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ON CALCULATION OF GENERALIZED DENSITIES

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ABSTRACT. In the paper continuous variants of densities of sets of positive integers are introduced, some of their properties are studied and formula for their calculation is proved.

1. Introduction

Asymptotic and logarithmic densities are standard means used to evaluate sizes of sets of positive integers. These concepts can be generalized to the so called concept of weighted densities defined as follows. Denote $\mathbb{R}_0^+$, $\mathbb{N}$ the set of all nonnegative real numbers and positive integers respectively and let $w\colon \mathbb{N} \to \mathbb{R}_0^+$ be a (weight) function with $w(1) > 0$ which satisfies

$$\sum_{n=1}^{\infty} w(n) = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{w(n)}{\sum\limits_{i=1}^{n} w(i)} = 0. \tag{1}$$

For $A \subset \mathbb{N}$ we define the lower and upper densities of A with respect to the weight function w, or w-density of A as follows.

$$\underline{d}_w(A) = \liminf_{n\to\infty} \frac{\sum\limits_{i\in A,\, i\le n} w(i)}{\sum\limits_{i\in\mathbb{N},\, i\le n} w(i)}, \qquad \overline{d}_w(A) = \limsup_{n\to\infty} \frac{\sum\limits_{i\in A,\, i\le n} w(i)}{\sum\limits_{i\in\mathbb{N},\, i\le n} w(i)}.$$

To calculate densities of sets is a standard task occurring frequently in papers on density theory, see e.g. ([1], [2], [3], [4], [5], [6], [7], [8]). Usually the sets in question are written as infinite union of consecutive blocks of positive integers

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and there is no general formula for densities of such sets. The aim of the present paper is to derive such a formula in the case when blocks and gaps of sets are distributed in some sense regularly. In addition we will extend the whole theory by considering the following continuous variants of the above concepts. Denote by $\mathscr{F}$ the set of all functions $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ integrable in each bounded subinterval of $\mathbb{R}_0^+$ and such that

$$\int_0^\infty f(t)\,\mathrm{d}t = \infty \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\int_{x-1}^{x} f(t)\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t} = 0. \tag{2}$$

Similarly denote $\mathscr{A}$ the set of all functions $a\colon \mathbb{R}_0^+ \to [0,1]$ integrable in each bounded subinterval of $\mathbb{R}_0^+$. For $f \in \mathscr{F}$ and $a \in \mathscr{A}$ we define the lower and upper densities of the function a with respect to f or shortly f-density of a as follows:

$$\underline{d}_f(a) = \liminf_{x\to\infty} \frac{\int_0^x f(t)a(t)\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t}, \qquad \overline{d}_f(a) = \limsup_{x\to\infty} \frac{\int_0^x f(t)a(t)\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t}.$$

In order to prevent from substantial dramatic changes in values of the weight function f, in the first lemma we will assume that the set $\mathbb{R}_0^+$ can be written in the form of disjoint union of intervals — $\mathbb{R}_0^+ = \bigcup_{n=1}^{\infty} J_n$ (where $\sup\{x:\ x \in J_n\} = \inf\{x:\ x \in J_{n+1}\} = j_n$) — for which $\inf\{j_n - j_{n-1}:\ n \geq 2\} > 0$, such that the following proposition holds:

$$\exists c > 0\ \forall n \in \mathbb{N}\ \forall x \in J_n - \Omega_n:\ f(x) \geq c \sup\{f(y):\ y \in J_n\} \tag{3}$$

for Ω_n a subset of measure zero of J_n .

2. Results

The first lemma says that, under certain conditions on f, if a function a is sufficiently close to the characteristic function χ_A of a set A then their f-densities equal.

Lemma 2.1. *Suppose that for $f \in \mathscr{F}$ both* (2) *and* (3) *hold. Let $A = \bigcup_{n=1}^{\infty} (c_n, d_n]$ where $0 < c_n < d_n < c_{n+1}$ and $\min\{d_n - c_n, c_{n+1} - d_n\} > \delta$ hold for every $n \in \mathbb{N}$*

and some constant $\delta > 0$. *Let* $a \in \mathscr{A}$ *satisfy* $a(t) = 0$ *for* $t \in \mathbb{R} - \bigcup_{n=1}^{\infty} (c'_n, d'_n]$ *where*

$$\lim_{n\to\infty} \left(|c_n - c'_n| + |d'_n - d_n|\right) = 0 \tag{4}$$

and

$$\lim_{n\to\infty} \left(\int_{c'_n}^{d'_n} a(t)\,\mathrm{d}t - (d_n - c_n) \right) = 0. \tag{5}$$

Then

$$\underline{d}_f(a) = \underline{d}_f(\chi_A) \qquad \text{and} \qquad \overline{d}_f(a) = \overline{d}_f(\chi_A).$$

P r o o f. We will only prove that $\underline{d}_f(a) = \underline{d}_f(\chi_A)$. Similar steps lead to proving the second equality. For $n \in \mathbb{N}$ denote $I_n = (c_n, d_n] \cup (c'_n, d'_n]$, $K_n = I_n - (c_n, d_n]$ and notice that by (4), I_n and I_{n+1} are disjoint intervals for all sufficiently large values of n. First we will show that (5) implies

$$\lim_{n\to\infty} \int_{I_n} |a(t) - \chi_A(t)|\,\mathrm{d}t = 0. \tag{6}$$

Denoting $x_n = \int_{K_n} (a(t) - \chi_A(t))\,\mathrm{d}t$ and $y_n = \int_{c_n}^{d_n} (a(t) - \chi_A(t))\,\mathrm{d}t$ we have for all sufficiently large values of n

$$\int_{c'_n}^{d'_n} a(t)\,\mathrm{d}t - (d_n - c_n) = \int_{I_n} (a(t) - \chi_A(t))\,\mathrm{d}t = x_n + y_n$$

and

$$\int_{I_n} |a(t) - \chi_A(t)|\,\mathrm{d}t = \int_{K_n} (a(t) - \chi_A(t))\,\mathrm{d}t - \int_{c_n}^{d_n} (a(t) - \chi_A(t))\,\mathrm{d}t = x_n - y_n.$$

Notice that (4) implies $\lim_{n\to\infty} x_n = 0$ and (6) follows from the fact that

$$\lim_{n\to\infty} x_n + y_n = 0 \quad \text{and} \quad \lim_{n\to\infty} x_n = 0 \qquad \text{implies} \quad \lim_{n\to\infty} x_n - y_n = 0.$$

Realize that if we divide interval J_n (see(3)) into finite number of disjoint intervals, say R_i, $i \in \{1, 2, \dots, k\}$, then for every $x \in R_i - \Omega_n$ it holds that $f(x) \geq c \sup\{f(y) :\ y \in R_i)\}$. Thus

$$\mathbb{R}_0^+ = \bigcup_{n=1}^{\infty} R_n,$$

where (for $|R_n| = \int\limits_{R_n} 1\,\mathrm{d}t$) we have $|R_n| \leq \left(\inf\{c_{n+1} - d_n :\ n \in \mathbb{N}\}\right)/2$ and $\inf\big\{|R_n| :\ n \in \mathbb{N}\big\} > 0$ (and (3) holds when writing R_n instead of J_n).

Now we will prove that, putting $r_n = \sup\{x :\ x \in R_n\}$,

$$\liminf_{x\to\infty} \frac{\int\limits_0^x f(t)a(t)\,\mathrm{d}t}{\int\limits_0^x f(t)\,\mathrm{d}t} = \liminf_{n\to\infty} \frac{\int\limits_0^{r_n} f(t)a(t)\,\mathrm{d}t}{\int\limits_0^{r_n} f(t)\,\mathrm{d}t}. \tag{7}$$

Let $x \in R_n$ and $\frac{\int_0^x f(t)a(t)\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t} < \frac{\int_0^{r_n} f(t)a(t)\,\mathrm{d}t}{\int_0^{r_n} f(t)\,\mathrm{d}t}$. Then

$$\begin{aligned}
&\frac{\int\limits_0^{r_n} f(t)a(t)\,\mathrm{d}t}{\int\limits_0^{r_n} f(t)\,\mathrm{d}t} - \frac{\int\limits_0^{x} f(t)a(t)\,\mathrm{d}t}{\int\limits_0^{x} f(t)\,\mathrm{d}t}\\
&= \frac{\left(\int\limits_0^x f(t)a(t)\,\mathrm{d}t + \int\limits_x^{r_n} f(t)a(t)\,\mathrm{d}t\right)\int\limits_0^x f(t)\,\mathrm{d}t - \int\limits_0^x f(t)a(t)\,\mathrm{d}t\left(\int\limits_0^x f(t)\,\mathrm{d}t + \int\limits_x^{r_n} f(t)\,\mathrm{d}t\right)}{\int\limits_0^x f(t)\,\mathrm{d}t \int\limits_0^{r_n} f(t)\,\mathrm{d}t}\\
&\leq \frac{\int\limits_x^{r_n} f(t)\,\mathrm{d}t \int\limits_0^x f(t)(1-a(t))\,\mathrm{d}t}{\int\limits_0^x f(t)\,\mathrm{d}t \int\limits_0^{r_n} f(t)\,\mathrm{d}t} \leq \frac{\int\limits_x^{r_n} f(t)\,\mathrm{d}t}{\int\limits_0^{r_n} f(t)\,\mathrm{d}t}.
\end{aligned}$$

As the sequence $(|R_n|)_{n=1}^{\infty}$ is bounded, we get (7) by the second equality in (2).

Now we have to prove that $\liminf\limits_{n\to\infty} \frac{\int_0^{r_n} f(t)a(t)\,\mathrm{d}t}{\int_0^{r_n} f(t)\,\mathrm{d}t} = \liminf\limits_{n\to\infty} \frac{\int_0^{r_n} f(t)\chi_A(t)\,\mathrm{d}t}{\int_0^{r_n} f(t)\,\mathrm{d}t}$. Let $n_0 \in \mathbb{N}$ be such that I_n is an interval and $\inf\{x :\ x \in I_{n+1}\} - \sup\{x :\ x \in I_n\} > \sup\big\{|R_n| :\ n \in \mathbb{N}\big\}$ holds for all $n \geq n_0$. Due to the first equality in (2) we have $\underline{d}_f(a) = \underline{d}_f(a^*)$, where $a^* \in \mathscr{A}$ can differ from a on a bounded subinterval of $\mathbb{R}_0^+$. Thus we can assume that $a(t) = \chi_A(t)$ for $t \in \big[0, \sup\{x :\ x \in I_{n_0}\}\big]$. To complete the proof, we shall show that

$$\lim_{n\to\infty} \frac{\int\limits_0^{r_n} f(t)(a(t) - \chi_A(t))\,\mathrm{d}t}{\int\limits_0^{r_n} f(t)\,\mathrm{d}t} = 0. \tag{8}$$

Denoting $\mathbb{I}_n = \bigcup_{j=1}^{\infty} I_j \cap [0, r_n]$ we have

$$\left| \int_0^{r_n} f(t)(a(t) - \chi_A(t))\, \mathrm{d}t \right| \le \sum_{m\colon R_m \cap \mathbb{I}_n \neq \emptyset} \int_{R_m} f(t)|a(t) - \chi_A(t)|\, \mathrm{d}t.$$

Suppose that $R_m \cap \mathbb{I}_n \neq \emptyset$. Then we obtain

$$\int_{R_m} f(t)|a(t) - \chi_A(t)|\, \mathrm{d}t \le \sup\bigl\{f(y) :\ y \in R_m\bigr\} \int_{R_m} |a(t) - \chi_A(t)|\, \mathrm{d}t$$
$$\le \sup\bigl\{f(y) :\ y \in R_m\bigr\} \int_{I_j} |a(t) - \chi_A(t)|\, \mathrm{d}t$$

for such (uniquely determined) $j \in \mathbb{N}$ for which $R_m \cap I_j \neq \emptyset$. Further

$$\begin{aligned}
&\sup\bigl\{f(y) : y \in R_m\bigr\} \int_{I_j} |a(t) - \chi_A(t)|\, \mathrm{d}t \\
&\qquad \le \inf\bigl\{f(y) :\ y \in R_m - \Omega_m\bigr\} \frac{1}{c} \int_{I_j} |a(t) - \chi_A(t)|\, \mathrm{d}t \\
&\qquad = \frac{\int_{I_j} |a(t) - \chi_A(t)|\, \mathrm{d}t}{|R_m| c} \int_{R_m} \inf\bigl\{f(y) :\ y \in R_m - \Omega_m\bigr\} \\
&\qquad \le \frac{\int_{I_j} |a(t) - \chi_A(t)|\, \mathrm{d}t}{|R_m| c} \int_{R_m} f(t)\, \mathrm{d}t.
\end{aligned}$$

We have

$$\lim_{n\to\infty} \frac{\left| \int_0^{r_n} f(t)(a(t) - \chi_A(t))\, \mathrm{d}t \right|}{\int_0^{r_n} f(t)\, \mathrm{d}t} \le \lim_{n\to\infty} \frac{\sum_{m\colon R_m \cap \mathbb{I}_n \neq \emptyset} \frac{\int_{I_j} |a(t)-\chi_A(t)|\, \mathrm{d}t}{|R_m| c} \int_{R_m} f(t)\, \mathrm{d}t}{\sum_{m\colon R_m \cap [0, r_n] \neq \emptyset} \int_{R_m} f(t)\, \mathrm{d}t}$$

and so (8) follows. Indeed — take into account the first equality in (2) and (6). □

The next lemma says that if f and g are close then f- and g-densities of a are equal.

Lemma 2.2. *Let $f, g \in \mathscr{F}$ be such that $\lim\limits_{x\to\infty} \frac{\int_0^x |f(t)-g(t)|\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t} = 0$. Then for every $a \in \mathscr{A}$ we get $\underline{d}_f(a) = \underline{d}_g(a)$ and $\overline{d}_f(a) = \overline{d}_g(a)$.*

P r o o f. We will prove only the equality for the lower density (we would use analogical steps when proving the second equality). First notice that

$$\lim_{x\to\infty} \frac{\int_0^x g(t)\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t} = \lim_{x\to\infty} \frac{\int_0^x (g(t)-f(t))\,\mathrm{d}t + \int_0^x f(t)\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t} = 1.$$

Now, taking into account

$$\int_0^x |f(t)-g(t)|\,\mathrm{d}t \geq \int_0^x |f(t)-g(t)|a(t)\,\mathrm{d}t \geq \int_0^x (f(t)-g(t))a(t)\,\mathrm{d}t,$$

we have

$$\begin{aligned}\liminf_{x\to\infty} \frac{\int_0^x f(t)a(t)\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t} &= \liminf_{x\to\infty} \frac{\int_0^x g(t)a(t)\,\mathrm{d}t + \int_0^x (f(t)-g(t))a(t)\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t} \\ = \liminf_{x\to\infty} \frac{\int_0^x g(t)a(t)\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t} &= \liminf_{x\to\infty} \frac{\int_0^x g(t)a(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t} \frac{\int_0^x g(t)\,\mathrm{d}t}{\int_0^x f(t)\,\mathrm{d}t} = \liminf_{x\to\infty} \frac{\int_0^x g(t)a(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}.\end{aligned}$$

$\square$

Let a weight function $w\colon \mathbb{N} \to \mathbb{R}_0^+$ with $w(1) > 0$ satisfy (1). Then one can easily verify that the function f_w defined by $f_w(0) = w(1)$ and $f_w(t) = w(n)$ for $n \in \mathbb{N}$ and $t \in (n-1, n]$ belongs to $\mathscr{F}$.

Theorem 2.1. *Let $w\colon \mathbb{N} \to \mathbb{R}_0^+$ with $w(1) > 0$ satisfy (1) and $g \in \mathscr{F}$ be such that $\lim\limits_{x\to\infty} \frac{\int_0^x |f_w(t)-g(t)|\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t} = 0$. Let $A = \bigcup\limits_{n=1}^{\infty} (c_n, d_n] \cap \mathbb{N}$ where $c_n < d_n < c_{n+1}$ are positive integers. Then for every $a \in \mathscr{A}$ fulfilling the conditions of* Lemma 2.1 *we have $\underline{d}_w(A) = \underline{d}_g(a)$ and $\overline{d}_w(A) = \overline{d}_g(a)$.*

P r o o f. The proof follows from previous lemmas and the inequality for $x > 0$

$$\left|\frac{\int_0^x \chi_A(t) f_w(t)\,\mathrm{d}t}{\int_0^x f_w(t)\,\mathrm{d}t} - \frac{\int_0^x a(t) g(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}\right|$$

$$\leq \left|\frac{\int_0^x \chi_A(t) f_w(t)\,\mathrm{d}t}{\int_0^x f_w(t)\,\mathrm{d}t} - \frac{\int_0^x a(t) f_w(t)\,\mathrm{d}t}{\int_0^x f_w(t)\,\mathrm{d}t}\right| + \left|\frac{\int_0^x a(t) f_w(t)\,\mathrm{d}t}{\int_0^x f_w(t)\,\mathrm{d}t} - \frac{\int_0^x a(t) g(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}\right|.$$

$\square$

The following theorem could be used in calculations in papers cited in Introduction.

Theorem 2.2. *Let $A = \bigcup_{n=0}^{\infty} (c_n, d_n]$ for $c_n, d_n \in \mathbb{R}_0^+$, $c_n < d_n < c_{n+1}$ and $\lim_{n\to\infty} c_n = \infty$. Let $f \in \mathscr{F}$ and assume that there exist proper limits*

$$b = \lim_{n\to\infty} \frac{\int_0^{c_{n+1}} f(t)\,\mathrm{d}t}{\int_0^{d_n} f(t)\,\mathrm{d}t}, \qquad a = \lim_{n\to\infty} \frac{\int_0^{d_n} f(t)\,\mathrm{d}t}{\int_0^{c_n} f(t)\,\mathrm{d}t}$$

and $ab > 1$. Then, for $\alpha(t) = \chi_A(t)$, we obtain

$$\underline{d}_f(\alpha) = \frac{a-1}{ab-1}, \qquad \overline{d}_f(\alpha) = \frac{(a-1)b}{ab-1}.$$

P r o o f. Let ϵ be a positive real number such that $(a-\epsilon)(b-\epsilon) > 1$ and let for all $n \in \mathbb{N}$ greater than $n_0 \in \mathbb{N}$ the following inequalities hold:

$$b - \epsilon < \frac{\int_0^{c_{n+1}} f(t)\,\mathrm{d}t}{\int_0^{d_n} f(t)\,\mathrm{d}t} < b + \epsilon, \qquad a - \epsilon < \frac{\int_0^{d_n} f(t)\,\mathrm{d}t}{\int_0^{c_n} f(t)\,\mathrm{d}t} < a + \epsilon. \tag{9}$$

Put $I_{i,m} = \frac{\int_0^{c_i} f(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t}$ for $m > i > n_0$. Observe that

$$I_{i,m} = \frac{\int_0^{c_i} f(t)\,\mathrm{d}t}{\int_0^{d_i} f(t)\,\mathrm{d}t} \frac{\int_0^{d_i} f(t)\,\mathrm{d}t}{\int_0^{c_{i+1}} f(t)\,\mathrm{d}t} \cdots \frac{\int_0^{d_{m-1}} f(t)\,\mathrm{d}t}{\int_0^{c_m} f(t)\,\mathrm{d}t} \frac{\int_0^{c_m} f(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t}$$

and so the following inequalities hold:

$$\frac{1}{a+\epsilon}\left(\frac{1}{(a+\epsilon)(b+\epsilon)}\right)^{m-i} < I_{i,m} < \frac{1}{a-\epsilon}\left(\frac{1}{(a-\epsilon)(b-\epsilon)}\right)^{m-i}. \tag{10}$$

Let now m be a positive integer which is greater than n_0 and for which the following inequality holds:

$$\frac{\int_0^{d_{n_0}} f(t)\alpha(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} < \epsilon.$$

Using the previous inequality, we get

$$\frac{\int_0^{d_m} f(t)\alpha(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} = \frac{\int_{d_{n_0}}^{d_m} f(t)\alpha(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} + E_0, \tag{11}$$

where $0 < E_0 < \epsilon$. Further we can write

$$\begin{aligned}\frac{\int_{d_{n_0}}^{d_m} f(t)\alpha(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} &= \frac{\int_{c_{n_0+1}}^{d_m} f(t)\alpha(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} = \frac{\sum_{i=n_0+1}^{m}\left(\int_0^{d_i} f(t)\,\mathrm{d}t - \int_0^{c_i} f(t)\,\mathrm{d}t\right)}{\int_0^{d_m} f(t)\,\mathrm{d}t} \\ &= \sum_{i=n_0+1}^{m} \frac{\int_0^{d_i} f(t)\,\mathrm{d}t}{\int_0^{c_i} f(t)\,\mathrm{d}t} \frac{\int_0^{c_i} f(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} - \sum_{i=n_0+1}^{m} \frac{\int_0^{c_i} f(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} \\ &= (a-1)\sum_{i=n_0+1}^{m} \frac{\int_0^{c_i} f(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} + \sum_{i=n_0+1}^{m} E_i \frac{\int_0^{c_i} f(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t}\end{aligned}$$

for $|E_i| < \epsilon$ (see (9)). Combining (11) and above derived equalities, we obtain

$$\frac{\int_0^{d_m} f(t)\alpha(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} = (a-1)\sum_{i=n_0+1}^{m} \frac{\int_0^{c_i} f(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} + \underbrace{\sum_{i=n_0+1}^{m} E_i \frac{\int_0^{c_i} f(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} + E_0}_{E}$$

and so, taking into account (10), we have

$$\frac{\int_0^{d_m} f(t)\alpha(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} < \underbrace{(a-1)\sum_{i=n_0+1}^{m} \frac{1}{a-\epsilon}\left(\frac{1}{(a-\epsilon)(b-\epsilon)}\right)^{m-i}}_{S_1} + E \tag{$\clubsuit$}$$

and

$$\frac{\int_0^{d_m} f(t)\alpha(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} > \underbrace{(a-1)\sum_{i=n_0+1}^{m} \frac{1}{a+\epsilon}\left(\frac{1}{(a+\epsilon)(b+\epsilon)}\right)^{m-i}}_{S_2} + E. \tag{$\clubsuit$}$$

As the function $g_m(x,y) = \sum\limits_{i=n_0+1}^{m} \frac{1}{x}\left(\frac{1}{xy}\right)^{m-i}$ is continuous at the point (a,b), we have

$$\lim_{\epsilon\to 0}(S_1 - S_2) = 0. \tag{$\spadesuit$}$$

Consider furthermore that

$$(a-1)g_m(a,b) \in (S_2, S_1). \tag{$\diamondsuit$}$$

Now we will estimate the absolute value of the number E. The right inequality in (10) yields

$$|E| < \epsilon\left(\sum_{i=0}^{\infty} \frac{1}{a-\epsilon}\left(\frac{1}{(a-\epsilon)(b-\epsilon)}\right)^{i} + 1\right) = \epsilon\left(\frac{b-\epsilon}{(a-\epsilon)(b-\epsilon)-1} + 1\right)$$

and so

$$\lim_{\epsilon\to 0}|E| = 0. \tag{$\heartsuit$}$$

When we take all (♣), (♢), (♡) and (♠) together and when we realize that $\frac{a-1}{a}\sum_{i=0}^{\infty}\left(\frac{1}{ab}\right)^i = \frac{(a-1)b}{ab-1}$, we get

$$\overline{d}_f(\alpha) = \lim_{m\to\infty}\frac{\int_0^{d_m} f(t)\alpha(t)\,\mathrm{d}t}{\int_0^{d_m} f(t)\,\mathrm{d}t} = \frac{(a-1)b}{ab-1}.$$

Now let us prove that $\underline{d}_f(\alpha) = \frac{a-1}{ab-1}$:

$$\begin{aligned}\underline{d}_f(\alpha) = \liminf_{m\to\infty}\frac{\int_0^{c_m} f(t)\alpha(t)\,\mathrm{d}t}{\int_0^{c_m} f(t)\,\mathrm{d}t} &= \liminf_{m\to\infty}\frac{\int_0^{d_{m-1}} f(t)\alpha(t)\,\mathrm{d}t}{\int_0^{c_m} f(t)\,\mathrm{d}t}\frac{\int_0^{d_{m-1}} f(t)\,\mathrm{d}t}{\int_0^{d_{m-1}} f(t)\,\mathrm{d}t}\\ &= \overline{d}_f(\alpha)\frac{1}{b} = \frac{a-1}{ab-1}.\end{aligned}$$

□

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