$$g_1(K) = g_1(K^b) \qquad \text{for each} \quad K \in \operatorname{conv} G.$$

Lemma 4.5. *Let $X \in \operatorname{conv}_0 \mathscr{A}$. Then $g_2(X)$ is bounded.*

Proof. This is a consequence of 4.3. □

Lemma 4.6. *Let $Y \in \operatorname{conv}^b G$. Put $g_1(Y) = X$. Then $g_2(X) = Y$.*

Proof. In view of $g_1(Y) = X$ we have $X \subseteq Y$. Then according to 4.3 we obtain $g_2(X) = \langle \delta X \rangle^0 \subseteq \langle \delta Y \rangle^0$. The relation $Y \in \operatorname{conv}^b G$ yields $\langle \delta Y \rangle^0 = Y$, hence $g_2(X) \subseteq Y$.

Let $(y_n) \in Y$. Then $y_n \geqq 0$ for each $n \in \mathbb{N}$. Further, (y_n) is bounded. Hence there is $m \in \mathbb{N}$ such that $y_n \leqq mu$ for each $n \in \mathbb{N}$. Therefore, according to Riesz Theorem, for each $n \in \mathbb{N}$ there exist elements z_n^i $(i = 1, 2, \ldots, m)$ such that $0 \leqq z_n^i \leqq u$ for $i = 1, 2, \ldots, m$ and $y_n = z_n^1 + \cdots + z_n^m$. Then $z_n^i \leqq y_n$ for each $n \in \mathbb{N}$ and each $i \in \{1, 2, \ldots, m\}$. We obtain $(z_n^i) \in A^{\mathbb{N}}$ and $(z_n^i) \in Y$, hence $(z_n)^i \in X$ for $i = 1, 2, \ldots, m$. This yields $(y_n) \in \langle X \rangle \subseteq g_2(X)$. Thus $Y \subseteq g_2(X)$. Summarizing, we obtain $g_2(X) = Y$. □

Corollary 4.7. $g_2(\operatorname{conv}_0 \mathscr{A}) = \operatorname{conv}^b G$.

Lemma 4.8. *Let $X \in \operatorname{conv}_0 \mathscr{A}$. Put $g_2(X) = Y$. Then $g_1(Y) = X$.*

Proof. We have $g_2(X) = \langle \delta X \rangle^0 \supseteq X$. Hence $Y \supseteq X$. Further, $g_1(Y) = Y \cap A^{\mathbb{N}} \supseteq X \cap A^{\mathbb{N}} = X$. Thus $g_1(Y) \supseteq X$.

Let $(z_n) \in g_1(Y)$. Since $X \in \operatorname{conv}_0 \mathscr{A}$, $\delta X = X$ and $Y = \langle X \rangle^0$. Also, $(z_n) \in Y$ and $(z_n) \in A^{\mathbb{N}}$. Thus there exists $(v_n) \in \langle X \rangle$ with $(v_n) \sim (z_n)$. The first relation yields that there is $m \in \mathbb{N}$ having the property that for each $n \in \mathbb{N}$ there exist $x_n^1, \dots, x_n^m$ with $v_n = x_n^1 + \dots + x_n^m$ and such that $(x_n^1), \dots, (x_n^m)$ belong to X. Further, there is $k \in \mathbb{N}$ such that for each $n \geqq k$, $v_n = x_n$, hence

$$z_n = x_n^1 + \dots + x_n^m. \tag{3}$$

In view of (3), $x_n^i \leqq z_n$ for $i = 1, \dots, m$. Thus, $x_n^1, \dots, x_n^m \in A$ for each $n \in \mathbb{N}$. From this and from the relation (3) we obtain

$$z_n = x_n^1 \oplus \dots \oplus x_n^m \qquad \text{for} \quad n \geqq k.$$

In view of the properties of elements of $\operatorname{conv}_0 \mathscr{A}$ we conclude that $(x_n) \in X$, hence $g_1(Y) \subseteq X$. □

Corollary 4.9. $g_1(\operatorname{conv}^b G) = \operatorname{conv}_0 \mathscr{A}$.

Theorem 4.10. *The mapping g_2 is an isomorphism of the partially ordered set $\operatorname{conv}_0 \mathscr{A}$ onto the partially ordered set $\operatorname{conv}^b G$ and $g_1 = g_2^{-1}$. The partially ordered set $\operatorname{conv} \mathscr{A}$ is isomorphic to the partially ordered set $\operatorname{conv}^b G$.*

Proof. The first assertion is a consequence of 4.1, 4.4, 4.6 and 4.9. Applying 3.6 we conclude that the second assertion is valid. □

It is obvious that

a) the least element $\alpha(0)$ of $\operatorname{conv} G$ belongs to $\operatorname{conv}^b G$;

b) if $\alpha \in \operatorname{conv}^b G$, then the interval $[\alpha(0), \alpha]$ of $\operatorname{conv} G$ is a subset of $\operatorname{conv}^b G$.

From this and from [6, Corollary 6.3] we infer:

Theorem 4.11. *Each interval of $\operatorname{conv} \mathscr{A}$ is a Brouwerian lattice.*

If $\{X_i\}_{i \in I}$ is a nonempty subset of $\operatorname{conv}_0 \mathscr{A}$, then we obviously have $\bigcap\limits_{i \in I} X_i \in \operatorname{conv}_0 \mathscr{A}$, hence

$$\bigcap_{i \in I} X_i = \bigwedge_{i \in I} X_i$$

in the lattice $\operatorname{conv}_0 \mathscr{A}$. Thus each interval of $\operatorname{conv}_0 \mathscr{A}$ is a complete lattice. The same holds for $\operatorname{conv} \mathscr{A}$.

5. On the power of the system $\operatorname{conv} \mathscr{A}$

As above, let $\mathscr{A}$ be an MV-algebra with $\mathscr{A} = \Gamma(G, u)$, where (G, u) is a unital lattice-ordered group.

Theorem 5.1. *The lattice* $\operatorname{conv} \mathscr{A}$ *has no atom.*

Proof. By way of contradiction, assume that $\operatorname{conv} \mathscr{A}$ has an atom. In view of $\operatorname{conv} \mathscr{A} \simeq \operatorname{conv}_0 \mathscr{A}$, we obtain that $\operatorname{conv}_0 \mathscr{A}$ possesses an atom X_0. Let g_2 be as in Section 6. According to 4.10, $g_2(X_2)$ is an atom of $\operatorname{conv}^b G$. Now the remark b) after 4.10 yields that $g_2(X_0)$ is, at the same time, an atom in $\operatorname{conv} G$. But according to [6, Theorem 5.13], $\operatorname{conv} G$ has no atom. We arrived at a contradiction. □

Since $\operatorname{conv} G$ has the least element, from 5.1 we conclude:

Corollary 5.2. *Either* $\operatorname{conv} \mathscr{A}$ *is a one-element set, or it is infinite.*

Example 5.3. Let Z be the additive group of all reals with the natural linear order. Let $0 < m \in Z$. Then m is a strong unit of Z and we can construct the MV-algebra $\mathscr{A}_m = \Gamma(Z, m)$. It is easy to verify that $\operatorname{conv} Z$ is the one-element set $\{\alpha(0)\}$; hence $\operatorname{Conv} Z = \operatorname{conv} Z$. The set A_m (the underlying set of $\mathscr{A}_m$) is finite; $\operatorname{conv} \mathscr{A}_m$, $\operatorname{conv}_0 \mathscr{A}_m$ and $\operatorname{Conv} \mathscr{A}_m$ are one-element sets.

Example 5.4. Let R be the additive group of all reals with the natural linear order. Then $\operatorname{Conv} R$ has exactly two elements, namely, the discrete convergence and the o-convergence. (Cf. [3].) Both these convergence are bounded.

Let $0 < r \in R$. The element r is a strong unit of R; put $\mathscr{A}_r = \Gamma(A, r)$. Then according to [7], $\operatorname{Conv} \mathscr{A}_r$ has exactly two elements. This yields that $\operatorname{conv} \mathscr{A}_r$ fails to be a one-element set. Therefore, in view of 5.2, $\operatorname{conv} \mathscr{A}_r$ is infinite.

REFERENCES

[1] BIRKHOFF, G.: *Lattice Theory* (3rd ed.). Amer. Math. Soc. Colloq. Publ. 25, Amer. Math. Soc., Providence, R.I., 1967.

[2] CIGNOLI, R.—D'OTTAVIANO, I. M. L.—MUNDICI, D.: *Algebraic Foundations of Many Valued Reasoning*, Kluwer Academic Publ., Dordrecht, 2000.

[3] HARMINC, M.: *The cardinality of the system of all convergences on an abelian lattice-ordered group*, Czechoslovak Math. J. **37** (1987), 533–546.

[4] HARMINC, M.: *Sequential convergences on lattice-ordered groups*, Czechoslovak Math. J. **39** (1989), 232–238.

[5] JAKUBÍK, J.: *Lattice-ordered groups having a largest convergence*, Czechoslovak Math. J. **39** (1989), 717–729.

[6] JAKUBÍK, J.: *Sequential convergences in ℓ-groups without Urysohn's xiom*, Czechoslovak Math. J. **42** (1992), 101–116.

[7] JAKUBÍK, J.: *Sequential convergences on MV-algebras*, Czechoslovak Math. J. **45** (1995), 709–726.

Received 2. 1. 2006
Revised 3. 10. 2006

Matematický ústav SAV
Grešákova 6
SK–040 01 Košice
SLOVAKIA
E-mail: kstefan@saske.sk