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WEAK MV-ALGEBRAS

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ABSTRACT. In a recent paper [CHAJDA, I.—KÜHR, J.: *A non-associative generalization of MV-algebras*, Math. Slovaca **57**, (2007), 301–312], authors introduced and studied a non-associative generalization of MV-algebras called NMV-algebras. In contrast to MV-algebras, sections (i.e. principal filters) in NMV-algebras which are proper (i.e. are not MV-algebras), do not admit a structure of an NMV-algebra with respect to the operations defined in a natural way. The aim of the paper is to present a new class of algebras generalizing MV-algebras but sharing the above property.

1. Introduction

Recall that MV-algebras were introduced in 50'ties by C. C. Chang as an algebraic semantics of the Łukasiewicz many valued propositional logic (see [6], [7]). More precisely, an MV-algebra is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities:
(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
(MV2) $x \oplus y = y \oplus x$
(MV3) $x \oplus 0 = x$
(MV4) $\neg\neg x = x$
(MV5) $x \oplus 1 = 1$ (where $1 := \neg 0$)
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

A typical example of an MV-algebra can be obtained as follows: consider any abelian lattice ordered group $(G, +, -, 0, \wedge, \vee)$ and take $0 < u \in G$. Then the interval $[0, u] = \{x \in G : \ 0 \leq x \leq u\}$ with the operations defined by

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$x \oplus y := (x + y) \wedge u$ and $\neg x := u - x$ becomes an MV-algebra. Denoting such an MV-algebra as $\Gamma(G, u)$, D. Mundici [15] (see also [8]) proved that for every MV-algebra $\mathscr{A}$ there exists an abelian l-group G and $0 < u \in G$ with $\mathscr{A} \cong \Gamma(G, u)$.

Given an MV-algebra $\mathscr{A}$, the relation $\leq$ defined by

(1) $x \leq y \iff \neg x \oplus y = 1,$

is known to be a lattice order on A with $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$, the top or the bottom element of which is 1 or 0, respectively.

Moreover, for any MV-algebra $\mathscr{A}$ and $p \in A$, one can define on the interval $[p, 1]$ (usually called *section*) a structure of an MV-algebra (called *section MV-algebra* on $[p, 1]$) in a natural way as follows:

(2) $x \oplus_p y = \neg(\neg x \oplus p) \oplus y, \qquad \neg_p x = \neg x \oplus p.$

In the recent years a non-commutative generalization of MV-algebras was introduced and studied by G. Georgescu and A. Iorgulescu [11] as pseudo MV-algebras and independently by J. Rachůnek under the name GMV-algebras ([16]).

In principle, these are algebras with a binary operation $\oplus$ and two unary operations $\sim$ and $\neg$ (negations) coinciding whenever $\oplus$ is commutative. More precisely, given any (not necessarily commutative) l-group G and $0 < u \in G$, then upon defining $x \oplus y := (x + y) \wedge u$, $\neg x := u - x$, $\sim x = -x + u$, the resulting algebra $\Gamma(G, u) = \big([0, u], \oplus, \neg, \sim, 0\big)$ becomes a GMV-algebra.

Similarly as for a commutative case, A. Dvurečenskij [9] proved that all GMV-algebras are of the form $\Gamma(G, u)$ for any l-group G.

Another important approach to generalize MV-algebras by omitting associativity (MV1) but keeping commutativity (MV2) was done by I. Chajda and J. Kühr [3]. More precisely, they considered algebras $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the axioms (MV2)–(MV6), where the axiom (MV1) was substituted by two more axioms
(WA) $\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1$
(here so-called *weak associativity*)
and

(H) $\neg x \oplus (x \oplus y) = 1.$

These algebras are called *NMV-algebras* (non-associative MV-algebras) ([3]). Clearly, every MV-algebra fulfils the axioms (WA) and (H), but the converse is not true.

To clarify the role of the axiom (WA), its validity enables to prove that the relation $\leq$ defined by (1) remains transitive (and hence being the order relation). From a logical point of view, such a property is quite natural since in all reasonable logics the set of truth values should be partially ordered.

We have seen that the sections in MV-algebras form MV-algebras as given by (2). However, this property is not true for NMV-algebras: I. Chajda recently proved that given an NMV-algebra $\mathscr{A}$, the sections $[p,1]$ have a structure of an NMV-algebra as defined by (2) iff $\oplus$ is associative. In other words, an NMV-algebra shares the mentioned property iff it is an MV-algebra.

The aim of this paper is to find a new class of generalized MV-algebras admitting the same structure on sections.

2. Weak MV-algebras

Definition 1. An algebra $(A, \oplus, \neg, 0)$ is called a weak MV-algebra of type $(2,1,0)$ or *WMV-algebra* briefly if it satisfies the axioms ($1 := \neg 0$):
(WMV1) $\neg\neg x = x$
(WMV2) $\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1$
(WMV3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.
(WMV4) $x \oplus 0 = 0 \oplus x = x$
(WMV5) $\neg y \oplus (\neg x \oplus y) = 1$
(WMV6) $p \leq x \leq y \implies \neg y \oplus p \leq \neg x \oplus p$, where $\leq$ is defined by (1).

Note that applying (WMV4) and (WMV5), every WMV-algebra satisfies the identity

$$x \oplus 1 = 1 \oplus x = 1.$$

To show that the relation defined by (1) is a partial order on A for any WMV-algebra $\mathscr{A}$, we use the same arguments as in the case of NMV-algebras: putting $x = 0$ in (WMV5), with respect to (WMV4), we obtain the identity $\neg y \oplus y = 1$, so $\leq$ is reflexive. The antisymmetry of $\leq$ is guaranteed just by (WMV1), (WMV4) and (WMV3). Finally, if $x \leq y$ and $y \leq z$ then $\neg x \oplus y = \neg y \oplus z = 1$, hence (WMV2) together with (WMV4) entail $\neg x \oplus z = 1$. Altogether, $\leq$ is a partial order on A.

In the sequel we will discuss a structure of sections in WMV-algebras. To this aim, we need the following terminology.

Given a poset $(P, \leq)$, we denote $\mathrm{L}(x,y) = \{a \in P : \ a \leq x, \ a \leq y\}$ and $\mathrm{U}(x,y) = \{a \in P : \ a \geq x, \ a \geq y\}$ for any $x, y \in P$. A poset $(P, \leq)$ is called *upwards* (*downwards*) *directed* if $\mathrm{U}(x,y) \neq \emptyset$ ($\mathrm{L}(x,y) \neq \emptyset$) holds for all $x, y \in P$. Directed poset is both upwards and downwards directed.

In his unpublished thesis [17], V. Snášel introduced the concept of a λ-semilattice as a natural generalization of semilattices:
an algebra $(L, \cup)$ of type (2) is called a *λ-semilattice* if it satisfies the identities

(S1) $x \cup x = x$ (idempotency)

(S2) $x \cup y = y \cup x$ (commutativity)

(S3) $x \cup ((x \cup y) \cup z) = (x \cup y) \cup z$ (weak associativity).

Remark that λ-semilattices were studied by J. Ježek and R. Quackenbush under the name *directoids*, see [12].

Similarly as for lattices, by a λ-*lattice* ([18]) we mean a structure $(L, \cup, \cap)$, where $(L, \cup)$ and $(L, \cap)$ are two λ-semilattices connected by absorption laws

(Ab) $x \cap (x \cup y) = x$, $x \cup (x \cap y) = x$.

It is quite easy to show that given a λ-semilattice $(L, \cup)$, the relation $\leq$ defined by

(3) $x \leq y \iff x \cup y = y$

is partial order on L, where $x \cup y \in \mathrm{U}(x, y)$ for all $x, y \in L$.

An order relation defined by (3) will be referred as an *induced order*. Thus $(L, \leq)$ is an upwards directed poset.

Remark that given a λ-lattice $\mathscr{L} = (L, \cup, \cap)$, the induced order $\leq$ defined by (3) coincides with that given by $x \leq y \iff x \cap y = x$. Hence we can refer to $\leq$ as the induced order on a λ-lattice $\mathscr{L}$.

Given a λ-semilattice $\mathscr{L} = (L, \cup)$ with the induced ordering $\leq$, a mapping $f\colon L \to L$ is called an *antitone involution* on $\mathscr{L}$ if

(i) $f(f(x)) = x$

(ii) $x \leq y \implies f(x) \geq f(y)$.

A structure $(L, \cup, f)$ is then said to be a λ-*semilattice with an antitone involution* f.

By a λ-*lattice with an antitone involution* f we mean a structure $(L, \cup, \cap, f)$, where $(L, \cup, f)$ is a λ-semilattice with an antitone involution f.

The next statement shows that each WMV-algebra can be viewed as a λ-semilattice of WMV-algebras:

Theorem 1. *Let $(A, \oplus, \neg, 0)$ be a WMV-algebra, $p \in A$, $x, y \in [p, 1]$. Then upon defining*

$$x \oplus_p y := \neg(\neg x \oplus p) \oplus y,$$

$$\neg_p x := \neg x \oplus p,$$

the structure $([p, 1], \oplus_p, \neg_p, p)$ is a WMV-algebra. Moreover, putting

$$x \cup y := \neg(\neg x \oplus y) \oplus y,$$

$$x \cap_p y := \neg_p(\neg_p x \cup \neg_p y),$$

the algebra $([p, 1], \cap_p, \cup, \neg_p)$ is a λ-lattice with an antitone involution.

Proof. First we show that $(A, \cup)$ is a λ-semilattice. Indeed, putting $y = z = 0$ in (WMV2) we obtain $\neg x \oplus x = 1$, thus $x \cup x = \neg(\neg x \oplus x) \oplus x = \neg 1 \oplus x = 0 \oplus x = x$ due to (WMV4). Clearly (WMV3) just means that $x \cup y = y \cup x$. The axiom (WMV2) can be rewritten to $\neg x \oplus ((x \cup y) \cup z) = 1$. Thus

$$\begin{aligned} x \cup ((x \cup y) \cup z) &= \neg(\neg x \oplus ((x \cup y) \cup z)) \oplus ((x \cup y) \cup z) \\ &= \neg 1 \oplus ((x \cup y) \cup z) = 0 \oplus ((x \cup y) \cup z) = (x \cup y) \cup z. \end{aligned}$$

Further, the axiom (WMV5) is equivalent to $y \leq \neg x \oplus y$, hence for each $p \in A$, $\cup$ is a binary operation on $[p, 1]$.

To show that $([p, 1], \cap_p)$ is a λ-semilattice, we compute

$$x \cap_p x = \neg_p(\neg_p x \cup \neg_p x) = \neg_p \neg_p x = \neg(\neg x \oplus p) \oplus p = x \cup p = x$$

for each $x \in [p, 1]$. The commutativity of $\cap_p$ is easily seen from its definition.

Let us prove the identity $x \cap_p ((x \cap_p y) \cap_p z) = (x \cap_p y) \cap_p z$ for all $x, y, z \in [p, 1]$. To simplify expressions, denote by $P = (x \cap_p y) \cap_p z$ and $x \bullet p = \neg x \oplus p$. Then we have $x \cup y = (x \bullet y) \bullet y = (y \bullet x) \bullet x$ and

$$P = (((((x \bullet p) \cup (y \bullet p)) \bullet p) \bullet p) \cup (z \bullet p)) \bullet p = ((((x \bullet p) \cup (y \bullet p)) \cup p) \cup (z \bullet p)) \bullet p.$$

According to (WMV5), $x \bullet p, y \bullet p \geq p$, hence $((x \bullet p) \cup (y \bullet p)) \cup p = (x \bullet p) \cup (y \bullet p)$ and $P = (((x \bullet p) \cup (y \bullet p)) \cup (z \bullet p)) \bullet p$. This yields

$$P \bullet p = (((x \bullet p) \cup (y \bullet p)) \cup (z \bullet p)) \cup p = ((x \bullet p) \cup (y \bullet p)) \cup (z \bullet p) \geq x \bullet p$$

by (WMV2), hence also

$$x \cap_p P = ((x \bullet p) \cup (P \bullet p)) \bullet p = (P \bullet p) \bullet p = P \cup p = P.$$

Finally, we show the validity of absorption laws: we have

$$x \cap_p (x \cup y) = ((x \bullet p) \cup ((x \cup y) \bullet p)) \bullet p = (x \bullet p) \bullet p = x \cup p = x,$$

since $x \bullet p \geq (x \cup y) \bullet p$ by (WMV6). Applying (WMV6) again we obtain

$$x \cap_p y = ((x \bullet p) \cup (y \bullet p)) \bullet p \leq (x \bullet p) \bullet p = x,$$

which gives

$$x \cup (x \cap_p y) = ((((x \bullet p) \cup (y \bullet p)) \bullet p) \bullet x) \bullet x = 1 \bullet x = x,$$

and we are done.

Now we show that $([p, 1], \oplus_p, \neg_p, p)$ is a WMV-algebra.

(WMV1) We have $\neg_p \neg_p x = (x \bullet p) \bullet p = x \cup p = x$.

(WMV2) Given $x, y \in [p, 1]$, we compute

$$\neg_p x \oplus_p y = ((x \bullet p) \bullet p) \bullet y = (x \cup p) \bullet y = x \bullet y = \neg x \oplus y.$$

Hence, the validity of (WMV2) for $[p,1]$ is a conclusion of (WMV2) for $\mathscr{A}$.

(WMV3) Again, take $x, y \in [p,1]$. Then

$$\begin{aligned}\neg_p(\neg_p x \oplus_p y) \oplus_p y &= (((((x \bullet p) \bullet p) \bullet y) \bullet p) \bullet p) \bullet y \\ &= (((x \bullet y) \bullet p) \bullet p) \bullet y = ((x \bullet y) \cup p) \bullet y.\end{aligned}$$

Clearly, due to (WMV6), $p \le y \le x \bullet y$, hence

$$\neg_p(\neg_p x \oplus_p y) \oplus_p y = x \cup y = y \cup x = \neg_p(\neg_p y \oplus_p x) \oplus_p x.$$

(WMV4) For $x \in [p,1]$ we derive $x \oplus_p y = (x \bullet p) \bullet p = x \cup p = x$, $p \oplus_p x = (p \bullet p) \bullet x = 1 \bullet x = x$.

(WMV5) If $x, y \in [p,1]$, then by (WMV2), $\neg_p y \oplus_p (\neg_p x \oplus_p y) = \neg_p y \oplus_p (x \bullet y) = ((y \bullet p) \bullet p) \bullet (x \bullet y) = (y \cup p) \bullet (x \bullet y) = y \bullet (x \bullet y) = 1$.

(WMV6) Let $p \le q \le x \le y$. Then we conclude $\neg_p y \oplus_p p = \neg y \oplus p$, $\neg_p x \oplus_p p = \neg x \oplus p$. Hence $\neg y \oplus p \le \neg x \oplus p$ by (WMV6); moreover, denoting $\le_p$ the order on $[p,1]$ given by $x \le_p y$ iff $\neg_p x \oplus_p y = 1$, we have $\neg_p x \oplus_p y = \neg x \oplus y$, thus $\le$ is the same as $\le_p$ and we are done. $\square$

3. λ-semillatices with section antitone involutions

By a *λ-semilattice with section antitone involutions* we mean a λ-semilattice $(S, \cup)$ with the top element 1, where every section $[p,1]$ has an antitone involution ${}^p\colon x \mapsto x^p$. If, moreover, $(S, \cup)$ has a least element 0, we speak about a *bounded* λ-semilattice with section antitone involutions.

Thus a λ-semilattice with section antitone involutions is a structure of type $(S, \cup, ({}^a)_{a \in S}, 1)$.

In order to overcome the difficulties with many partial unary operations ^{p}, one can define a total binary operation $\bullet$ on S by

(4) $x \bullet y := (x \cup y)^y$.

Clearly $x \bullet y$ is well defined since $x \cup y \in [y,1]$.

The following easy lemma shows that λ-semilattices with section antitone involutions can be axiomatized by identities:

Lemma 1. *A λ-semilattice $(S, \cup)$ with the top element 1 is a λ-semilattice with section antitone involutions iff there is a binary operation $\bullet$ on S having the properties:*

(i) $(((x \cup p) \cup y) \bullet p) \bullet ((x \cup p) \bullet p) = 1$,

(ii) $x \cup y = (x \bullet y) \bullet y$,

(iii) $((x \bullet y) \bullet y) \bullet y = x \bullet y$.

Proof. Let $(S, \cup)$ be a λ-semilattice with section antitone involutions with the induced order $\leq$ and let $\bullet$ be defined by (4). Observe that $x \leq y$ iff $x \bullet y = 1$.

Now given $x, y, p \in S$, clearly $\alpha = (x \cup p) \cup y \geq x \cup p \geq p$. Since the involution ^{p} on $[p, 1]$ is antitone, we have $\alpha^p \leq (x \cup p)^p$, thus $\alpha^p \bullet (x \cup p)^p = 1$, which is just (i).

To prove (ii), we compute $(x \bullet y) \bullet y = ((x \cup y)^y \cup y)^y = (x \cup y)^{yy} = x \cup y$. Finally, applying (ii) we get $((x \bullet y) \bullet y) \bullet y = (x \cup y) \bullet y = ((x \cup y) \cup y)^y = (x \cup y)^y = x \bullet y$.

Conversely, assume that $\bullet$ satisfies the conditions (i)–(iii). For $x \in [p, 1]$, define $x^p := x \bullet p$. By (ii) and (iii), $(x \bullet p) \cup p = ((x \bullet p) \bullet p) \bullet p = x \bullet p$, thus $x^p = x \bullet p \geq p$ and so $x^p \in [p, 1]$. Further, $x^{pp} = (x \bullet p) \bullet p = x \cup p = x$ by (ii), hence $x \mapsto x^p$ is an involution on $[p, 1]$. To show that $x \mapsto x^p$ is antitone, assume $p \leq x \leq y$. Then $(x \cup p) \cup y = x \cup y = y$, $x \cup p = x$, which due to (i) gives $y^p \leq x^p$ as desired. Moreover, in view of (ii) and (iii) $x \bullet y = ((x \bullet y) \bullet y) \bullet y = (x \cup y) \bullet y = (x \cup y)^y$. $\square$

Lemma 1 shows that λ-semilattices with section antitone involutions can be considered as algebras $(S, \cup, 1, \bullet)$ of type $(2, 0, 2)$, where $(S, \cup, 1)$ is a λ-semilattice with a top element 1 satisfying the identities (i)–(iii).

By a λ-lattice with section antitone involutions we mean a structure $(A, \cup, \cap, 1, \bullet)$, where $(A, \cup, \cap, 1)$ is λ-lattice with a top element 1 and $(A, \cup, 1, \bullet)$ is a λ-semilattice with section antitone involutions.

We show that λ-lattices with section antitone involutions fulfill from a congruence point of view as much as we can hope:

Theorem 2. *The variety of all λ-lattices with sectional antitone involutions is regular and arithmetical.*

Proof. Let $\mathscr{V}$ be the variety of λ-lattices with section antitone involutions.

$\mathscr{V}$ is regular: Let

$$t_1(x, y, z) = ((x \bullet y) \cap (y \bullet x)) \cap z,$$

$$t_2(x, y, z) = ((x \bullet y) \bullet z) \cup ((y \bullet x) \bullet z).$$

We show that $t_1(x, y, z) = t_2(x, y, z) = z$ iff $x = y$.

Obviously, $t_1(x, x, z) = t_2(x, x, z) = z$. Conversely, assume that $t_1(x, y, z) = t_2(x, y, z) = z$. Then $z \leq x \bullet y, y \bullet x$ and $z \geq (x \bullet y) \bullet z, (y \bullet x) \bullet z$. But by

Lemma 1(iii) we have $(x \bullet y) \bullet z, (y \bullet x) \bullet z \geq z$, so that $(x \bullet y) \bullet z = z = (y \bullet x) \bullet z$, whence $x \bullet y = (x \bullet y) \cup z = ((x \bullet y) \bullet z) \bullet z = z \bullet z = 1$, so $x \leq y$. Similarly $y \leq x$, and hence $x = y$.

$\mathscr{V}$ is arithmetical: Let

$$m(x, y, z) = (((x \bullet y) \bullet z) \cap ((z \bullet y) \bullet x)) \cap (x \cup z).$$

Prove that $m(x, y, y) = m(x, y, x) = m(y, y, x) = x$. We have

$$m(x, y, y) = (((x \bullet y) \bullet y) \cap ((y \bullet y) \bullet x)) \cap (x \cup y) = ((x \cup y) \cap x) \cap (x \cup y) = x,$$
$$m(x, y, x) = (((x \bullet y) \bullet x) \cap ((x \bullet y) \bullet x)) \cap (x \cup x) = ((x \bullet y) \bullet x) \cap x = x$$

since $(x \bullet y) \bullet x \geq x$ by Lemma 1(ii) and (iii), and $m(y, y, x) = (((y \bullet y) \bullet x) \cap ((x \bullet y) \bullet y)) \cap (y \cup x) = (x \cap (x \cup y)) \cap (y \cup x) = x$. □

We shall prove that there is a 1–1 correspondence between WMV-algebras and bounded λ-semilattices with section antitone involutions:

Theorem 3.

(a) *Let* $(A, \oplus, \neg, 0)$ *be a WMV-algebra. Define*
$x \cup y := \neg(\neg x \oplus y) \oplus y$ *and*
$x \bullet y := \neg x \oplus y$.
Then $\Phi(A) = (A, \cup, \bullet, 0, 1)$ *is a bounded* λ*-semilattice with section antitone involutions.*

(b) *Let* $(S, \cup, \bullet, 0, 1)$ *be a bounded* λ*-semilattice with section antitone involutions. If we define*
$x \oplus y := (x \bullet 0) \bullet y$,
$\neg x := x \bullet 0$,
then $\Psi(S) = (S, \oplus, \neg, 0)$ *is a WMV-algebra.*

(c) *Moreover, we have* $\Psi(\Phi(A)) = A$ *and* $\Phi(\Psi(S)) = S$.

P r o o f.

(a) We already know from Theorem 1 that $(A, \cup)$ is a bounded λ-semilattice. We show that the conditions (i)–(iii) of Lemma 1 are satisfied. It is immediately seen from the definition of induced order $\leq$ on A that

(5) $x \leq y \iff \neg x \oplus y = 1 \iff x \bullet y = 1.$

Thus given $x, y, p \in A$, we have $p \leq x \cup p \leq (x \cup p) \cup y$, which due to (WMV6) yields

$$((x \cup p) \cup y) \bullet p = \neg((x \cup p) \cup y) \oplus p \leq \neg(x \cup p) \oplus p = (x \cup p) \bullet p.$$

This inequality is with respect to (5) just (i). The identity (ii) is just the transcription of the axiom (WMV3). Finally, to prove (iii) we compute

$$((x \bullet y) \bullet y) \bullet y = \neg(\neg(\neg x \oplus y) \oplus y) \oplus y = (\neg x \oplus y) \cup y = \neg x \oplus y = x \bullet y,$$

since $y \leq \neg x \oplus y$ by (WMV5).

(b) Assume conversely that $(S, \cup, \bullet, 0, 1)$ is a bounded λ-semilattice with section antitone involutions.

Observe that by (ii), $\neg x \oplus y = ((x \bullet 0) \bullet 0) \bullet y = (x \cup 0) \bullet y = x \bullet y$.
(WMV1) $\neg\neg x = (x \bullet 0) \bullet 0 = x \cup 0 = x$ by (ii),
(WMV2) $\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = x \bullet ((((x \bullet y) \bullet y) \bullet z) \bullet z) = x \bullet ((x \cup y) \cup z) = 1$ since $x \leq (x \cup y) \cup z$.
(WMV3) $\neg(\neg x \oplus y) \oplus y = (x \bullet y) \bullet y = x \cup y = y \cup x = \neg(\neg y \oplus x) \oplus x$ by (ii),
(WMV4) $x \oplus 0 = (x \bullet 0) \bullet 0 = x \cup 0 = x$, $0 \oplus x = (0 \bullet x) \bullet x = 1 \bullet x = 1^x = x$ as for each $x \in A$ the involution ^{x} on $[x, 1]$ exchanges the elements $x \leftrightarrow 1$.
(WMV5) $\neg x \oplus y = x \bullet y \geq y$, hence $\neg y \oplus (\neg x \oplus y) = 1$.
(WMV6) Rewriting (i) in terms of $\oplus$ and $\neg$, we get

$$\neg((x \cup p) \cup y) \oplus p \leq \neg(x \cup y) \oplus p,$$

which is just (WMV6).

Now, let $(A, \oplus, \neg, 0)$ be a WMV-algebra and denote $\Phi(A) = (A, \cup, \bullet, 0, 1)$, $\Psi(\Phi(A)) = (A, \oplus', \neg', 0)$. We have $x \oplus' y = (x \bullet 0) \bullet y = \neg(\neg x \oplus 0) \oplus y = x \oplus y$, $\neg' x = x \bullet 0 = \neg x \oplus 0 = \neg x$. Hence $\Psi(\Phi(A)) = A$.

Conversely, let $(S, \cup, \bullet, 0, 1)$ be a bounded λ-semilattice with section antitone involutions. Denote $\Psi(S) = (S, \oplus, \neg, 0)$ and $\Phi(\Psi(S)) = (S, \cup', \bullet', 0, 1')$. Clearly $x \cup' y = \neg(\neg x \oplus y) \oplus y = (x \bullet y) \bullet y = x \cup y$, $x \bullet' y = \neg x \oplus y = x \bullet y$ and $1' = \neg 0 = 0 \bullet 0 = 1$, thus $\Phi(\Psi(S)) = S$. $\square$

Corollary. *The variety of all WMV-algebras is regular and arithmetical.*

REFERENCES

[1] ABBOTT, J. C.: *Semi-boolean algebra*, Mat. Vesnik **4** (1967), 177–198.

[2] BAHLS, P.—COLE, J.—GALATOS, N.—JIPSEN, P.—TSINAKIS, C.: *Cancellative residuated lattices*, Algebra Universalis **50** (2003), 83–106.

[3] CHAJDA, I.—KÜHR, J.: *A non-associative generalization of MV-algebras*, Math. Slovaca **57**, (2007), 301–312.

[4] CHAJDA, I.—HALAŠ, R.—KÜHR, J.: *Distributive lattices with sectionally antitone involutions*, Acta Sci. Math. (Szeged) **71** (2005), 19–33.

[5] CHAJDA, I.—HALAŠ, R.—KÜHR, J.: *Implication in MV-algebras*, Algebra Universalis **52** (2004), 377–382.

[6] CHANG, C. C.: *Algebraic analysis of many-valued logic*, Trans. Amer. Math. Soc. **88** (1958), 467–490.

[7] CHANG, C. C.: *A new proof of the completeness of the Łukasiewicz axioms*, Trans. Amer. Math. Soc. **93** (1959), 74–80.

[8] CIGNOLI, R. L. O.—D'OTTAVIANO, I. M. L.—MUNDICI, D.: *Algebraic Foundations of Many-valued Reasoning*, Kluwer Acad. Publ., Dordrecht-Boston-London, 2000.

[9] DVUREČENSKIJ, A.: *Pseudo MV-algebras are intervals in l-groups*, J. Aust. Math. Soc. **70** (2002), 427–445.

[10] GALATOS, N.—TSINAKIS, C.: *Generalized MV-algebras*, J. Algebra **283** (2005), 254–291.

[11] GEORGESCU, G.—IORGULESCU, A.: *Pseudo MV-algebras*, Mult.-Valued Log. **6** (2001), 95–135.

[12] JEŽEK, J.—QUACKENBUSH, R.: *Directiods: algebraic models of up-directed sets*, Algebra Universalis **27** (1990), 49–69.

[13] KARÁSEK, J.: *Rotations of λ-lattices*, Math. Bohem. **121** (1996), 293–300.

[14] MANGANI, P.: *Su certe algebre connesse con logiche a piú valori*, Boll. Unione Mat. Ital. **8** (1973), 68–78.

[15] MUNDICI, D.: *Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus*, J. Funct. Anal **65**, (1986), 15–63.

[16] RACHŮNEK, J.: *A non-commutative generalization of MV-algebras*, Czechoslovak Math. J. **52** (2002), 255–273.

[17] SNÁŠEL, V.: *λ-lattices*. Ph.D. Thesis, Masaryk Univ., Brno, 1991.

[18] SNÁŠEL, V.: *λ-lattices*, Math. Bohem. **122** (1997), 267–272.

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