

BRICKS AND PSEUDO MV-ALGEBRAS ARE EQUIVALENT

N. V. SUBRAHMANYAM

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ABSTRACT. We will show that the bricks (of Bosbach) and the pseudo MV-algebras are each term equivalent to the class of semigroups with a pair of unary operations $\hat{}$ and $\tilde{}$ satisfying the equations: $(\hat{a}a)\tilde{b} = b = b(a\tilde{a})\tilde{}$ and $a(\tilde{b}a)\hat{} = (b\tilde{a})\hat{b}$ and also show that a brick is an interval $[0, u]$ of the positive cone of a unital lattice ordered group. We further extend the notion of implications to a pseudo MV-algebra and study the algebra of such implications.

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1. Introduction

The concept of a brick is due to Bruno Bosbach [3] and the definition of a pseudo MV-algebra, which is a generalization of Chang's MV-algebra ([6]) is due to G. Georgescu and A. Iorgulescu [8]. J. Rachůnek also has introduced a noncommutative generalization of Chang's MV-algebra, which is equivalent to a pseudo MV-algebra and called by him ([9]) a generalized MV-algebra (or simply, a GMV-algebra). The main object of this paper is to show that bricks and pseudo MV-algebras are term equivalent.

As is well known, a Chang's MV-algebra has several equivalent definitions (for example, see [2] and [5]) which are much more compact than the original definition due to Chang [6]. We present below the notion of a *U-algebra*, which can be used to define a *pseudo MV-algebra* in terms of fewer axioms (see Theorem 3.3 below).

We also extend to pseudo MV-algebras the notion of implication in MV-algebras due to Hajda, Halaš and Kühr [5] by defining the term functions

$$a \circ b := \bar{a} \oplus b \quad \text{and} \quad a \square b := a \oplus \tilde{b}$$

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to be the *pseudo MV implications* in a pseudo MV-algebra $(A; \oplus, ^-, \sim, 0, 1)$ and show that $(A; \odot, \square)$ is a cone algebra and conversely every cone algebra is a subalgebra of $(A; \odot, \square)$ for some pseudo MV-algebra $(A; \oplus, ^-, \sim, 0, 1)$.

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We recall the definition of a pseudo MV-algebra due to Georgescu and Iorgulescu as quoted in [7]:

DEFINITION 2.1. By a pseudo MV-algebra is meant an algebra $(A; \oplus, ^-, \sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ which, together with an additional binary operation \odot defined by

$$(A0) \quad b \odot a = (\bar{a} \oplus \bar{b})^\sim,$$

satisfies the following axioms:

$$(A1) \quad (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

$$(A2) \quad x \oplus 0 = x = 0 \oplus x$$

$$(A3) \quad x \oplus 1 = 1 = 1 \oplus x$$

$$(A4) \quad \tilde{1} = 0 = \bar{1}$$

$$(A5) \quad (\bar{x} \oplus \bar{y})^\sim = (\tilde{x} \oplus \tilde{y})^-$$

$$(A6) \quad x \oplus (\tilde{x} \odot y) = y \oplus (\tilde{y} \odot x) = (x \odot \bar{y}) \oplus y = (y \odot \bar{x}) \oplus x$$

$$(A7) \quad x \odot (\bar{x} \oplus y) = (x \oplus \tilde{y}) \odot y$$

$$(A8) \quad (\bar{x})^\sim = x$$

The axiom (A6) above is superfluous and so we include also a proof of the following simple lemma.

LEMMA 2.2.

- (i) $(\bar{x})^\sim = (\tilde{x})^- = x$
- (ii) $\bar{0} = 1 = \tilde{0}$
- (iii) $x \odot 0 = 0 \odot x = 0; x \odot 1 = 1 \odot x = x$
- (iv) $x \odot \bar{x} = \tilde{x} \odot x = 0$
- (v) $(x \odot y)^- = \bar{y} \oplus \bar{x}; (x \odot y)^\sim = \tilde{y} \oplus \tilde{x}$
- (vi) $(x \odot y) \odot z = x \odot (y \odot z)$
- (vii) $(\tilde{x} \odot x)^\sim \odot y = y = y \odot (x \odot \bar{x})^-$
- (viii) $x \odot (\tilde{y} \odot x)^- = (y \odot \bar{x})^\sim \odot y$

Proof.

- (i) By (A5), (A4), (A2) and (A8) with $y = 1$.
- (ii) By (A4) and (i) above.
- (iii) (A0), (ii), (A3) and (A4) $\implies x \odot 0 = 0 \odot x = 0$ and
(A0), (A4), (A2) and (i) $\implies x \odot 1 = 1 \odot x = x$.
- (iv) By (A7) and (iii) with $x = 0, y = 0$ respectively.
- (v) By (A5) and (i).
- (vi) By (A1), (i) and (v).
- (vii) By (i) and (iii).
- (viii) By (A7), (v) and (i).

□

COROLLARY 2.3. *If $\mathcal{A} = (A; \oplus, -, \sim, 0, 1)$ is a pseudo MV-algebra, then $d(\mathcal{A}) := (A; \odot, \sim, -, 1, 0)$, called the dual of \mathcal{A} , is also a pseudo MV-algebra and $d(d(\mathcal{A})) = \mathcal{A}$.*

We will show that the last three equations of the above Lemma 2.2, are sufficient to define a pseudo MV-algebra. To identify such semigroups, we introduce the following definition:

DEFINITION 2.4. A semigroup $(A; \cdot)$ together with unary operations $\hat{}$ and \sim satisfying the equations:

$$(\alpha) \quad (\hat{a}a)\hat{b} = b = b(a\hat{a})\sim$$

$$(\beta) \quad a(\hat{b}a)\sim = (b\hat{a})\sim b,$$

is called a *U-algebra*.

Hence we at once have the following from (vi), (vii) and (viii) of Lemma 2.2.

COROLLARY 2.5. *If $(A; \oplus, -, \sim, 0, 1)$ is a pseudo MV-algebra, then $(A; \odot, \sim, -, 1, 0)$ is a U-algebra.*

We now assume that $(A; \cdot, \hat{}, \sim)$ is a U-algebra; and observe that any equation which is valid in this algebra remains valid if written in the reverse order with the symbols $\hat{}$ and \sim interchanged. This is the *principle of duality* which we use frequently.

LEMMA 2.6.

- (i) $\hat{a}a = b\hat{b}$ and we denote this common value by 0.
- (ii) $\hat{0} = \check{0}$ and we write this as 1.
- (iii) $a1 = 1a = a$.
- (iv) $a0 = 0a = 0$.

- (v) $a\check{b} = 0 \iff \hat{b}a = 0$.
- (vi) $a \leq b \iff a\check{b} = 0$ defines a partial order in A with 0 as the least element.
- (vii) $\hat{1} = 0 = \check{1}$.
- (viii) $(\hat{a})^\sim = (\check{a})^\sim = a$.
- (ix) 1 is the greatest element of A .

P r o o f.

(i) By Definition 2.4(α),

$$\begin{aligned}
 \hat{a}a &= (\hat{a}a)^\sim(\hat{a}a) \\
 &= ((\hat{a}a)(b\check{b})^\sim)^\sim(\hat{a}a) \\
 &= (b\check{b})[(\hat{a}a)^\sim(b\check{b})]^\sim && \text{by Definition 2.4}(\beta) \\
 &= (b\check{b})(b\check{b})^\sim = b\check{b} && \text{by Definition 2.4}(\alpha) \\
 &:= 0.
 \end{aligned}$$

(ii) By Definition 2.4(α) and (i) above, $\hat{0}$ is a left identity and $\check{0}$ is a right identity of the semigroup $(A; \cdot)$ and hence $\hat{0} = \check{0} := 1$.

(iii) By (ii) above.

(iv) We have $0\check{a} = \hat{a}a\check{a} = \hat{a}0 = \hat{a}\{(a1)^\sim a\} = \hat{a}\{(a\check{0})^\sim a\} = \hat{a}\{0(\hat{a}0)^\sim\}$ (by Definition 2.4(β)) $= (\hat{a}0)(\hat{a}0)^\sim = 0$ by Definition 2.4(α). Hence $a0 = a\check{a}\check{a} = 0\check{a} = 0$ and $0a = \hat{a}\hat{a}a = \hat{a}0 = 0$.

(v) $a\check{b} = 0 \implies \hat{b}a = \hat{b}(\hat{0}a) = \hat{b}((a\check{b})^\sim a) = \hat{b}(b(\hat{a}b)^\sim)$ (by Definition 2.4(β)) $= (\hat{b}b)(\hat{a}b)^\sim = 0(\hat{a}b)^\sim = 0$ by (iv) above, and $\hat{b}a = 0 \implies a\check{b} = 0$ by duality.

(vi) $a \leq a$ since $a\check{a} = 0$; and if $a \leq b$ and $b \leq c$, then $\hat{b}a = 0$ and $b\check{a} = 0$ so that $a = b$ by Definition 2.4(β). Assume now that $a \leq b$ and $b \leq c$; then $\hat{b}a = 0$ and $b\check{c} = 0$. Hence, by Definition 2.4(β), $a\check{c} = (a(\hat{b}a)^\sim)^\sim\check{c} = ((b\check{a})^\sim b)^\sim\check{c} = (b\check{a})^\sim(b\check{c}) = (b\check{a})^\sim 0 = 0$. Hence $a \leq c$ so that \leq is a partial ordering on A . Finally, $0 \leq a$ for all $a \in A$, since $0\check{a} = 0$ for all $a \in A$.

(vii) Since $\hat{0}(\hat{0})^\sim = 0$, we have $\check{1} = (\hat{0})^\sim \leq 0$ and hence by (vi) above, $\check{1} = 0$; and by duality, $\hat{1} = 0$.

(viii) $(\hat{a})^\sim = 1(\hat{a}1)^\sim = (a\check{1})^\sim a = \hat{0}a = a$ and dually $(\check{a})^\sim = a$.

(ix) $a\check{1} = a0 = 0$; hence, $a \leq 1$. □

We now recall the definition of a brick due to B o s b a c h [3]:

DEFINITION 2.7. An algebra $(B; *, :, 1)$ of type $(2, 2, 0)$ is called a *brick* if it satisfies the following equations:

- (1) $(a * a) * b = b = b : (a : a)$
- (2) $a * (b : c) = (a * b) : c$

$$(3) \ a : (b * a) = (b : a) * b$$

$$(4) \ 1 : (a * 1) = a$$

Bosbach has proved that if $(B; *, :, 1)$ is brick, then $(B; \cap)$ is a meet semilattice when $a \cap b$ is defined to be $a : (b * a) = (b : a) * b$ and that 1 is the greatest element of this semilattice (see [3, p. 65]).

We now prove:

THEOREM 2.8. *If $\mathcal{A} = (A; \cdot, \hat{\cdot}, \check{\cdot})$ is a U-algebra, then $(A; \cdot, 1)$, where $1 = \hat{0} = \check{0}$, is a monoid and if we define $a * b = \hat{a}b$ and $a : b = a\check{b}$, then $(A; *, :, 1)$ is a brick and this is denoted by $\mathcal{B}(\mathcal{A})$. Conversely, if $\mathcal{A} = (A; *, :, 1)$ is a brick, then $(A; \cdot, \hat{\cdot}, \check{\cdot})$ is a U-algebra if we define $ab = \check{a} * b$; $\hat{a} = a * 1$ and $\check{a} = 1 : a$, and this is denoted by $\mathcal{U}(\mathcal{A})$.*

Proof. Assume that $(A; \cdot, \hat{\cdot}, \check{\cdot})$ is a U-algebra; then

$$\begin{aligned} (a * a) * b &= (\hat{a}a)\hat{b} = b = b(a\check{a})^\sim = b : (a : a) \\ (a * b) : c &= (\hat{a}b)\check{c} = \hat{a}(b\check{c}) = a * (b : c) \\ a : (b * a) &= a(\hat{b}a)^\sim = (b\check{a})^\sim b = (b : a) * b \end{aligned}$$

by Definition 2.4 and

$$1 : (a * 1) = (\hat{a})^\sim = a \quad \text{by Lemma 2.6(viii).}$$

Now assume $(A; *, :, 1)$ is a brick. Then $\check{a} * b = (1 : a) * (1 : (b * 1)) = ((1 : a) * 1) : (b * 1) = (a \cap 1) : \hat{b} = a : \hat{b}$. Hence, $(ab)c = (\check{a} * b) : \hat{c} = \check{a} * (b : \hat{c}) = a(b\check{c})$ so that $(A; \cdot)$ is a semigroup.

By (3) and (4) of the Definition 2.7, $(\hat{a})^\sim = (\check{a})^\sim = a$ and hence $\hat{a}b = (\check{a})^\sim * b = a * b$ and $a\check{b} = \check{a} * \check{b} = (\check{a}) * (1 : b) = (\check{a})^\sim : b = a : b$. Now the equations (1) and (4) translate directly into the equations (α) and (β) of Definition 2.4. \square

If $\mathcal{A} = (A; *, :, 1)$ is a brick, then by Theorem 2.8, $\mathcal{U}(\mathcal{A}) = (A; \cdot, \hat{\cdot}, \check{\cdot})$ is a U-algebra and hence $\mathcal{B}(\mathcal{U}(\mathcal{A})) = (A; \otimes, \odot, \textcircled{1})$ is again a brick, where $a \otimes b = \hat{a}b$, $a \odot b = a\check{b}$, $ab = \check{a} * b$, $\hat{a} = a * 1$ and $\check{a} = 1 : a$. Then $a \otimes b = \hat{a}b = (\check{a})^\sim * b = a * b$ and $a \odot b = a\check{b} = a : (\check{b})^\sim = a : b$; hence $1 = \textcircled{1} \odot (1 \otimes \textcircled{1}) = \textcircled{1} : (1 * \textcircled{1}) = 1 \cap \textcircled{1} = \textcircled{1}$. Hence $\mathcal{B}(\mathcal{U}(\mathcal{A})) = \mathcal{A}$. Similarly, if $\mathcal{A} = (A; \cdot, \hat{\cdot}, \check{\cdot})$ is a U-algebra, then $\mathcal{B}(\mathcal{A}) = (A; *, :, 1)$ is a brick and hence $\mathcal{U}(\mathcal{B}(\mathcal{A})) = (A; \odot, \textcircled{\odot}, \textcircled{\odot})$ is a U-algebra, where $a * b = \hat{a}b$, $a : b = a\check{b}$, $a \odot b = a^\odot * b$, $a^\odot = a * 1$ and $a^\odot = 1 : a$. Then $a^\odot = 1 : a = 1\check{a} = \check{a}$, $a^\odot = a * 1 = \hat{a}1 = \hat{a}$ and hence $a \odot b = \check{a} * b = (\hat{a})^\sim b = ab$. Hence $\mathcal{U}(\mathcal{B}(\mathcal{A})) = \mathcal{A}$.

Hence we have the following corollary:

COROLLARY 2.9. *Bricks and U-algebras are term equivalent.*

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We again assume that $(A; \cdot, \hat{\cdot}, \check{\cdot})$ is a U-algebra. Then by the above Theorem 2.8 and Lemma 2.6(v), A is a meet semilattice with $\inf \{a, b\} = a : (b * a) = a(\hat{b}a)^\sim$. We now prove a crucial lemma:

LEMMA 3.1.

- (i) $a \leq b \iff \check{b} \leq \check{a} \iff \hat{b} \leq \hat{a}$
- (ii) $(\hat{a}\hat{b})^\sim = (\check{a}\check{b})^\sim$
- (iii) $a \cup b = (\hat{a} \cap \hat{b})^\sim = (\check{a} \cap \check{b})^\sim$

Proof.

(i) $a \leq b \implies a\check{b} = 0$ and $\hat{b}a = 0 \implies (\check{a})^\sim\check{b} = 0$ and $\hat{b}(\hat{a})^\sim = 0 \implies \check{b} \leq \check{a}$ and $\hat{b} \leq \hat{a} \implies (\check{a})^\sim \leq (\check{b})^\sim$ and $(\hat{a})^\sim \leq (\hat{b})^\sim$ and each $\implies a \leq b$.

(ii) Since $(\check{a}\check{b})^\sim \check{a}\check{b} = 0$, we have $\hat{a}\hat{b}(\check{a}\check{b})^\sim = 0$ by repeated use of Lemma 2.6(v) and hence $(\check{a}\check{b})^\sim \leq (\hat{a}\hat{b})^\sim$. The reverse inequality is obtained by duality.

(iii) By (i), $a \leq c, b \leq c \iff \hat{c} \leq \hat{a} \cap \hat{b} \iff (\hat{a} \cap \hat{b})^\sim \leq (\hat{c})^\sim = c$; hence $a \cup b$ exists and equals $(\hat{a} \cap \hat{b})^\sim$. By duality, $a \cup b$ also equals $(\check{a} \cap \check{b})^\sim$. \square

THEOREM 3.2. *If $(A; \cdot, \hat{\cdot}, \check{\cdot})$ is a U-algebra, then define $a \oplus b = (\hat{b}\hat{a})^\sim = (\check{b}\check{a})^\sim$; then $(A; \oplus, \check{\cdot}, \hat{\cdot}, 0, 1)$ is a pseudo MV-algebra.*

Proof. Note that the additional binary operation \odot in the Definition 2.1 coincides with the semigroup operation \cdot of the U-algebra $(A; \cdot, \hat{\cdot}, \check{\cdot})$. Now by Definition 2.4(β), $x(\check{x} \oplus y) = x(\hat{y}x)^\sim = (y\check{x})^\sim y = (x \oplus \hat{y})y$ and this proves (A7).

We have already seen that the U-algebra A is a meet semilattice with $a \cap b = a(\hat{b}a)^\sim = (b\check{a})^\sim b$. Hence by Lemma 3.1(iii) we can compute $a \cup b$ in two ways:

$a \cup b = (\hat{a} \cap \hat{b})^\sim = ((\hat{b}a)^\sim \hat{b})^\sim = b \oplus (\hat{b}a)$ and $a \cup b = (\check{a} \cap \check{b})^\sim = (\check{a}(\check{b}\check{a})^\sim)^\sim = (b\check{a}) \oplus a$. From this (A6) follows since $a \cup b = b \cup a$. The remaining axioms are easy to verify. \square

By Lemma 2.2 and the equation (A0) of Definition 2.1, a pseudo MV-algebra is clearly term equivalent to its dual. Now combining the Corollary 2.3 and Corollary 2.5 with the Theorem 2.8, we obtain:

THEOREM 3.3. *If $(A; \odot, \hat{\cdot}, \check{\cdot}, 1, 0)$ is a pseudo MV-algebra, then $(A; \odot, \hat{\cdot}, \check{\cdot})$ is a U-algebra. Conversely, if $(A; \odot, \hat{\cdot}, \check{\cdot})$ is a U-algebra, then there exist $0, 1 \in A$ such that $(A; \odot, \hat{\cdot}, \check{\cdot}, 1, 0)$ is a pseudo MV-algebra.*

Hence by Theorem 3.2 it follows that bricks and pseudo MV-algebras are term equivalent and a U-algebra defines a pseudo MV-algebra.

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We now recall the definitions and some crucial results concerning cone algebras introduced by Bosbach [3].

DEFINITION 4.1. An algebra $(C, *, :)$ of type $(2, 2)$ is called a cone algebra if it satisfies the following equations:

- (1) $(a * a) * b = b = b : (a : a)$
- (2) $(a * b) : c = a * (b : c)$
- (3) $a : (b * a) = (b : a) * b$
- (4) $(a * b) * (a * c) = (b * a) * (b * c)$ and
- (5) $(c : a) : (b : a) = (c : b) : (a : b)$.

Also, a cone algebra satisfying the equation $a * b = b : a$ is said to be *symmetric*.

Clearly, if $(C, *, :)$ is a cone algebra, then so is the algebra (C, \otimes, \odot) where $a \otimes b = b : a$ and $a \odot b = b * a$, and hence *any equation valid in a cone algebra remains valid, if written in the reverse order with $*$ and $:$ interchanged*.

The most important example of a cone algebra is the positive cone G^+ of a (not necessarily abelian) lattice ordered group $(G; +, \leq)$ with operations defined by

$$a * b = (-a + b) \cup 0 \quad \text{and} \quad a : b = (a - b) \cup 0.$$

Bosbach has defined [4, p. 107] a right residuation groupoid as a binary algebra $(A; \circ)$ satisfying the axioms: $(a \circ a) \circ b = b$, $(a \circ b) \circ (a \circ c) = (b \circ a) \circ (b \circ c)$, $a \circ (b \circ b) = c \circ c$ and $a \circ b = c \circ c = b \circ a \implies a = b$. Also, an algebra $(A; *, :)$ is called a residuation groupoid if and only if both $(A; *)$ and $(A, :)$ are right residuation groupoids. For instance, in the example given above, the cone algebra $(G^+; *, :)$ is a residuation groupoid, called the residuation groupoid of the ℓ -group cone G^+ . In fact, Bosbach has shown [3, Statements 1, 6, p. 59] that every cone algebra is a residuation groupoid. We need the following results due to Bosbach [3] in the sequel:

- (α) Every brick is a cone algebra [3, p. 65].
- (β) A cone algebra $(C; *, :)$ is the cone algebra of some ℓ -group cone if and only if
- (AC) given $a, b \in C$, there exists $x \in C$ such that $a * x = b$ and $x * a = 0$.
([3, Statement 1.20, p. 61])

Note: We use 0 and 1 respectively where Bosbach has used 1 and 0.

- (γ) *First Embedding Theorem:* Necessary and sufficient condition for an algebra $(R; *, :)$ of type $(2, 2)$ to admit an extension $(S; *, :)$ which is the residuation groupoid of some ℓ -group cone is that $(R; *, :)$ is a cone algebra [3, p. 64].

Now let C be a cone algebra; then by the first embedding theorem of Bosbach [3], C is a subalgebra of the cone algebra of G^+ of some lattice ordered group G . Now let \widehat{C} be the subsemigroup of $(G^+, +)$ generated by C .

LEMMA 4.2. $(\widehat{C}; *, :)$ is the cone algebra of an ℓ -group cone contained in G^+ .

Proof. For each integer $k > 0$, let $C_k = \left\{ \sum_{i=1}^k a_i : a_1, \dots, a_k \in C \right\}$. Since $0 \in C$, $C_k \subseteq C_l$ if $k \leq l$; and clearly, $\widehat{C} = \bigcup_{k=1}^{\infty} C_k$. Now by using the identity

$$(a + b) * (c + d) = (b * (a * c)) + [(a * c) * b] * ((c * a) * d)) \quad (\#)$$

which is valid in every ℓ -group cone, it is clear that $C_k * C_k \subseteq C_k \implies C_{2k} * C_{2k} \subseteq C_{2k}$. For $k = 1$, $C_k = C$ and $C * C \subseteq C$ and hence by induction we get $C_{2^n} * C_{2^n} \subseteq C_{2^n}$ for all $n \geq 0$. If $a, b \in \widehat{C}$, then for some $n \geq 0$, $a, b \in C_{2^n}$ and hence $a * b \in C_{2^n} \subseteq \widehat{C}$. Hence \widehat{C} is stable under $*$; and by duality, \widehat{C} is stable also under $:$ and hence $(\widehat{C}; *, :)$ is a cone algebra $\subseteq G^+$. Finally if $a, b \in \widehat{C}$ then $a + b \in \widehat{C}$, $a * (a + b) = b$ and $(a + b) * a = 0$; hence by Theorem β , \widehat{C} is an ℓ -group cone. \square

LEMMA 4.3. Let C and \widehat{C} be as in the Lemma 4.2; then $(\widehat{C}; +)$ is abelian if and only if C is symmetric.

Proof. Assume C is symmetric and let $a, b \in C$; then by $(\#)$,

$$(a + b) * (b + a) = (b * (a * b)) + \{(a * b) * b\} * \{(b * a) * a\} = 0$$

since $a * b \leq b$ and $(a * b) * b = (b * a) * a$. Hence $b + a \leq a + b$ and hence by symmetry, $a + b = b + a$ for all $a, b \in C$. Since $(\widehat{C}; +)$ is generated by C , $(\widehat{C}; +)$ is abelian. The converse is clear. \square

If $(C; *, :, 1)$ is a brick, then by (α) and (γ) , $(C, *, :)$ is a subalgebra of \widehat{C} , the cone algebra of some ℓ -group cone; and now we show:

LEMMA 4.4. If C is a brick, then \widehat{C} is the positive cone of a unital ℓ -group.

Proof. Let G be a lattice ordered group with $\widehat{C} = G^+$. If $g \in G$, then $|g| \in G^+ = \widehat{C}$ and hence by Lemma 4.2, $|g| = s_1 + s_2 + \dots + s_n$ for some $s_1, s_2, \dots, s_n \in C$. Since C is a brick, each $s_i \leq u$ where u is the greatest element of C and hence $|g| \leq nu$. Hence G is a unital ℓ -group with strong order unit u . \square

LEMMA 4.5. If $a \in \widehat{C}$ and $c \in C$, then $a * c \in C$.

Proof. By the identity $(\#)$ and induction. \square

THEOREM 4.6. *C is a convex subalgebra of \widehat{C} .*

Proof. Assume $a \in \widehat{C}$, $c \in C$ and $a \leq c$; then by Lemma 4.5, $a * c \in C$ and hence $a = a \cap c = c : (a * c) \in C$. \square

COROLLARY 4.7. *Every brick is an interval of an ℓ -group cone.*

Hence, we have another proof of Dvurečenskij's theorem [7, Theorem 3.9].

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If $(M; \oplus, \bar{\cdot}, \odot)$ is an MV-algebra of Chang [6], then Hajda, Halaš and Kühr have defined ([5]) the implication in the MV-algebra M as the term function $x \circ y := \bar{x} \oplus y$, which itself extends the concept of implication in a Boolean algebra, defined and intensively studied by Abbott [1]. We now extend this concept to pseudo MV-algebras in the following.

DEFINITION 5.1. Let $(A; \oplus, \bar{\cdot}, \sim, 0, 1)$ be a pseudo MV-algebra; then we call the term functions

$$x \circ y = \bar{x} \oplus y \quad \text{and} \quad x \square y = x \oplus \tilde{y}$$

as the *pseudo MV implications* and the algebra $(A; \circ, \square, 0)$ is called the *pseudo MV-implication algebra*, or simply, a *pMV-implication algebra*, which is derived from $(A; \oplus, \bar{\cdot}, \sim, 0, 1)$.

By a *pMV-implication algebra* is meant the algebra derived from some pseudo MV-algebra and by an *algebra of pMV implications* is meant a *subalgebra* of a pMV-implication algebra.

LEMMA 5.2. *Every pMV-implication algebra is a brick and hence a cone algebra.*

Proof. Let $(A; \oplus, \bar{\cdot}, \sim, 0, 1)$ be a pseudo MV-algebra; then by (A0) of the Definition 2.1 and the Lemma 2.2, we have

$$\bar{a} \oplus a = (\tilde{a} \odot a)^{\sim} = \bar{0} = 1$$

and dually, $a \oplus \tilde{a} = 1$. Hence $(a \circ a) \circ b = (\bar{a} \oplus a)^{\sim} \oplus b = 0 \oplus b = b$ and similarly, $b \square (a \square a) = b$.

Also, $a \circ (b \square c) = \bar{a} \oplus (b \oplus \tilde{c}) = (\bar{a} \oplus b) \oplus \tilde{c} = (a \circ b) \square c$; and

$$\begin{aligned} a \square (b \circ a) &= a \oplus (\bar{b} \oplus a)^{\sim} = a \oplus (\tilde{a} \odot b) \\ &= (a \odot \bar{b}) \oplus b \quad \text{by (A6)} \\ &= (b \oplus \tilde{a})^{\sim} \oplus b = (b \square a) \circ b. \end{aligned}$$

Finally, $0 \square (a \circ 0) = 0 \oplus (\bar{a} \oplus 0)^{\sim} = (\bar{a})^{\sim} = a$. Hence the lemma. \square

LEMMA 5.3. *Every pseudo MV-algebra is a pMV-implication algebra.*

Proof. Let $(A; \oplus, \bar{\cdot}, \sim, 0, 1)$ be a pseudo MV-algebra and $(A; \circ, \square, 0)$ be the corresponding pMV-implication algebra (as in the Lemma 5.2). Then by the Lemma 5.2, $(A; \circ, \square, 0)$ is a brick and if we define

$$\hat{a} = a \circ 0, \check{a} = 0 \square a, \quad \text{and} \quad ab = \check{a} \circ b,$$

then $(A; \cdot, \hat{\cdot}, \check{\cdot})$ is a U-algebra and hence by Theorem 3.2, $(A; \cdot; \hat{\cdot}, \check{\cdot}, 0, 1)$ is a pseudo MV-algebra. Now $\hat{a} = a \circ 0 = \bar{a}$, $\check{a} = 0 \square a = \bar{a}$ and $ab = \check{a} \circ b = (\bar{a})^- \oplus b = a \oplus b$. Hence $(A; \oplus, \bar{\cdot}, \sim, 0, 1)$ is an algebra of pMV implications. \square

It is well known that an MV-algebra is a pseudo MV-algebra $(A; \oplus, \bar{\cdot}, \sim, 0, 1)$ where \oplus is commutative (and hence) $\bar{\cdot} = \sim$; so, unlike in [5], an MV-algebra also has two implication functions $a \circ b = \bar{a} \oplus b$ and $a \square b = a \oplus \bar{b}$ which do not coincide. But in this case, $a \circ b = b \square a$ for all a, b so that by Lemma 5.2 we obtain:

COROLLARY 5.4. *If $(A; \oplus, \bar{\cdot}, 0, 1)$ is an MV-algebra, then $(A; \circ, \square, 0)$ is a symmetric brick and hence $(A; \circ, \square)$ is a symmetric cone algebra.*

Hence by Lemma 5.3, we have

COROLLARY 5.5. *Every MV-algebra is an MV-implication algebra.*

Remark 5.6. The class of MV-implication algebras is a *proper* subclass of the class of weak implication algebras defined by Chajda and others in [5] and so that Corollary 5.5 should be regarded as a stronger result than their Theorem 1 of [5, p. 378]. We present below an example of a weak implication algebra, which is *not* a cone algebra, and hence is *not* an MV-implication algebra (see Corollary 5.4).

Example 5.7. Let A be the set consisting of four distinct elements $0, a, b, c$ and define the operation $*$ by the following table:

$*$	0	a	b	c
0	0	a	b	c
a	0	0	a	a
b	0	0	0	a
c	0	0	a	0

Since $(a * c) * (a * b) = 0 \neq a = (c * a) * (c * b)$, $(A; *)$ is *not* a cone algebra and it is a routine verification to show that $(A; *)$ is a weak implication algebra.

We next show that every cone algebra is an algebra of pMV implications by using the second embedding theorem of Bosbach [3]. First, we prove the following converse of Lemma 5.2, which also gives a second proof of Lemma 5.3.

LEMMA 5.8. *Every brick is a pMV-implication algebra.*

Proof. Let $(A; *, :, 1)$ be a brick, then if we define $\hat{a} = a * 1$, $\check{a} = 1 : a$ and $ab = \hat{a} * b$, we know that $(A; \cdot, \hat{\cdot}, \check{\cdot}, 1, 0)$ is a pseudo MV-algebra (cf. Theorem 3.3). Now the implication functions in this pseudo MV-algebra are precisely $\hat{a}b = a * b$ and $a\check{b} = a : b$. \square

LEMMA 5.9. *Let $(G; +, \leq)$ be a lattice ordered group; then $(G^+, *, :)$ can be embedded into a brick.*

Proof. Let $(\mathbb{Z}; +, \leq)$ be the ℓ -group of integers under the natural order and $K = \mathbb{Z} \times G$. Then $(K; +, \leq)$ is an ℓ -group with respect to componentwise addition and the lexicographic ordering. Write $\theta = (0, 0)$ and $u = (1, 0)$; then the interval $[\theta, u]$ is a brick with respect to the operations $*$ and $:$ defined as usual in the ℓ -group cone K^+ and has u as the greatest element. If $(m, a) \in K$, then $(m, a) \in [\theta, u] \iff (0, 0) \leq (m, a) \leq (1, 0) \iff m = 0 \text{ and } a \in G^+ \text{ or } m = 1 \text{ and } -a \in G^+$. Also, it is easily verified that $a \mapsto (0, a)$ is a monomorphism of G^+ into the brick $[\theta, u]$. \square

Since by Bosbach's first embedding theorem, every cone algebra is a subalgebra of the cone algebra of some ℓ -group cone, we get:

COROLLARY 5.10 (Bosbach's Second Embedding Theorem). *Every cone algebra can be embedded into some brick.*

By Lemma 5.8 we now obtain:

THEOREM 5.11. *Every cone algebra is an algebra of pMV implications.*

COROLLARY 5.12. *Every symmetric cone algebra is an algebra of MV implications.*

Remark 5.13. Since every subalgebra of a cone algebra is again a cone algebra and a weak implication algebra is not necessarily a cone algebra, given a weak implication algebra A , there *may not exist* an MV-algebra M such that A is contained in the algebra of MV implications.

Remark 5.14. In the proof of Lemma 5.9, $u = (1, 0)$ is in the center of K and hence for the U-algebra $(A; \cdot, \hat{\cdot}, \check{\cdot})$ obtained from the brick $[\theta, u]$, $\hat{\cdot} = \check{\cdot}$ whether or not \cdot is commutative. Hence $\hat{\cdot} = \check{\cdot}$ does not imply $(A; \cdot)$ is abelian. However, if $(A; \cdot)$ is abelian in any U-algebra $(A; \cdot, \hat{\cdot}, \check{\cdot})$, then $\hat{\cdot} = \check{\cdot}$.

Proof. Assume $(A; \cdot)$ is abelian; then $0 = \hat{a}a = a\hat{a} \implies (\check{a})\hat{a} = 0$ so that $\check{a} \geq \hat{a}$ and by duality, $\hat{a} \geq \check{a}$ so that $\hat{a} = \check{a}$ for all $a \in A$. \square

Remark 5.15. Let G be the set of all ordered pairs (a, b) where a is a positive rational number and b is a rational number and define

$$(a, b) + (c, d) = (ac, ad + b).$$

Then $(G; +)$ is a nonabelian group with $(1, 0)$ as the identity element and the inverse of (a, b) as $(a^{-1}, -ba^{-1})$. Clearly, G is countable. If we now define

$$(a, b) \leq (c, d) \iff a < c \text{ or } a = c \text{ and } b \leq d,$$

then $(G; +, \leq)$ is a countable, nonabelian lattice ordered group. Now, if we use this ℓ -group $(G; +, \leq)$ in Lemma 5.9, the interval $[\theta, u]$ is a countable, noncommutative, pseudo MV-algebra in which $\hat{} = \check{}$.

Hence:

PROPOSITION 5.16. *There exists a non-Archimedean, countable pseudo MV-algebra $(A; \oplus, ^-, \sim, 0, 1)$ in which $^- = \sim$. (See [7, Theorem 4.2, Example 4.4].)*

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C/o Dr. K. V. Krishna

Department of Mathematics

Indian Institute of Technology Guwahati

Guwahati-781 039

INDIA

E-mail: nvs.math@gmail.com