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OSCILLATION OF FOURTH ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS I

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ABSTRACT. Oscillatory and asymptotic behaviour of solutions of a class of nonlinear fourth order neutral difference equations of the form

$$\Delta^2(r(n)\Delta^2(y(n)+p(n)y(n-m)))+q(n)G(y(n-k))=0$$

and

(NH) $\Delta^2(r(n)\Delta^2(y(n)+p(n)y(n-m)))+q(n)G(y(n-k))=f(n)$

are studied under the assumption $\sum\limits_{n=0}^{\infty}\frac{n}{r(n)}=\infty$, for various ranges of $p(n)$. Sufficient conditions are obtained for the existence of bounded positive solutions of (NH).

1. Introduction

In [3], Kusano and Naito have studied oscillatory behaviour of solutions of a class of fourth order nonlinear differential equations of the form

$$(r(t)y'')''+yF(y^2,t)=0,$$

where r and F are continuous and positive on $[0,\infty)$ and $[0,\infty)\times[0,\infty)$ respectively under the assumption that

(H$_1$) $\int\limits_0^{\infty}\frac{t}{r(t)}\,\mathrm{d}t=\infty$.

The object of this paper is to study the oscillatory and asymptotic behaviour of solutions of a class of fourth order nonlinear neutral difference equations of the form

$$\Delta^2(r(n)\Delta^2(y(n)+p(n)y(n-m)))+q(n)G(y(n-k))=0, \tag{1}$$

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where Δ is the forward difference operator defined by $\Delta y(n) = y(n+1) - y(n)$, p, q are real valued functions defined on $N(n_0) = \{n_0, n_0+1, \dots\}$, $n_0 \geq 0$, such that $q(n) \geq 0$, function $G \in C(\mathbb{R}, \mathbb{R})$ is non-decreasing and $uG(u) > 0$ for $u \neq 0$ and $m > 0$, $k \geq 0$ are integers, under the discrete analogue of the assumption (H_1) as

(A_1) $r(n)$ is a real valued function such that $r(n) > 0$ and $\sum\limits_{n=0}^{\infty} \frac{n}{r(n)} = \infty$.

The associated forced equation

$$\Delta^2(r(n)\Delta^2(y(n) + p(n)y(n-m))) + q(n)G(y(n-k)) = f(n), \tag{2}$$

where $f(n)$ is a real valued function is also studied under the assumption (A_1). Different ranges of $p(n)$ and different type of forcing functions is considered. In recent papers [4], [5], [8], Parhi and Tripathy have discussed oscillation and asymptotic behaviour of solutions of higher order neutral difference equations of the form

$$\Delta^m(y(n) + p(n)y(n-m)) + q(n)G(y(n-k)) = 0 \tag{3}$$

and

$$\Delta^m(y(n) + p(n)y(n-m)) + q(n)G(y(n-k)) = f(n). \tag{4}$$

If $r(n) \equiv 1$, then (A_1) is satisfied and Eqs. (1) and (2) reduce to (3) and (4) respectively for $m = 4$. However, Eqs. (1) and (2) cannot be termed in general as the particular cases of (3) and (4) in view of (A_1). Therefore it is interesting to study Eqs. (1) and (2) under (A_1). A close observation reveals that the nature of the function r influences the behaviour of solutions of (1) and (2). This influence is quite explicit in case of unforced equation (2). Necessary and sufficient conditions for oscillation of (1)/(2) are obtained in this paper.

Thandapani and Arockiasamy [7], has considered the fourth order non-linear difference equation of the form

$$\Delta^2\left(r_n\Delta^2\left(y_n + p_n y_{n-k}\right)\right) + f\left(n, y_{\sigma(n)}\right) = 0, \qquad n \in N(n_0), \tag{5}$$

where $f\colon N(n_0) \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $uf(n,u) > 0$ for all $u \neq 0$, $\{r_n\}$ and $\{p_n\}$ are positive real sequences, $\{\sigma_n\}$ is an increasing sequence of integers and k is a non negative integer. They have obtained necessary and sufficient conditions for (5) when $0 \leq p_n < p < 1$ for all $n \in N(n_0)$. Clearly, if we consider $f\left(n, y_{\sigma(n)}\right) = q(n)G\left(y\left(n-k\right)\right)$, then the work in [7] is a particular case of the present work as the range of $p(n)$ is concerned. Here an attempt is made to study oscillatory and asymptotic behaviour of solutions of (1) under various ranges of $p(n)$. Also forced equation is considered for different ranges of $p(n)$.

By a solution of Eq. (1)/Eq. (2) on $N(n_0)$ we mean, a real valued function $y(n)$ defined on $N(-\rho) = \{-\rho, -\rho+1, \dots\}$ which satisfies (1)/(2) for $n \geq n_0 \geq 0$, where $\rho = \max\{m, k\}$. If

$$y(n) = A_n, \qquad n = -\rho, -\rho + 1, \dots, 0, 1, 2, 3 \dots, \tag{6}$$

are given, then (1) admits a unique solution satisfying the initial condition (6). A solution $y(n)$ of (1) is said to be oscillatory if for every integer $N > 0$, there exists an $n \ge N$ such that $y(n)y(n+1) \le 0$. Otherwise, it is called non oscillatory.

Equation (1) may be regarded discrete analogue of

$$\left(r(t)\left(y(t) + p(t)y(t-\tau)\right)''\right)'' + q(t)G(y(t-\sigma)) = 0, \qquad t \ge 0.$$

Oscillatory and asymptotic behaviour of solutions of this equation and the associated forced equation is studied in [6].

Here is some preparatory results, which are useful in establishing the results of the work.

Lemma 1.1. *Let* (A_1) *hold. Let* u *be a real-valued function on* $[0,\infty)$ *such that* $\Delta^2(r(n)\Delta^2 u(n)) \le 0$ *for large* n. *If* $u(n) > 0$ *ultimately, then one of the cases* (a) *and* (b) *holds for large* n *and if* $u(n) < 0$ *ultimately, then one of the cases* (b), (c), (d) *and* (e) *holds for large* n, *where*

(a) $\Delta u(n) > 0$, $\Delta^2 u(n) > 0$ *and* $\Delta(r(n)\Delta^2 u(n)) > 0$,
(b) $\Delta u(n) > 0$, $\Delta^2 u(n) < 0$ *and* $\Delta(r(n)\Delta^2 u(n)) > 0$,
(c) $\Delta u(n) < 0$, $\Delta^2 u(n) < 0$ *and* $\Delta(r(n)\Delta^2 u(n)) > 0$,
(d) $\Delta u(n) < 0$, $\Delta^2 u(n) < 0$ *and* $\Delta(r(n)\Delta^2 u(n)) < 0$,
(e) $\Delta u(n) < 0$, $\Delta^2 u(n) > 0$ *and* $\Delta(r(n)\Delta^2 u(n)) > 0$.

Proof. Since $\Delta^2(r(n)\Delta^2 u(n)) \le 0$, then $u(n)$, $\Delta u(n)$, $\Delta^2 u(n)$ and $\Delta(r(n)\Delta^2 u(n))$ are monotonic and hence there are eight cases. Let $u(n) > 0$ for $n \ge n_0 > 0$. It is enough to show that (c), (d), (e) and the following cases viz;

(f) $\Delta u(n) < 0$, $\Delta^2 u(n) > 0$ and $\Delta(r(n)\Delta^2 u(n)) < 0$,
(g) $\Delta u(n) > 0$, $\Delta^2 u(n) > 0$ and $\Delta(r(n)\Delta^2 u(n)) < 0$,
(h) $\Delta u(n) > 0$, $\Delta^2 u(n) < 0$ and $\Delta(r(n)\Delta^2 u(n)) < 0$

do not hold. It seems that the cases (c) and (d) do not occur due to $u(n) < 0$ for large n.

In case (e), $\Delta^2 u(n) > r(n_1)\Delta^2 u(n_1)/r(n)$, for $n \ge n_1 > n_0$. Hence

$$n\Delta^2 u(n) > r(n_1)\Delta^2 u(n_1)\left(\frac{n}{r(n)}\right). \tag{7}$$

Taking sum to the inequality (7), we have

$$\sum_{s=n_1}^{n-1} s\Delta^2 u(s) > r(n_1)\Delta^2 u(n_1)\sum_{s=n_1}^{n-1}\frac{s}{r(s)}.$$

Using summation by parts, we obtain

$$\begin{aligned} n\Delta u(n) &> n_1\Delta u(n_1) + \sum_{s=n_1}^{n-1} \Delta u(s+1) + r(n_1)\Delta^2 u(n_1) \sum_{s=n_1}^{n-1} \frac{s}{r(s)} \\ &= n_1\Delta u(n_1) + u(n+1) - u(n_1+1) + r(n_1)\Delta^2 u(n_1) \sum_{s=n_1}^{n-1} \frac{s}{r(s)} \\ &> n_1\Delta u(n_1) - u(n_1+1) + r(n_1)\Delta^2 u(n_1) \sum_{s=n_1}^{n-1} \frac{s}{r(s)} \end{aligned}$$

that is, $\Delta u(n) > 0$ for large n due to (A_1), a contradiction. As $\Delta(r(n)\Delta^2 u(n))$ is monotonic decreasing, then for $n \geq n_1 > n_0$,

$$\Delta(r(n)\Delta^2 u(n)) \leq \Delta\left(r(n_1)\Delta^2 u(n_1)\right).$$

Summing the above inequality from n_2 to $(n-1)$, we obtain

$$r(n)\Delta^2 u(n) \leq (n-n_2)\Delta\left(r(n_1)\Delta^2 u(n_1)\right), \qquad n \geq n_2 > n_1.$$

Consequently, in each of the cases (f) and (g), $\Delta^2 u(n) < 0$ for large n, a contradiction. In case (h), $\Delta^2\left(r(n)\Delta^2 u(n)\right) \leq 0$ implies that $\Delta\left(r(n)\Delta^2 u(n)\right) \leq \Delta\left(r(n_1)\Delta^2 u(n_1)\right)$, that is,

$$\sum_{s=n_1}^{n-1} \Delta\left(r(s)\Delta^2 u(s)\right) \leq (n-n_1)\Delta\left(r(n_1)\Delta^2 u(n_1)\right),$$

that is,

$$\begin{aligned} r(n)\Delta^2 u(n) &\leq r(n_1)\Delta^2 u(n_1) + (n-n_1)\Delta\left(r(n_1)\Delta^2 u(n_1)\right) \\ &< -Ln, \qquad n \geq n_2 > n_1, \end{aligned}$$

where $L > 0$ is a constant. Hence

$$\sum_{s=n_2}^{n-1} \Delta^2 u(s) < -L \sum_{s=n_2}^{n-1} \frac{s}{r(s)}$$

that is,

$$\Delta u(n) < \Delta u(n_2) - L \sum_{s=n_2}^{n-1} \frac{s}{r(s)}$$

for large n, a contradiction. Next, assume that $u(n) < 0$ for $n \geq n_0 > 0$. The case (a) does not occur because in this case $u(n) > 0$ ultimately. In each of the cases (f) and (g), $\Delta^2 u(n) < 0$ for large n, a contradiction. Similar contradiction can be obtained for the case (h). Thus the lemma is proved. □

Remark. If $0 \leq p(n) < 1$, then the cases (a) and (b) hold ultimately. In [1] (or in [7]), the lemma holds for $0 \leq p(n) < 1$. With different ranges of $p(n)$, Lemma 1.1 strengthens [1, Lemma].

Lemma 1.2. *Let the conditions of* Lemma 1.1 *hold. If* $u(n) > 0$ *ultimately, then* $u(n) > R_N(n-1)\Delta(r(n)\Delta^2 u(n))$, *where*

$$R_N(n) = \sum_{t=N}^{n-1} \sum_{s=N}^{t-1} \frac{(s-N)}{r(s)}.$$

The proof of the lemma can be followed from [1].

Lemma 1.3. ([2, p. 184]) *If* $q(n) \geq 0$ *for* $n \geq 0$ *and*

$$\liminf_{n\to\infty} \sum_{s=n-k}^{n-1} q(s) > \frac{k^{k+1}}{(k+1)^{k+1}},$$

then $\Delta x(n) + q(n)x(n-k) \leq 0$, $n \geq 0$, *cannot have an eventually positive solution.*

2. Oscillations of homogeneous equations

In this section, sufficient conditions are obtained for the oscillation and asymptotic behaviour of all solutions of Eq. (1). We need the following assumptions for our use in the sequel.

(A_2) There exists $\lambda > 0$ such that $G(u) + G(\nu) \geq \lambda G(u+\nu)$ for $u > 0$ and $\nu > 0$.

(A_3) $G(u)G(\nu) = G(u\nu)$.

(A_3^1) $G(u)G(\nu) \geq G(u\nu)$.

(A_4) $\int\limits_0^{\pm c} \frac{\mathrm{d}u}{G(u)} < \infty$ for all $c > 0$.

(A_5) $\sum\limits_{n=N+\rho}^{\infty} G(R_N(n-1))Q(n) = \infty$, $N \geq 0$,
where $Q(n) = \min\{q(n), q(n-m)\}$ for $n \geq m$.

(A_6) $G(u) = -G(u), u \in \mathbb{R}$.

(A_7) $\sum\limits_{n=m}^{\infty} Q(n) = \infty$.

(A_8) $\liminf\limits_{|x|\to 0} \frac{G(x)}{x} \geq \gamma > 0$.

(A$_9$) $\liminf_{n\to\infty} \sum_{s=n-k}^{n-1} G(R_N(s-k-1))q(s) > \frac{k^{k+1}}{G(1-p)(k+1)^{k+1}}$.

(A$_{10}$) $\sum_{n=N+k}^{\infty} G(R_N(n-k-1))q(n) = \infty$.

(A$_{11}$) $\frac{G(x_1)}{x_1^\alpha} \geq \frac{G(x_2)}{x_2^\alpha}$ for $x_1 \geq x_2 > 0$ and $\alpha \geq 1$,

(A$_{12}$) $\sum_{n=N+k}^{\infty} R_N^\alpha(n-k-1))Q(n) = \infty$,

(A$_{13}$) $\sum_{n=0}^{\infty} q(n) = \infty$.

Theorem 2.1. *Let $0 \leq p(n) \leq p < 1$. Suppose that* (A$_1$), (A$_3$), (A$_8$) *and* (A$_9$) *hold. Then every solution of* (1) *oscillates.*

Remark 1. (A$_9$) implies that (A$_{10}$) holds. Indeed, if $\sum_{s=N+k}^{\infty} G(R_N(s-k-1))q(s) = \alpha < \infty$, then for $n > N+2k$

$$\sum_{s=n-k}^{n-1} G\left(R_N(s-k-1)\right)q(s) = \left(\sum_{s=N+k}^{n-1} - \sum_{s=N+k}^{n-k}\right) G\left(R_N(s-k-1)\right)q(s)$$

implies that

$$\liminf_{n\to\infty} \sum_{s=n-k}^{n-1} G\left(R_N(s-k-1)\right)q(s) \leq \alpha - \alpha = 0,$$

a contradiction to (A$_9$).

Proof of the Theorem. Suppose that $y(n)$ is a non-oscillatory solution of (1). Let $y(n) > 0$ for $n \geq n_0 > 0$. Setting

$$z(n) = y(n) + p(n)y(n-m) \tag{8}$$

we obtain

$$0 < z(n) \leq y(n) + py(n-m) \tag{9}$$

and

$$\Delta^2\left(r(n)\Delta^2 z(n)\right) = -q(n)G(y(n-k)) \leq 0 \tag{10}$$

for $n \geq n_0 + \rho$. Then one of the cases (a) and (b) of Lemma 1.1 holds. In each case, $z(n)$ is increasing and hence for $n \geq n_0 + 2\rho$,

$$\begin{aligned}(1-p(n))z(n) &< z(n) - p(n)z(n-m)\\ &= y(n) - p(n)p(n-m)y(n-2m) < y(n)\end{aligned}$$

that is, $(1-p(n))z(n) \geq (1-p)z(n)$ implies that $y(n) > 1 - pz(n)$. From (10) we obtain for $n \geq N+k > n_0 + 2\rho + k$

$$\begin{aligned}
0 &\geq \Delta^2(r(n)\Delta^2 z(n)) + q(n)G((1-p)z(n-k)) \\
&\geq \Delta^2(r(n)\Delta^2 z(n)) + q(n)G(1-p)G(z(n-k)) \\
&\geq \Delta^2(r(n)\Delta^2 z(n)) + q(n)G(1-p)G(R_N(n-k-1)\Delta(r(n-k)\Delta^2 z(n-k))) \\
&\geq \Delta^2(r(n)\Delta^2 z(n)) + q(n)G(1-p)G(R_N(n-k-1)) \cdot \\
&\qquad\qquad \cdot G(\Delta(r(n-k)\Delta^2 z(n-k))) \qquad (11)
\end{aligned}$$

using Lemma 1.2 and (A_3). Let $\lim_{n\to\infty} \Delta(r(n)\Delta^2 z(n)) = \alpha$. If $0 < \alpha < \infty$, then $\Delta(r(n)\Delta^2 z(n)) > \beta > 0$, $n \geq n_1 > N+k$. From (11) we obtain

$$G(1-p)q(n)G\left(R_N(n-k-1)\right)G(\beta) \leq -\Delta^2\left(r(n)\Delta^2 z(n)\right)$$

for $n \geq n_2 > n_1 + k$. Hence

$$\sum_{n=n_2}^{\infty} G(R_N(n-k-1)q(n) < \infty,$$

a contradiction to (A_{10}). Thus $\alpha = 0$. Then using (A_8) we have

$$G(\Delta(r(n)\Delta^2 z(n))) \geq \gamma\Delta(r(n)\Delta^2 z(n))$$

for $n \geq n_3 > n_2$. Consequently, (11) becomes

$$\begin{aligned}
0 \quad \geq \quad & \Delta^2\left(r(n)\Delta^2 z(n)\right) \\
& +\gamma q(n)G\left((1-p)G\left(R_N(n-k-1)\Delta\left(r(n-k)\Delta^2 z(n-k)\right)\right)\right),
\end{aligned}$$

for $n \geq n_3+k$. This shows that the inequality $\Delta u(n) + \gamma G(1-p)G(R_N(n-k-1)) \cdot$ $\cdot\, q(n)u(n-k) \leq 0$ admit a positive solution $(\Delta(r(n)\Delta^2 z(n))$, a contradiction due to (A_9) and Lemma 1.3. Hence $y(n) < 0$ for $n \geq n_0$. Putting $x(n) = -y(n)$ we obtain $x(n) > 0$ for $n \geq n_0$ and

$$\Delta^2\left(r(n)\Delta^2(x(n) + p(n)x(n-m))\right) + q(n)G(x(n-k)) = 0.$$

Proceeding as above we arrive at a contradiction. Hence the theorem is proved. □

Example 1. Consider

$$\Delta^2\left[(n/2)\Delta^2\left(y(n) + \frac{1}{3}\left(1+(-1)^n\right)y(n-2)\right)\right] + \frac{32}{3}(n+1)y^{\frac{1}{3}}(n-3) = 0, \qquad n > 0.$$

Clearly, all the conditions of Theorem 2.1 are satisfied. Hence all solutions of the equation are oscillatory. In particular, $y(n) = (-1)^{3n} = (-1)^n$ is an oscillatory solution of the equation.

Theorem 2.2. *Let $0 \le p(n) \le p < \infty$ and $m \le k$. If* (A_1), (A_2), (A_3), (A_4) *and* (A_5) *hold, then* (1) *is oscillatory.*

Proof. Suppose for contrary that $y(n)$ is a non-oscillatory solution of (1) such that $y(n) > 0$ for $n \ge n_0 > 0$. The proof for the case $y(n) < 0$, $n \ge n_0$, is similar. Setting $z(n)$ as in (8), we obtain (9) and (10) for $n \ge n_0 + \rho$. From Lemma 1.1, it follows that one of the cases (a) and (b) holds. The use of (A_2) and (A_3) yields

$$\begin{aligned}
0 &= \Delta^2\left[r(n)\Delta^2 z(n)\right] + q(n)G(y(n-k)) + G(p)\Delta^2\left[r(n-m)\Delta^2 z(n-m)\right]\\
&\quad + G(p)q(n-m)G(y(n-m-k))\\
&\ge \Delta^2\left[r(n)\Delta^2 z(n)\right] + G(p)\Delta^2\left[r(n-m)\Delta^2 z(n-m)\right] + \lambda Q(n)G(y(n-k)\\
&\quad + py(n-k-m))\\
&\ge \Delta^2\left[r(n)\Delta^2 z(n)\right] + G(p)\Delta^2\left[r(n-m)\Delta^2 z(n-m)\right] + \lambda Q(n)G(z(n-k))
\end{aligned}$$

for $n \ge n_1 > n_0 + 2\rho$. Hence by Lemma 1.2 we obtain

$$\begin{aligned}
0 &\ge \Delta^2\left[r(n)\Delta^2 z(n)\right] + G(p)\Delta^2\left[r(n-m)\Delta^2 z(n-m)\right]\\
&\quad +\lambda Q(n)G(R_N(n-k-1)\Delta(r(n-k)\Delta^2 z(n-k)))\\
&= \Delta^2\left[r(n)\Delta^2 z(n)\right] + G(p)\Delta^2\left[r(n-m)\Delta^2 z(n-m)\right]\\
&\quad +\lambda Q(n)G(R_N(n-k-1))G(\Delta(r(n-k)\Delta^2 z(n-k))),
\end{aligned}$$

for $n \ge N + \rho > n_1$. Consequently,

$$\begin{aligned}
&\lambda Q(n)G(R_N(n-k-1))\\
&\le \frac{-\Delta^2\left[r(n)\Delta^2 z(n)\right] - G(p)\Delta^2\left[r(n-m)\Delta^2 z(n-m)\right]}{G\left(\Delta\left(r(n-k)\Delta^2 z(n-k)\right)\right)}\\
&\le -\frac{\Delta^2\left[r(n)\Delta^2 z(n)\right]}{G\left(\Delta\left(r(n)\Delta^2 z(n)\right)\right)} - G(p)\frac{\Delta^2\left[r(n-m)\Delta^2 z(n-m)\right]}{G\left(\Delta\left(r(n-m)\Delta^2 z(n-m)\right)\right)}\\
&\le \int\limits_{\Delta w(n+1)}^{\Delta w(n)} \frac{\mathrm{d}u}{G(u)} + G(p)\int\limits_{\Delta w(n-m+1)}^{\Delta w(n-m)} \frac{\mathrm{d}\nu}{G(\nu)},
\end{aligned}$$

where $w(n) = r(n)\Delta^2 z(n)$, $\Delta w(n+1) \le u \le \Delta w(n)$ and $\Delta w(n-m+1) \le \nu \le \Delta w(n-m)$ implies that

$$\begin{aligned}
&\lambda \sum_{n=N+\rho}^{t-1} Q(n)G(R_N(n-k-1))\\
&\le \sum_{n=N+\rho}^{t-1}\left(\int\limits_{\Delta w(n+1)}^{\Delta w(n)} \frac{\mathrm{d}u}{G(u)} + G(p)\int\limits_{\Delta w(n-m+1)}^{\Delta w(n-m)} \frac{\mathrm{d}\nu}{G(\nu)}\right)
\end{aligned}$$

$$= \int_{\Delta w(t)}^{\Delta w(N+\rho)} \frac{\mathrm{d}u}{G(u)} + G(p) \int_{\Delta w(t-m)}^{\Delta w(N+\rho-m)} \frac{\mathrm{d}\nu}{G(\nu)}.$$

Since $\Delta w(n)$ is decreasing, then

$$\lambda \sum_{n=N+\rho}^{\infty} Q(n)G(R_N(n-k-1)) \leq \lim_{t\to\infty} \left(\int_{\Delta w(t)}^{\Delta w(N+\rho)} \frac{\mathrm{d}u}{G(u)} + G(p) \int_{\Delta w(t-m)}^{\Delta w(N+\rho)} \frac{\mathrm{d}\nu}{G(\nu)} \right) < \infty,$$

a contradiction to (A_5). Hence the theorem is proved. □

Theorem 2.3. $0 \leq p(n) \leq p < \infty$. *Assume that* (A_1), (A_2), (A_3^1), (A_6) *and* (A_7) *hold. Then every solution of* (1) *oscillates.*

Proof. Let $y(n)$ be a non-oscillatory solution of (1). Let $y(n) > 0$ for $n \geq n_0 > 0$. The proof for the case $y(n) < 0$, $n \geq n_0$, can similarly be dealt with. Setting $z(n)$ as in (8), we obtain (9) and (10) for $n \geq n_0 + \rho$. From Lemma 1.1, it follows that one of the cases (a) and (b) holds. Hence $z(n) > \beta > 0$ for $n \geq n_1 \geq n_0 + \rho$. Proceeding as in the proof of Theorem 2.2 we obtain

$$\begin{aligned} 0 &\geq \Delta^2 \left[r(n)\Delta^2 z(n)\right] + G(p)\Delta^2 \left[r(n-m)\Delta^2 z(n-m)\right] + \lambda Q(n)G(z(n-k)) \\ &\geq \Delta^2 \left[r(n)\Delta^2 z(n)\right] + G(p)\Delta^2 \left[r(n-m)\Delta^2 z(n-m)\right] + \lambda Q(n)G(\beta), \end{aligned}$$

for $n \geq n_2 > n_1 + 2\rho$. Hence $\sum\limits_{n=n_2}^{\infty} Q(n) < \infty$, a contradiction. This completes the proof of the theorem. □

Remark 2. (A_3^1) and (A_6) need not imply (A_3). Indeed, if

$$G(u) = \left(\left(\alpha + \beta|u|^{\lambda}\right)|u|^{\mu}\right) \operatorname{sgn} u, \qquad \lambda > 0,\ \ \mu > 0,\ \ \alpha \geq 0,\ \ \beta \geq 1,$$

then (A_3^1) and (A_6) are satisfied but does not (A_3).

Remark 3. The prototype of G satisfying (A_2), (A_3^1) and (A_6) is $G(u) = \left(\left(a + b|u|^{\lambda}\right)|u|^{\mu}\right) \operatorname{sgn} u$, $a \geq 1$, $b \geq 1$, $\lambda \geq 0$ and $\mu \geq 0$.

Remark 4. In Theorem 2.3, G could be super linear, linear or sub-linear. However, (A_7) implies (A_5) because $\Delta R_N(n) > 0$, for $n \geq N_1 > N$.

Theorem 2.4. *Let* $0 \leq p(n) \leq p < \infty$ *and* $m \leq k$. *If* (A_1), (A_2), (A_3), (A_{11}) *and* (A_{12}) *hold, then* (1) *is oscillatory.*

Proof. Proceeding as in Theorem 2.2, we obtain

$$\Delta^2 \left[r(n)\Delta^2 z(n)\right] + G(p)\Delta^2 \left[r(n-m)\Delta^2 z(n-m)\right] + \lambda Q(n)G(z(n-k)) \leq 0 \quad (12)$$

for $n \geq n_1 > n_0 + 2\rho$. Since $z(n)$ is increasing, then $z(n) > K > 0$ for $n \geq n_2 > n_1$. Using (A_{11}) and Lemma 1.2 we obtain

$$\begin{aligned} G(z(n-k)) &= (G(z(n-k))/z^\alpha(n-k))\, z^\alpha(n-k) \\ &\geq (G(K)/K^\alpha)\, z^\alpha(n-k) \\ &> \left(\frac{G(K)}{K^\alpha}\right) R_N^\alpha(n-k-1)\left[\Delta\left(r(n-k)\Delta^2 z(n-k)\right)\right]^\alpha \end{aligned}$$

and hence the inequality (12) yields

$$\begin{aligned} &\lambda\left(\frac{G(K)}{K^\alpha}\right) R_N^\alpha(n-k-1)Q(n) \\ &< \frac{-\Delta^2\left[r(n)\Delta^2 z(n)\right] - G(p)\Delta^2\left[r(n-m)\Delta^2 z(n-m)\right]}{\left[\Delta\left(r(n-k)\Delta^2 z(n-k)\right)\right]} \end{aligned}$$

that is,

$$\begin{aligned} &\lambda\left(\frac{G(K)}{K^\alpha}\right) R_N^\alpha(n-k-1)Q(n) \\ &< -\frac{\Delta^2\left[r(n)\Delta^2 z(n)\right]}{\left[\Delta r(n)\Delta^2 z(n)\right]^\alpha} - \frac{G(p)\Delta^2\left[r(n-m)\Delta^2 z(n-m)\right]}{\left[\Delta r(n-m)\Delta^2 z(n-m)\right]^\alpha} \\ &\leq -\frac{\Delta^2 w(n)}{u^\alpha} - \frac{G(p)\Delta^2 w(n-m)}{\nu^\alpha} \\ &= -\int_{\Delta w(n)}^{\Delta w(n+1)} \frac{\mathrm{d}u}{u^\alpha} - G(p)\int_{\Delta w(n-m)}^{\Delta w(n+1-m)} \frac{\mathrm{d}\nu}{\nu^\alpha}, \end{aligned}$$

where $w(n) = r(n)\Delta^2 z(n)$. Consequently,

$$\begin{aligned} &\frac{\lambda(G(K))}{K^\alpha}\sum_{n=n_2}^{t-1} R_N^\alpha(n-k-1)Q(n) \\ &< -\sum_{n=n_2}^{t-1}\left(\int_{\Delta w(n)}^{\Delta w(n+1)} \frac{\mathrm{d}u}{u^\alpha} + G(p)\int_{\Delta w(n-m)}^{\Delta w(n+1-m)} \frac{\mathrm{d}\nu}{\nu^\alpha}\right) \\ &= -\int_{\Delta w(n_2)}^{\Delta w(t)} \frac{\mathrm{d}u}{u^\alpha} - G(p)\int_{\Delta w(n_2-m)}^{\Delta w(t-m)} \frac{\mathrm{d}\nu}{\nu^\alpha}. \end{aligned}$$

Since $\lim_{t\to\infty} \Delta w(t)$ exists, it follows that $\sum_{n=n_2}^{\infty} R_N^\alpha(n-k-1)Q(n) < \infty$, a contradiction to ($A_{12}$). Thus the theorem is proved. □

Example 2. Every solution of

$$\Delta^2\left[n\Delta^2\left(y(n)+3(1+(-1)^n)y(n-3)\right)\right]+32(n+1)y^3(n-4)=0$$

oscillates by Theorem 2.4. In particular, $y(n)=(-1)^n$ is an oscillatory solution of the given equation.

Theorem 2.5. *Let $-1<p\le p(n)\le 0$. Suppose that* (A$_1$), (A$_3$), (A$_4$) *and* (A$_{13}$) *hold. Then every solution of* (1) *oscillates or tends to zero as $n\to\infty$.*

Proof. Let $y(n)$ be a non-oscillatory solution of (1). In view of (A$_3$), it is enough to consider $y(n)>0$ for $n\ge n_0>0$. Setting $z(n)$ as in (8) we obtain (10), for $n>n_0+\rho$. Hence $z(n)>0$ or <0 for $n\ge n_1>n_0+\rho$. Assume that $z(n)>0$ for $n\ge n_1$. From Lemma 1.1, it follows that one of the cases (a) and (b) holds. Hence $z(n)>R_N(n-1)\Delta[r(n)\Delta^2 z(n)]$ for $n\ge N>n_1$ by Lemma 1.2. Clearly, $z(n)\le y(n)$. As $\Delta[r(n)\Delta^2 z(n)]$ is monotonic decreasing, then for $n\ge n_2>N+k$,

$$\Delta^2\left[r(n)\Delta^2 z(n)\right]\le -q(n)G(R_N(n-k-1))G\left[\Delta(r(n)\Delta^2 z(n))\right]$$

due to (10). Following to Theorem 2.2 we get

$$\sum_{n=n_2}^{\infty} q(n)G(R_N(n-k-1))<\infty.$$

Since $R_N(n)>0$ and non-decreasing, it shows that $\sum\limits_{n=n_2}^{\infty} q(n)<\infty$, a contradiction to (A$_{13}$). Hence $z(n)<0$ for $n\ge n_1$. Consequently, $y(n)<-p(n)y(n-m)$ $<y(n-m)$ implies that $y(n)$ is bounded and so is $z(n)$. Here one of the cases (b)–(e) holds by Lemma 1.1. Let the case (b) hold. If $\limsup\limits_{n\to\infty} z(n)=\alpha$, then $-\infty<\alpha\le 0$. Assume that $\alpha=0$. Then $\limsup\limits_{n\to\infty} z(n)=0\ge\limsup\limits_{n\to\infty}(y(n)+py(n-m)\ge\limsup\limits_{n\to\infty} y(n)+\liminf\limits_{n\to\infty}(py(n-m))=\limsup\limits_{n\to\infty} y(n)+p\limsup\limits_{n\to\infty} y(n-m)=(1+p)\limsup\limits_{n\to\infty} y(n)$, that is, $\lim\limits_{n\to\infty} y(n)=0$. If $-\infty<\alpha<0$, then there exists $n^*>0$ and $\beta<0$ such that $z(n)<\beta<0$ for $n\ge n_3>\max\{n_2,n^*\}$. Hence $y(n)>py(n-m)$ for $n\ge n_0$ implies that $y(n-k)>p^{-1}\beta>0$, for $n\ge n_3+\rho$. Consequently, (10) yields

$$q(n)G(p^{-1}\beta)\le -\Delta^2\left[r(n)\Delta^2 z(n)\right]$$

that is, $\sum\limits_{n=n_3+\rho}^{\infty} q(n)<\infty$, a contradiction. In each of the cases (c) and (d), $\lim\limits_{n\to\infty} z(n)=-\infty$, a contradiction to the fact that $z(n)$ is bounded. Let the case (e) hold. Clearly,

$$\Delta^2 z(n)>\left(\frac{r(n_1)\Delta^2 z(n_1)}{r(n)}\right)\qquad\text{for}\quad n>n_1.$$

Thus

$$\sum_{s=n_1}^{n-1} s\Delta^2 z(s) > r(n_1)\Delta^2 z(n_1) \sum_{s=n_1}^{n-1} \frac{s}{r(s)}.$$

Applying summation by parts we get $\Delta z(n) > 0$ for large n due to bounded $z(n)$ and (A_1), a contradiction. Hence the theorem is proved. $\square$

Theorem 2.6. *Let $-\infty < p_1 \le p(n) \le p_2 < -1$. If (A_1) and (A_{13}) hold, then every bounded solution of (1) oscillates or tends to zero as $n \to \infty$.*

Proof. Suppose for contrary that $y(n)$ is a bounded non-oscillatory solution of (1) such that $y(n) > 0$ for $n \ge n_0$. Setting $z(n)$ as in (8), we obtain (10) for $n \ge n_0 + \rho$ and hence $z(n) > 0$ or < 0 for $n \ge n_1 > n_0 + 2\rho$. If $z(n) > 0$ for $n \ge n_1$, then one of the cases (a) and (b) of Lemma 1.1 holds and $y(n) > -p(n)y(n-m) > y(n-m)$. Hence $\liminf\limits_{n\to\infty} y(n) > 0$. From (10), it follows that $\sum\limits_{n=n_2}^{\infty} q(n) < \infty$, $n_2 > n_1$, a contradiction to (A_{13}). Thus $z(n) < 0$ for $n > n_1$. Since $z(n)$ is bounded, none the cases (c), (d) and (e) of Lemma 1.1 occurs. Considering the case (b) and if $-\infty < \lim\limits_{n\to\infty} z(n) < 0$, then proceeding as in the proof of Theorem 2.5, the contradiction is obtained. If $\lim\limits_{n\to\infty} z(n) = 0$, then $0 = \liminf\limits_{n\to\infty} z(n) \le \liminf\limits_{n\to\infty}(y(n)+p_2 y(n-m)) \le \limsup\limits_{n\to\infty} y(n) + \liminf\limits_{n\to\infty}(p_2 y(n-m)) = (1+p_2)\limsup\limits_{n\to\infty} y(n)$, that is, $\limsup\limits_{n\to\infty} y(n) = 0$. Hence $\lim\limits_{n\to\infty} y(n) = 0$. If $y(n) < 0$, for $n \ge n_0$, then setting $x(n) = -y(n) > 0$ for $n \ge n_0$, Eq. (1) becomes

$$\Delta^2\left[r(n)\Delta^2(x(n) + p(n)x(n-m))\right] + q(n)\tilde{G}(x(n-k)) = 0,$$

where $\tilde{G}(u) = -G(-u)$. Proceeding as above we obtain $\lim\limits_{n\to\infty} x(n) = 0$ and hence $\lim\limits_{n\to\infty} y(n) = 0$. This completes the proof of the theorem. $\square$

Example 3. Consider

$$\Delta^2\left[2^{-n}\Delta^2(y(n) - (2+2^{-n})y(n-1))\right] + \left(\frac{27}{1024}\mathrm{e}^n + \frac{441}{8192}y^3(n-2)\right) = 0, \qquad n \ge 0.$$

From Theorem 2.6, it follows that every solution of the equation oscillates or tends to zero as $n \to \infty$. In particular, $y(n) = 2^{-(n+1)}$ is such a solution.

3. Oscillation of forced equations

This section deals with the oscillation of all solutions of (2). In the following, we obtain sufficient conditions of oscillation of solutions of forced equation (2). Let

(A_{14}) there exist a real valued function $F(n)$ such that it changes sign and
$$\Delta^2(r(n)\Delta^2 F(n)) = f(n);$$

(A_{15}) there exist a real valued function $F(n)$ such that $F(n)$ changes sign with
$$-\infty < \liminf_{n\to\infty} F(n) < 0 < \limsup_{n\to\infty} F(n) < \infty \quad \text{and} \quad \Delta^2(r(n)\Delta^2 F(n)) = f(n);$$

(A_{16}) there exist a real valued function $F(n)$ such that $F(n)$ does not change and sign
$$\lim_{n\to\infty} F(n) = 0 \qquad \text{and} \qquad \Delta^2(r(n)\Delta^2 F(n)) = f(n);$$

(A_{17}) there exist a real valued function $F(n)$ such that
$$\lim_{n\to\infty} F(n) = 0 \qquad \text{and} \qquad \Delta^2(r(n)\Delta^2 F(n)) = f(n);$$

Remark 5. If $\lim\limits_{n\to\infty} F(n) = \alpha \neq 0$ in (A_{16}), then we may proceed as follows:

We set $\tilde{F}(n) = F(n) - \alpha$ to obtain $\Delta^2\tilde{F}(n) = \Delta^2 F(n)$ and hence $\lim\limits_{n\to\infty} \tilde{F}(n) = 0$. If $\tilde{F}(n)$ changes sign, then it comes under (A_{14}). If $\tilde{F}(n)$ does not change sign, then it comes under (A_{16}).

(A_{18}) $\sum\limits_{n=k}^{\infty} Q(n)G(F^+(n-k)) = \infty = \sum\limits_{n=k}^{\infty} Q(n)G(F^-(n-k))$, where $F^+(n) = \max\{F(n), 0\}$ and $F^-(n) = \max\{-F(n), 0\}$;

(A_{19}) $\sum\limits_{n=k}^{\infty} q(n)G(F^+(n-k)) = \infty = \sum\limits_{n=k}^{\infty} q(n)G(F^-(n+m-k))$;

(A_{20}) $\sum\limits_{n=k}^{\infty} q(n)G(F^-(n-k)) = \infty = \sum\limits_{n=k}^{\infty} q(n)G(F^+(n+m-k))$.

Theorem 3.1. *Let $0 \le p(n) \le p < \infty$. Suppose that* ($A_1$), ($A_2$), ($A_3^1$), ($A_6$), ($A_{14}$) *and* ($A_{18}$) *hold. Then all solutions of* (2) *oscillate.*

Proof. Let $y(n)$ be a non-oscillatory of (2). Hence $y(n) > 0$ or < 0 for $n \ge n_0 > 0$. Suppose that $y(n) > 0$ for $n \ge n_0$. Setting $z(n)$ as in (8) and
$$w(n) = z(n) - F(n) \tag{13}$$
we obtain
$$\Delta^2(r(n)\Delta^2 w(n)) = -q(n)G(y(n-k)) \le 0, \tag{14}$$

for $n \geq n_0+\rho$. Thus $w(n)>0$ or <0, for $n \geq n_1 > n_0+2\rho$. Since $F(n)$ changes sign, then $w(n)>0$ for $n \geq n_1$ by (13). Hence one of the cases (a) and (b) of Lemma 1.1 holds for large n and $z(n) \geq F^+(n)$. For $n \geq n_2 > n_1$, we have

$$\begin{aligned} 0 &= \Delta^2\left[r(n)\Delta^2 w(n)\right] + G(p)\Delta^2\left[r(n-m)\Delta^2 w(n-m)\right] + q(n)G(y(n-k)) \\ &\quad + G(p)q(n-m)G(y(n-m-k)) \\ &\geq \Delta^2\left[r(n)\Delta^2 w(n)\right] + G(p)\Delta^2\left[r(n-m)\Delta^2 w(n-m)\right] + \lambda Q(n)G(y(n-m) \\ &\quad + py(n-m-k)) \\ &\geq \Delta^2\left[r(n)\Delta^2 w(n)\right] + G(p)\Delta^2\left[r(n-m)\Delta^2 w(n-m)\right] + \lambda Q(n)G(z(n-k)) \\ &\geq \Delta^2\left[r(n)\Delta^2 w(n)\right] + G(p)\Delta^2\left[r(n-m)\Delta^2 w(n-m)\right] + \lambda Q(n)G(F^+(n-k)) \end{aligned} \tag{15}$$

Hence

$$\sum_{n=n_2+k}^{\infty} Q(n)G(F^+(n-k)) < \infty,$$

a contradiction to (A_{18}). If $y(n)<0$ for $n \geq n_0$, set $x(n) = -y(n)$ to obtain $x(n)>0$ for $n \geq n_0$ and

$$\Delta^2(r(n)\Delta^2(x(n)+p(n)x(n-m))) + q(n)\tilde{G}(x(n-k)) = \tilde{f}(n),$$

where $\tilde{f}(n) = -f(n)$. If $\tilde{F}(n) = -F(n)$, then $\tilde{F}(n)$ changes sign $\tilde{F}^+(n) = F^-(n)$ and $\Delta^2(r(n)\Delta^2\tilde{F}(n)) = \tilde{f}(n)$. Proceeding as above we obtain a contradiction. Hence the theorem is proved. □

Theorem 3.2. *Let $-1 < p \leq p(n) \leq 0$. Suppose that* (A_1), (A_{15}), (A_{19}) *and* (A_{20}) *hold. Then every solution of* (2) *oscillates.*

Proof. Proceeding as in proof of Theorem 3.1 we obtain $w(n)>0$ or <0, for $n \geq n_1 > n_0+\rho$ when $y(n)>0$ for $n \geq n_0$. Let $w(n)>0$ for $n \geq n_1$. Hence one of the cases (a) and (b) of Lemma 1.1 holds. Further $w(n)>0$ implies that $y(n) \geq y(n)+p(n)y(n-m) > F(n)$ and hence $y(n) \geq F^+(n)$. From (14) we obtain $\sum_{n=n_1+k}^{\infty} q(n)G(F^+(n-k)) < \infty$, a contradiction. Hence $w(n)<0$, for $n \geq n_1$. Then one of the cases (b)–(e) of Lemma 1.1 holds. Let the case (b) hold. $w(n)<0$ implies that $y(n) > F^-(n+m))$ for $n \geq n_1$. Consequently, (14) gives

$$\sum_{n=n_1+k}^{\infty} q(n)G(F^-(n+m-k)) < \infty,$$

a contradiction. If $y(n)$ is unbounded, then there exists a subsequence $\{n'_j\}$ of $\{n\}$ such that $n'_j \to \infty$ and $y(n'_j) \to \infty$ as $j \to \infty$ and $y(n'_j) = \max\{y(n) :\ n_1 \leq$

$n \le n'_j\}$. For $n'_j > n_1$, we obtain $w(n'_j) = y(n'_j) + p(n'_j)y(n'_j - m) - F(n'_j) \ge (1+p)y(n'_j) - F(n'_j)$. Since $F(n)$ is bounded and $(1+p) > 0$, then $w(n'_j) > 0$ for large n'_j, a contradiction. Hence $y(n)$ is bounded, that is, $w(n)$ is bounded. Consequently, the cases (c) and (d) of Lemma 1.1 fail to hold. On the other hand, $w(n)$ is bounded and (A_1) implies that the case (e) of Lemma 1.1 does not hold. If $y(n) < 0$ for $n \ge n_0$, then setting $x(n) = -y(n) > 0$ for $n \ge n_0$, Eq. (2) becomes

$$\Delta^2\left[(r(n)\Delta^2(x(n) + p(n)x(n-m))\right] + q(n)\tilde{G}(x(n-k)) = \tilde{f}(n),$$

where $\tilde{G}(u) = -G(u)$ and $\tilde{f}(n) = -f(n)$. If $\tilde{F}(n) = -F(n)$, then $\tilde{F}(n)$ changes sign with $-\infty < \liminf_{n\to\infty} \tilde{F}(n) < 0 < \limsup_{n\to\infty} \tilde{F}(n) < \infty$, $\tilde{F}(n) = F^-(n)$, $\tilde{F}^-(n) = F^+(n)$ and $\Delta^2(r(n)\Delta^2\tilde{F}(n) = \tilde{f}(n)$. Proceeding as above a contradiction is obtained. Thus the theorem is proved. □

Example 4. Consider

$$\Delta^2\left[(\mathrm{e}^{-n}\Delta^2(y(n) + p(n)y(n-1))\right] + q(n)y^3(n-2) = f(n), \qquad n \ge 0, \quad (16)$$

where $p(n) = 2(1 + (-1)^n)$, $q(n) = \left[\mathrm{e}^n + \left(8\mathrm{e}^{-1} + 4\mathrm{e}^{-2}\right)\mathrm{e}^{-n}\right]$ and $f(n) = \left(\mathrm{e}^n - 4\mathrm{e}^{-n}\right)(-1)^n$, $Q(n) = \min\{q(n), q(n-1)\} = \mathrm{e}^{n-1} + \left(8 + 4\mathrm{e}^{-1}\right)\mathrm{e}^{-n}$. If we define

$$F(n) = \left[\frac{\mathrm{e}^{2n}}{(e+1)^2(\mathrm{e}^2+1)^2} - \frac{1}{(\mathrm{e}^{-1}+1)^2}\right](-1)^n,$$

then $\Delta^2[\mathrm{e}^{-n}\Delta^2 F(n)] = (\mathrm{e}^n - 4\mathrm{e}^{-n})(-1)^n$. Hence

$$F^+(n-2) = \begin{cases} \frac{\mathrm{e}^{2n}}{\mathrm{e}^4(\mathrm{e}+1)^2(\mathrm{e}^2+1)^2} - \frac{1}{(\mathrm{e}^{-1}+1)^2}, & \text{if } n \text{ is even},\\ 0, & \text{if } n \text{ is odd}, \end{cases}$$

$$F^-(n-2) = \begin{cases} 0, & \text{if } n \text{ is even},\\ \frac{\mathrm{e}^{2n}}{\mathrm{e}^4(\mathrm{e}+1)^2(\mathrm{e}^2+1)^2} - \frac{1}{(\mathrm{e}^{-1}+1)^2}, & \text{if } n \text{ is odd}. \end{cases}$$

Consequently,

$$\sum_{n=2}^{\infty} Q(n)G(F^+(n-2)) = \sum_{n=2}^{\infty}\left[\mathrm{e}^{n-1} + (8 + 4\mathrm{e}^{-1})\mathrm{e}^{-n}\right]\left[F^+(n-2)\right]^3 = \infty$$

and

$$\sum_{n=2}^{\infty} Q(n)G(F^-(n-2)) = \sum_{n=2}^{\infty}\left[\mathrm{e}^{n-1} + (8 + 4\mathrm{e}^{-1})\mathrm{e}^{-n}\right]\left[F^-(n-2)\right]^3 = \infty.$$

From Theorem 3.1 it follows that all solutions of (16) oscillate. In particular $y(n) = (-1)^n$ is an oscillatory solution of (16).

Theorem 3.3. *Let $-\infty < p \le p(n) \le 0$. If* ($A_1$), ($A_3$), ($A_{15}$), ($A_{19}$) *and* ($A_{20}$) *hold, then every solution of* (2) *oscillates or tends to $\pm\infty$ as $n \to \infty$.*

Proof. Proceeding as in the proof of Theorem 3.2 we obtain a contradiction if $w(n) > 0$ for $n \ge n_1 > n_0 + \rho$. Hence $w(n) < 0$ for $n \ge n_1$. Then one of the cases (b)–(e) of Lemma 1.1 holds. Let the case (b) hold. Clearly, $py(n-m) < F(n)$ due to $w(n) < 0$, that is, $y(n) > (-p^{-1})F^{-}(n+m)$ for $n \ge n_1$. Using (A_3) it follows from (14) that

$$\sum_{n=n_1+k}^{\infty} q(n)G(F^{-}(n+m-k)) < \infty,$$

a contradiction. In each of the cases (c) and (d), $\lim_{n\to\infty} w(n) = -\infty$. However, if $-\infty < \lim_{n-\to\infty} w(n) \le 0$, then we obtain a contradiction due to (A_1). Hence for each of the cases (c)–(e), $\lim_{n\to\infty} w(n) = -\infty$. Consequently, $py(n-m) < w(n) + F(n)$ implies that $\limsup_{n\to\infty} y(n-m) \le \lim_{n\to\infty} w(n) + \limsup_{n\to\infty} F(n)$, that is, $p \liminf_{n\to\infty} y(n) = -\infty$ due to (A_{15}). Hence $\lim_{n\to\infty} y(n) = \infty$. The proof for the case $y(n) < 0$ for $n \ge n_0$ is similar. Thus the proof of the theorem is complete. $\square$

Corollary 3.4. *If the conditions of* Theorem 3.3 *are satisfied, then every bounded solution of* (2) *oscillates.*

Theorem 3.5. *Let $0 \le p(n) \le p < \infty$ and let* (A_1), (A_2), (A_3^1), (A_6) *and* (A_{16}) *hold. If $\sum_{n=k}^{\infty} Q(n)G(|F(n-k)|) = \infty$, then every bounded solution of* (2) *oscillates or tends to zero as $n \to \infty$.*

Proof. Proceeding as in Theorem 3.1 we obtain $w(n) > 0$ or < 0 for $n \ge n_1 > n_0 + \rho$. Let $w(n) > 0$ for $n \ge n_1$. Hence $z(n) > F(n)$. Suppose that $F(n) > 0$ for $n \ge n_2 > n_1$. From (15) and Lemma 1.1, it follows that $\sum_{n=n_2+k}^{\infty} Q(n)G(F(n-k)) = \infty$, a contradiction. Hence $F(n) < 0$ for $n \ge n_2$. From (14) we obtain $\sum_{n=n_2+k}^{\infty} Q(n)G(y(n-k)) < \infty$ due to Lemma 1.1. Thus $\liminf_{n\to\infty} y(n) = 0$, because $\sum_{n=n_2+k}^{\infty} Q(n)G(|F(n-k)|) < \infty$ implies that $\sum_{n=k}^{\infty} q(n) = \infty$. Since $w(n)$ is bounded and monotonic, then $\lim_{n\to\infty} w(n)$ exists and hence $\lim_{n\to\infty} z(n)$ exists. Thus $\lim_{n\to\infty} z(n) = 0$ ([4, Lemma 2.1]). Consequently, $z(n) \ge y(n)$ implies that $\lim_{n\to\infty} y(n) = 0$. Let $w(n) < 0$ for $n > n_1$. Then $y(n) \le z(n) < F(n)$ implies that $\lim_{n\to\infty} y(n) = 0$. Thus the theorem is proved. $\square$

Theorem 3.6. *Let $-1 < p \le p(n) \le 0$. (A_1), (A_{13}) and (A_{16}) hold, then every solution of* (2)*, oscillates or tends to zero as $n \to \infty$.*

P r o o f. Proceeding as in the proof of Theorem 3.1, we have $w(n) > 0$ or < 0 for $n \ge n_1 > n_0 + \rho$. Let $w(n) >$ for $n \ge n_1$. From (14) we obtain due to Lemma 1.1 that

$$\sum_{n=n_2+k}^{\infty} q(n)G(y(n-k)) < \infty, \qquad n_2 > n_1. \tag{17}$$

Hence $\liminf_{n\to\infty} y(n) = 0$. On the other hand, $\lim_{n\to\infty} w(n) = \infty$ in case (a) of Lemma 1.1. Then it follows that $\lim_{n\to\infty} z(n) = \infty$. However, $y(n) \ge z(n)$ implies that $y(n) \to \infty$ as $n \to \infty$, a contradiction. In case of Lemma 1.1(b), $\lim_{n\to\infty} w(n) = \alpha$, $0 < \alpha \le \infty$. The above contradiction is obtained if $\alpha = \infty$. Hence $0 < \alpha < \infty$. Consequently, $\lim_{n\to\infty} z(n) = \alpha$. From [4, Lemma 2.1], it follows that $\alpha = 0$, a contradiction. Thus $w(n) < 0$ for $n > n_1$. Following to Theorem 3.2 we obtain that $y(n)$ is bounded and hence $w(n)$ is bounded. In each of the cases (c) and (d) of Lemma 1.1, $\lim_{n\to\infty} w(n) = -\infty$, a contradiction. In each of the cases (b) and (e) of Lemma 1.1, (17) holds and hence $\liminf_{n\to\infty} y(n) = 0$. Consequently, $\lim_{n\to\infty} z(n)$ exists. Using [4, Lemma 2.1], we have $\lim_{n\to\infty} z(n) = 0$. Hence $0 = \lim_{n\to\infty} z(n) = \limsup_{n\to\infty} [y(n) + p(n)y(n-m)] \ge \limsup_{n\to\infty} y(n) + \liminf_{n\to\infty} [py(n-m)] = (1+p) \limsup_{n\to\infty} y(n)$, that is $\lim_{n\to\infty} y(n) = 0$. The proof of the theorem is complete. $\square$

Example 5. Consider

$$\Delta^2 \left[e^{-2n} \Delta^2 \left(y(n) + (e^{-n} - 1)y(n-1)\right)\right] + q(n)y^3(n-2) = e^{-3n}, \qquad n \ge 1 \tag{18}$$

$q(n) = e^{-6} + e^{-6}(e-1)(e^{-1}-1)^2(e^{-3}-1)^2 - e(e^{-2}-1)^2(e^{-4}-1)^2 e^{-n}$. Here $-1 < -e^{-1} < p(n) < 0$ and $f(n) = e^{-3n}$. If $F(n) = (e^{-3}-1)^{-2}(e^{-1}-1)^{-2}e^{-n}$, then $\Delta^2 \left[e^{-2n}\Delta^2 F(n)\right] = e^{-3n}$ and $\lim_{n\to\infty} F(n) = 0$. As all the conditions of Theorem 3.6 are satisfied, then every solution of (18) oscillates or tends to zero as $n \to \infty$. In particular, $y(n) = e^{-n}$ is a solution of (18) such that $y(n) \to 0$ as $n \to \infty$.

Theorem 3.7. *Let $-\infty < p(n) \le 0$. If (A_1), (A_{13}) and (A_{16}) hold, then every bounded solution of* (2) *oscillates or tends to zero as $n \to \infty$.*

The proof is similar to that of Theorem 3.6 and hence is omitted.

Corollary 3.8. *Suppose that the conditions of* Theorem 3.7 *are satisfied. Then every non-oscillatory solution of* (2) *which does not tend to zero as $n \to \infty$ is unbounded.*

Remark 6. Theorems 3.1–3.3, 3.5 and Corollary 3.4 do not hold for Eq. (1). However, Theorems 3.6 and 3.7 hold for Eq. (1).

4. Existence of positive solutions

In this section, sufficient conditions are obtained for the existence of bounded positive solutions of Eq. (2).

Theorem 4.1. *Let* $0 \le p(n) \le p < 1$. *Suppose that* (A_{15}) *holds with* $-\frac{1}{8}(1-p) < \liminf_{n\to\infty} F(n) < 0 < \limsup_{n\to\infty} F(n) < (1/2)(1-p)$ *and* G *is Lipschitzian on intervals of the form* $[a,b]$, $0 < a, b, \infty$. *If*

$$\sum_{n=0}^{\infty} \frac{(n+1)}{r(n)} \sum_{s=n}^{\infty} (s+1)q(s) < \infty, \tag{19}$$

then (2) *admits a positive bounded solution.*

P r o o f. It is possible to choose a positive integer N_1 such that

$$L \sum_{n=N_1}^{\infty} \frac{(n+1)}{r(s)} \sum_{s=n}^{\infty} (s+1)q(s) < \frac{1}{4}(1-p),$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $\left[\frac{1}{8}(1-p), 1\right]$. Let $X = l_\infty^{N_1}$, Banach space of all real valued functions $x(n)$, $n > N_1$, with supremum norm $\|x\| = \sup\left\{|x(n)| : \ n > N_1\right\}$. Define

$$S = \left\{x \in X : \ \tfrac{1}{8}(1-p) \le x(n) \le 1, \ \ n > N_1\right\}.$$

Hence S is a complete metric space, when the metric is induced by the norm on X. For $y \in S$, define

$$Ty(n) = \begin{cases} Ty(N_1+\rho), & N_1 \le n < N_1+\rho, \\ -p(n)y(n-m) + \frac{p+1}{2} + F(n), & \\ -\sum\limits_{i=n}^{\infty} \frac{(i-n+1)}{r(i)} \sum\limits_{s=i}^{\infty} (s-i+1)q(s)G(y(s-k)), & n \ge N_1+\rho. \end{cases}$$

Hence $Ty(n) < \frac{1+p}{2} + \frac{1-p}{2} = 1$ and $Ty(n) > -p + \frac{1+p}{2} - \frac{1-p}{8} - \frac{1-p}{4} = \frac{1-p}{8}$ for $n \ge N_1 + \rho$. Consequently $Ty \in S$, that is, $T\colon S \to S$. Further, for $x, y \in S$,

$$|Ty(n) - Tx(n)| \le p\|x-y\| + \frac{1-p}{4}\|x-y\| = \frac{1+3p}{4}\|x-y\|.$$

Hence $\|Ty - Tx\| \le \frac{1+3p}{4}\|x-y\|$, for every $x, y \in S$. Thus T is a contraction. Consequently, T has a unique fixed point y in S. Then $y(n)$ is a solution of (2) with $\frac{1}{8}(1-p) \le y(n) \le 1$. Thus the theorem is proved. □

Theorem 4.2. *Let $0 < p(n) \leq p < 1$. Suppose that* (A_{17}) *and* (19) *hold. If G is Lipschitzian on intervals of the form $[a, b]$, $0 < a < b < \infty$, then* (2) *admits a positive bounded solution.*

P r o o f. We may choose N_1 sufficiently large such that $|F(n)| < \frac{1-p}{20}$ and

$$L \sum_{n=N_1}^{\infty} \frac{(n+1)}{r(n)} \sum_{s=n}^{\infty} (s+1)q(s) < \frac{1-p}{20},$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $\left[\frac{1-b}{20}, 1\right]$. We set $X = l_{\infty}^{N_1}$ and

$$S = \left\{x \in X : \ \tfrac{1-p}{20} \leq x(n) \leq 1, \ \ n \geq N_1\right\}.$$

For $y \in S$, we set

$$Ty(n) = \begin{cases} Ty(N_1 + \rho), & N_1 \leq n \leq N_1 + \rho, \\ -p(n)y(n-m) + \frac{1+4p}{5} + F(n), & \\ -\sum\limits_{i=n}^{\infty} \frac{(i-n+1)}{r(i)} \sum\limits_{s=n}^{\infty} (s+1)q(s)G(y(s-k)), & n \geq N_1 + \rho. \end{cases}$$

Proceeding as in the proof of Theorem 4.1 we may show that T has a unique fixed point y in S and it is the required solution of (2). This completes the proof of the theorem. □

Remark 7. Theorems similar to Theorems 4.1 and 4.2 can be proved in other ranges of $p(n)$.

5. Summary

In [8], Eq. (3) is studied with even and odd m. When $r(n) \equiv 1$, the results for super linear case are hold to that of the results in [8]. Other than $r(n) \equiv 1$, the present work is more general than the works in [5] and [8]. Equations (1) and (2) are studied under the assumption $\sum\limits_{n=0}^{\infty} \frac{n}{r(n)} < \infty$ in a separate paper. It would be interesting to study neutral difference equations with quasi-differences of the form

$$\Delta(r_3(n)(\Delta r_2(n)(\Delta r_1(n)\Delta(y(n) + p(n)y(n-m))))) + q(n)G(y(n-k)) = f(n).$$

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