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THE GENERALIZED APPROXIMATE PERRON INTEGRAL

D. K. Ganguly — Ranu Mukherjee

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ABSTRACT. In this paper we introduce the concept of the generalized AP-integral and discuss the properties of this integral.

1. Introduction

The AP-integral or more precisely the approximately continuous Perron integral was first defined by Burkill [1] and its Riemann-type definition was given by Bullen [2]. Schwabik [5] presented a generalized version of the Perron integral leading to the new approach to a Generalized Ordinary Differential Equation. Kurzweil (1957) and Henstock (1961) independently found that Perron integration could be given by a generalized Riemann integral. Perron integration, being equivalent to special Denjoy integration, includes Lebesgue integration so that it was clear that the latter is included in generalized Riemann integration ([4]). The approximate Perron integral of Burkill (1931) has a generalized Riemann integral form that is easily obtained ([3]). A distinguishing feature of modern Mathematics is the establishment of patterns of behaviour in mathematical systems, leading to a unified set of proofs of results instead of a heterogeneous collection of proofs, one set for each system. In this paper, our aim is to unify the variety of generalized Riemann-type, Riemann-Stieltjes type and approximate Perron integrals by using functions of two variables from $[a, b] \times [a, b]$ to $\mathbb{R}$.

We now recall the following definitions presented in the book [4].

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Definition 1.1. A collection Δ of closed subintervals of $[a, b]$ is called an *approximate full cover* (*AFC*) of $[a, b]$ if for every $x \in [a, b]$ there exists a measurable set $D_x \subset [a, b]$ such that $x \in D_x$ and D_x has density 1 at x, with $[u, v] \in \Delta$ whenever $u, v \in D_x$ and $u \leq x \leq v$.

A division of $[a, b]$ obtained by $a = x_0 < x_1 < \cdots < x_n = b$ and $\{\xi_1, \xi_2, \ldots, \xi_n\}$ is called a Δ-*division* if Δ is an approximate full cover with $[x_{i-1}, x_i]$ coming from Δ or more precisely, $x_{i-1} \leq \xi_i \leq x_i$ and $x_{i-1}, x_i \in D_{\xi_i}$ for all i. We call ξ_i the *associated point* of $[x_{i-1}, x_i]$ and x_i $(i = 0, 1, \ldots, n)$ the *division points*.

A division of $[a, b]$ given by $a \leq y_1 \leq \zeta_1 \leq z_1 \leq y_2 \leq \zeta_2 \leq z_2 \leq \cdots \leq y_m \leq \zeta_m \leq z_m \leq b$ is called a Δ-*partial division* if Δ is an approximate full cover with $([y_i, z_i], \zeta_i) \in \Delta$, for $i = 1, 2, \ldots, m$.

The next Cousin-type lemma from [4] makes it possible to give a Riemann-type definition of the AP-integral.

Lemma 1.2. *If Δ is an approximate full cover of $[a, b]$, then there exists a Δ-division of $[a, b]$.*

Definition 1.3. A function $f \colon [a, b] \to \mathbb{R}$ is said *to be AP-integrable to a real number A* if for every $\epsilon > 0$ there is an AFC Δ of $[a, b]$ such that for every Δ-division $D = ([u, v], \xi)$ of $[a, b]$ we have

$$\left|(D) \sum f(\xi)(v - u) - A\right| < \epsilon$$

and we write A $= (AP) \int\limits_a^b f$.

Now the generalized approximate Perron integral is defined as follows:

Definition 1.4. A function $U \colon [a, b] \times [a, b] \to \mathbb{R}$ is said to be *generalized AP* (*GAP*)*-integrable to a real number A* if for every $\epsilon > 0$ there is an AFC Δ of $[a, b]$ such that for every Δ-division $D = ([\alpha, \beta], \tau)$ of $[a, b]$ we have

$$\left|(D) \sum \{U(\tau, \beta) - U(\tau, \alpha)\} - A\right| < \epsilon$$

and we write $A = (GAP) \int\limits_a^b U$.

The set of all functions U which are generalized approximate Perron integrable on $[a, b]$ is denoted by $GAP[a, b]$. We use the notation

$$S(U, D) = (D) \sum \{U(\tau, \beta) - U(\tau, \alpha)\}$$

for the Riemann-type sum corresponding to the function U and the Δ-division $D = ([\alpha, \beta], \tau)$ of $[a, b]$.

Note that the integral is uniquely determined.

Remark 1.5. If the AFC Δ in Definition 1.4 is replaced by an ordinary full cover, that is, the family of all interval-point pairs $([u,v],\xi)$ which are δ-fine for some $\delta(\xi) > 0$, then we have the definition of Henstock integral ([4]).

Setting $U(\tau,t) = f(\tau)t$ and $U(\tau,t) = f(\tau)g(t)$ where $f,g\colon [a,b] \to \mathbb{R}$ and $\tau,t \in [a,b]$, we obtain Riemann-type and Riemann-Stieltjes type integrals for the functions f, g and a given Δ-division D of $[a,b]$.

Considering $U(\tau,t) = f(\tau)t$ in Definition 1.4 we obtain the classical approximately continuous Perron integral.

This definition is given in a more general form because of the general form of the function U. Setting $U(\tau,t) = f(\tau)t$ and replacing AFC Δ by an ordinary full cover it becomes Perron's major and minor functions in the known form.

2. Fundamental properties

Theorem 2.1. *If $U,V \in GAP[a,b]$ and $c_1,c_2 \in \mathbb{R}$, then $(c_1U+c_2V) \in GAP[a,b]$ and*

$$(GAP)\int_a^b (c_1U + c_2V) = c_1(GAP)\int_a^b U + c_2(GAP)\int_a^b V.$$

Proof. Let $A = (GAP)\int_a^b U$ and $B = (GAP)\int_a^b V$. Then given $\epsilon > 0$ there is an approximate full cover Δ_1 of $[a,b]$ such that for every Δ_1-division $D_1 = ([\alpha',\beta'],\tau')$ of $[a,b]$ we have

$$\left|(D_1)\sum\{U(\tau',\beta') - U(\tau',\alpha')\} - A\right| < \epsilon/2. \tag{2.1}$$

Similarly, there is an approximate full cover Δ_2 of $[a,b]$ such that for every Δ_2-division $D_2 = ([\alpha'',\beta''],\tau'')$ of $[a,b]$ we have

$$\left|(D_2)\sum\{V(\tau'',\beta'') - V(\tau'',\alpha'')\} - B\right| < \epsilon/2. \tag{2.2}$$

Putting $\Delta = \Delta_1 \cap \Delta_2$, we get an approximate full cover of $[a,b]$.

Therefore, for any Δ-division $D = ([\alpha, \beta], \tau)$ of $[a, b]$ we have

$$\begin{aligned}
&\Big|(D)\sum\{(c_1U + c_2V)(\tau, \beta) - (c_1U + c_2V)(\tau, \alpha)\} - (c_1A + c_2B)\Big| \\
\le &\Big|(D)\sum c_1\{U(\tau, \beta) - U(\tau, \alpha)\} - c_1A\Big| \\
&+ \Big|(D)\sum c_2\{V(\tau, \beta) - V(\tau, \alpha)\} - c_2B\Big| \\
< &|c_1|\epsilon/2 + |c_2|\epsilon/2, \qquad \text{by (2.1) and (2.2).}
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this gives

$$(GAP)\int_a^b (c_1U + c_2V) = c_1(GAP)\int_a^b U + c_2(GAP)\int_a^b V.$$

□

Theorem 2.2 (Cauchy Criterion). $U \in GAP[a, b]$ *if and only if for every* $\epsilon > 0$ *there is an approximate full cover* Δ *of* $[a, b]$ *such that*

$$|S(U, D_1) - S(U, D_2)| < \epsilon$$

for any two Δ*-divisions* D_1, D_2 *of* $[a, b]$.

Proof. Let $U \in GAP[a, b]$. Then to each $\epsilon > 0$ there is an approximate full cover Δ of $[a, b]$ such that for any Δ-division D of $[a, b]$ we have

$$\Big|S(U, D) - (GAP)\int_a^b U\Big| < \epsilon/2.$$

Hence for any two Δ-divisions D_1, D_2 of $[a, b]$ we have

$$|S(U, D_1) - S(U, D_2)| < \epsilon.$$

Conversely, suppose that the Cauchy Criterion is satisfied. That is, for every $\epsilon > 0$ there is an approximate full cover Δ of $[a, b]$ such that for any two Δ-divisions D_1, D_2 of $[a, b]$ we have

$$|S(U, D_1) - S(U, D_2)| < \epsilon.$$

Let $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > \cdots > 0$ with $\epsilon_n \to 0$ as $n \to\propto$.

For each positive integer n, there is an approximate full cover Δ'_n of $[a, b]$ such that for any two Δ'_n-divisions D_n, D'_n of $[a, b]$ we have

$$|S(U, D_n) - S(U, D'_n)| < \epsilon_n.$$

Let $\Delta_1 = \Delta'_1$, $\Delta_n = \Delta'_1 \cap \Delta'_2 \cap \cdots \cap \Delta'_n$ $(n = 2, 3, \dots)$ and fix a Δ_n-division D_n of $[a, b]$.

Let $t_n = S(U, D_n)$. It is clear that $\Delta_{n+1} \subseteq \Delta_n$, $n = 1, 2, 3 \dots$.

Note that $\{t_n\}$ is a Cauchy sequence of real numbers and let $A = \lim_{n\to\infty} t_n$. Then given $\epsilon > 0$ there exists a positive integer N such that $|t_N - A| < \epsilon_N < \epsilon/2$. Then for any Δ_N-division D'_N of $[a, b]$ we have

$$|S(U, D'_N) - A| \leq |S(U, D'_N) - t_N| + |t_N - A| < \epsilon_N + \epsilon_N < \epsilon.$$

Therefore, $U \in GAP[a, b]$ and $(GAP) \int_a^b U = A$. $\square$

Theorem 2.3. *If $U \in GAP[a, b]$, then for every $[c, d] \subset [a, b]$ we have $U \in GAP[c, d]$.*

Proof. Since $U \in GAP[a, b]$, by Cauchy criterion, given $\epsilon > 0$ there exists an approximate full cover Δ of $[a, b]$ such that for any two Δ-divisions D, D' of $[a, b]$ we have

$$|S(U, D) - S(U, D')| < \epsilon.$$

We assume that $a < c < d < b$. Take any two Δ-divisions D_1, D_2 of $[c, d]$ and another Δ-division D_3 of $[a, c] \cup [d, b]$.

Then $D_1 \cup D_3$ forms a Δ-division of $[a, b]$. Then $S(U, D_1 \cup D_3) = S(U, D_1) + S(U, D_3)$.

Similarly, $S(U, D_2 \cup D_3) = S(U, D_2) + S(U, D_3)$ and $D_2 \cup D_3$ is a Δ-division of $[a, b]$. So, by Cauchy criterion, we have

$$\begin{aligned}|S(U, D_1) - S(U, D_2)| &= |\{S(U, D_1) + S(U, D_3)\} - \{S(U, D_2) + S(U, D_3)\}| \\ &= |S(U, D_1 \cup D_3) - S(U, D_2 \cup D_3)| < \epsilon.\end{aligned}$$

Hence $U \in GAP[c, d]$. $\square$

Theorem 2.4. *If $a < c < b$ and $U \in GAP[a, c]$, $U \in GAP[c, b]$, then $U \in GAP[a, b]$ and*

$$(GAP) \int_a^b U = (GAP) \int_a^c U + (GAP) \int_c^b U.$$

Proof. Let $\epsilon > 0$ be arbitrary. Let $A = (GAP) \int_a^c U$ and $B = (GAP) \int_c^b U$. By the assumption, there exists an approximate full cover Δ_1 of $[a, c]$ such that for any Δ_1-division $D_1 = ([\alpha, \beta], \tau)$ of $[a, c]$ we have

$$|S(U, D_1) - A| < \epsilon/2.$$

We may assume that when $\tau \in [a, c)$, the interval $[\alpha, \beta]$ of D_1 is contained in $[a, c)$.

Similarly, there exists an approximate full cover Δ_2 of $[c, b]$ such that for any Δ_2-division $D_2 = ([\alpha', \beta'], \tau')$ of $[c, b]$ we have

$$|S(U, D_2) - B| < \epsilon/2.$$

When $\tau' \in (c, b]$, the interval $[\alpha', \beta']$ of D_2 is contained in $(c, b]$. Let $D_{1,c}$ be an approximate neighbourhood of c contained in $[a, c]$. Furthermore, $([\alpha, c], c) \in \Delta_1$ whenever $\alpha \in D_{1,c}$ and $\alpha < c$.

Similarly, we can choose an approximate neighbourhood $D_{2,c}$ of c contained in $[c, b]$ and $([c, \beta], c) \in \Delta_2$ whenever $\beta \in D_{2,c}$ and $c < \beta$.

We define Δ as follows.

When $\tau \in [a, c)$ we choose the interval-point pair $([\alpha, \beta], \tau)$ from Δ_1 and we choose the interval-point pair $([\alpha, \beta], \tau)$ from Δ_2 if $\tau \in (c, b]$.

When $\tau =$ c, we choose the interval-point pair $([\alpha, \beta], c)$ such that $([\alpha, c], c) \in \Delta_1$ and $([c, \beta], c) \in \Delta_2$.

This creates our required Δ.

Then for every Δ-division $D = ([\alpha, \beta], \tau)$ of $[a, b]$ we can express the sum $S(U, D)$ as the sum of $S(U, D_1)$ and $S(U, D_2)$ in which $\tau \in [a, c]$ and $\tau \in [c, b]$ respectively and D_1 is a Δ_1-division of $[a, c]$ and D_2 is a Δ_2-division of $[c, b]$.

Then

$$\begin{aligned} |S(U, D) - (A + B)| &= |S(U, D_1) + S(U, D_2) - (A + B)| \\ &\le |S(U, D_1) - A| + |S(U, D_2) - B| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence $U \in GAP[a, b]$ and

$$(GAP)\int_a^b U = A + B = (GAP)\int_a^c U + (GAP)\int_c^b U.$$

$\square$

The following theorem has an important use in the theory of generalized Perron integral.

Theorem 2.5 (Saks-Henstock Lemma). *Let* $U\colon [a,b]\times[a,b] \to \mathbb{R}$ *be GAP-integrable over* $[a,b]$. *Then given* $\epsilon > 0$ *there is an approximate full cover* Δ *of* $[a,b]$ *such that for every* Δ*-division* $D = \big\{([\alpha_{j-1},\alpha_j],\tau_j) :\ j = 1,2,\dots,q\big\}$ *of* $[a,b]$ *we have*

$$\Big|\sum_{j=1}^{q}\{U(\tau_j,\alpha_j) - U(\tau_j,\alpha_{j-1})\} - (GAP)\int\limits_a^b U\Big| < \epsilon.$$

Then if $\big\{([\beta_j,\gamma_j],\zeta_j) :\ j = 1,2,\dots,m\big\}$ *represents a* Δ*-partial division of* $[a,b]$ *we have*

$$\Bigg|\sum_{j=1}^{m}\Bigg[\{U(\zeta_j,\gamma_j) - U(\zeta_j,\beta_j)\} - (GAP)\int\limits_{\beta_j}^{\gamma_j} U\Bigg]\Bigg| < \epsilon.$$

P r o o f. Without any loss of generality it can be assumed that $\beta_j < \gamma_j$ for $j = 1,2,\dots,m$. Denote $\gamma_0 = a$ and $\beta_{m+1} = b$. If $\gamma_j < \beta_{j+1}$ for some $j = 0,1,\dots,m$ then the integral $(GAP)\int\limits_{\gamma_j}^{\beta_{j+1}} U$ exists by Theorem 2.3 and therefore for every $\epsilon > 0$ there is an approximate full cover $\Delta_j \subseteq \Delta$ of $[\gamma_j,\beta_{j+1}]$ such that for every Δ_j-division D^j of $[\gamma_j,\beta_{j+1}]$ we have

$$\Big|S(U,D^j) - (GAP)\int\limits_{\gamma_j}^{\beta_{j+1}} U\Big| < \epsilon/2(m+1).$$

If $\gamma_j = \beta_{j+1}$, then we take $S(U,D^j) = 0$. Then $\sum\limits_{j=1}^{m}[U(\zeta_j,\gamma_j) - U(\zeta_j,\beta_j)] + \sum\limits_{j=1}^{m} S(U,D^j)$ represents an integral sum which corresponds to a certain Δ-division and hence

$$\Big|\sum_{j=1}^{m}[U(\zeta_j,\gamma_j) - U(\zeta_j,\beta_j)] + \sum_{j=1}^{m} S(U,D^j) - (GAP)\int\limits_a^b U\Big| < \epsilon/2.$$

Thus

$$
\begin{aligned}
&\Bigg|\sum_{j=1}^{m}\Bigg[\{U(\zeta_j,\gamma_j)-U(\zeta_j,\beta_j)\}-(GAP)\int_{\beta_j}^{\gamma_j}U\Bigg]\Bigg| \\
&=\Bigg|\sum_{j=1}^{m}[U(\zeta_j,\gamma_j)-U(\zeta_j,\beta_j)]+\sum_{j=1}^{m}S(U,D^j)-\sum_{j=1}^{m}S(U,D^j) \\
&\qquad\qquad -(GAP)\int_a^b U+\sum_{j=1}^{m}(GAP)\int_{\gamma_j}^{\beta_{j+1}}U\Bigg| \\
&\le\Bigg|\sum_{j=1}^{m}[U(\zeta_j,\gamma_j)-U(\zeta_j,\beta_j)]+\sum_{j=1}^{m}S(U,D^j)-(GAP)\int_a^b U\Bigg| \\
&\qquad\qquad +\sum_{j=1}^{m}\Bigg|S(U,D^j)-(GAP)\int_{\gamma_j}^{\beta_{j+1}}U\Bigg| \\
&<\epsilon/2+(m+1)\ \epsilon/2(m+1)<\epsilon/2+\epsilon/2=\epsilon.
\end{aligned}
$$

Hence the result is obtained. □

Using the Saks-Henstock lemma presented in Theorem 2.5 we obtain the following corollary of it.

Corollary 2.6. *If $U\in GAP[a,b]$, then for every $\epsilon>0$ there exists an approximate full cover Δ of $[a,b]$ such that for every Δ-division $D=\big\{([\alpha_{j-1},\alpha_j],\tau_j):\ j=1,2,\dots,q\big\}$ of $[a,b]$ we have*

$$
\sum_{j=1}^{q}\Bigg|[U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})]-(GAP)\int_{\alpha_{j-1}}^{\alpha_j}U\Bigg|<\epsilon.
$$

P r o o f. Since $U\in GAP[a,b]$, given $\epsilon>0$ there exists an approximate full cover Δ of $[a,b]$ such that for every Δ-division $D=\big\{([\alpha_{j-1},\alpha_j],\tau_j):\ j=1,2,\dots,q\big\}$ of $[a,b]$ we have

$$
\Bigg|\sum_{j=1}^{q}[U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})]-(GAP)\int_a^b U\Bigg|<\epsilon/2
$$

or,

$$\left|\sum_{j=1}^{q}[U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})]-\sum_{j=1}^{q}(GAP)\int\limits_{\alpha_{j-1}}^{\alpha_j}U\right|<\epsilon/2$$

or,

$$\left|\sum_{j=1}^{q}\left[\{U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})\}-(GAP)\int\limits_{\alpha_{j-1}}^{\alpha_j}U\right]\right|<\epsilon/2.$$

We decompose the division D into two systems D^+ and D^- consisting of non-negative and negative contributions of $\{U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})\}-(GAP)\int\limits_{\alpha_{j-1}}^{\alpha_j}U$ respectively.

By Saks-Henstock lemma, we obtain

$$(D^+)\sum_{j=1}^{q}\left[\{U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})\}-(GAP)\int\limits_{\alpha_{j-1}}^{\alpha_j}U\right]<\epsilon/2$$

and

$$(D^-)\sum_{j=1}^{q}\left[(GAP)\int\limits_{\alpha_{j-1}}^{\alpha_j}U-\{U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})\}\right]<\epsilon/2.$$

Hence

$$\begin{aligned}&\sum_{j=1}^{q}\left|[U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})]-(GAP)\int\limits_{\alpha_{j-1}}^{\alpha_j}U\right|\\ &=(D^+)\sum_{j=1}^{q}\left[\{U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})\}-(GAP)\int\limits_{\alpha_{j-1}}^{\alpha_j}U\right]\\ &+(D^-)\sum_{j=1}^{q}\left[(GAP)\int\limits_{\alpha_{j-1}}^{\alpha_j}U-\{U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})\}\right]\\ &<\epsilon/2+\epsilon/2=\epsilon.\end{aligned}$$

Hence the result is obtained. □

3. The indefinite generalized Perron integral

Definition 3.1. Let $U \in GAP[a,b]$. The function $\phi\colon [a,b] \to \mathbb{R}$ defined by $\phi(s) = (GAP)\int_a^s U$, $a < s \le b$, $\phi(a) = 0$ is called the *indefinite GAP-integral of U.*

According to Theorem 2.3 this definition makes sense.

Lemma 3.2. *Let $U\colon [a,b]\times[a,b] \to \mathbb{R}$ be such that $U \in GAP[a,b]$ and $c \in [a,b]$. Then*

$$\lim_{s\to c}\Big[(GAP)\int_a^s U - U(c,s) + U(c,c)\Big] = (GAP)\int_a^c U.$$

Proof. Let $\epsilon > 0$ be given. Since $U \in GAP[a,b]$, given $\epsilon > 0$ there exists an approximate full cover Δ of $[a,b]$ such that for every Δ-division D of $[a,b]$ we have

$$\Big|S(U,D) - (GAP)\int_a^b U\Big| < \epsilon.$$

We may suppose that c is always an associated point in Δ-division D of $[a,b]$. Let D_c be an approximate neighbourhood of c contained in $[a,b]$. Furthermore, $([s,c],c) \in \Delta$ whenever $s \in D_c$ and $s < c$.

Similarly, we can choose an approximate neighbourhood D_c of c contained in $[a,b]$ and $([c,s],c) \in \Delta$ whenever $s \in D_c$ and $c < s$.

Then Saks-Henstock lemma gives

$$\Big|U(c,s) - U(c,c) - (GAP)\int_c^s U\Big| < \epsilon \qquad \text{if} \quad c < s.$$

That is,

$$\Big|(GAP)\int_a^s U - U(c,s) + U(c,c) - (GAP)\int_a^c U\Big|$$
$$= \Big|(GAP)\int_c^s U - U(c,s) + U(c,c)\Big| < \epsilon.$$

This yields the relation.

The proof is similar if $s < c$. □

Theorem 3.3. *If $U \in GAP[a,b]$ and ϕ is the indefinite GAP-integral of U, then ϕ is approximately continuous at a point $s_0 \in [a,b]$ if and only if the function $U(s_0,\cdot)\colon [a,b] \to \mathbb{R}$ is continuous at the point s_0.*

Proof. Let $U \in GAP[a,b]$ and $s_0 \in [a,b]$. We assume that the function $U(s_0,\cdot)$ is continuous at the point s_0. We shall show that ϕ is approximately continuous at s_0. By the above lemma, we obtain

$$\lim_{s\to s_0}\left[(GAP)\int_a^s U - U(s_0,s) + U(s_0,s_0)\right] = (GAP)\int_a^{s_0} U.$$

i.e.

$$\lim_{s\to s_0}[\phi(s) - U(s_0,s) + U(s_0,s_0)] = \phi(s_0). \tag{3.1}$$

Again, since $U(s_0,\cdot)$ is continuous at the point s_0, for each $\epsilon > 0$ there exists a $\eta > 0$ such that

$$|s - s_0| < \eta \implies |U(s_0,s) - U(s_0,s_0)| < \epsilon/2.$$

Or, in other words, $\lim\limits_{s\to s_0} U(s_0,s) = U(s_0,s_0)$. Then (3.1) gives, $\lim\limits_{s\to s_0}\phi(s) = \phi(s_0)$. That is, ϕ is continuous at the point s_0. Thus ϕ is approximately continuous at the point s_0.

Conversely, suppose that ϕ is approximately continuous at a point $s_0 \in [a,b]$. That is, for every $\epsilon > 0$ there exists a set D_{s_0} of density 1 at s_0 such that

$$|\phi(s) - \phi(s_0)| < \epsilon/2 \qquad \text{for every} \quad s \in D_{s_0}. \tag{3.2}$$

Since $U \in GAP[a,b]$, given $\epsilon > 0$ there is an approximate full cover Δ of $[a,b]$ such that for every Δ-division D of $[a,b]$ we have

$$\left|S(U,D) - (GAP)\int_a^b U\right| < \epsilon/2.$$

We may suppose that s_0 is always an associated point in Δ-division D of $[a,b]$ and $|s - s_0| < \eta$ ($\eta > 0$). Let D_{s_0} be an approximate neighbourhood of s_0 contained in $[a,b]$. Furthermore, $([s,s_0],s_0) \in \Delta$ whenever $s \in D_{s_0}$ and $s < s_0$. Similarly, we can choose an approximate neighbourhood D_{s_0} of s_0 contained in $[a,b]$ and $([s_0,s],s_0) \in \Delta$ whenever $s \in D_{s_0}$ and $s_0 < s$.

Then by Saks-Henstock lemma applied to $([s, s_0], s_0)$ or $([s_0, s], s_0)$ we have

$$\left|U(s_0, s_0) - U(s_0, s) - (GAP)\int_s^{s_0} U\right| < \epsilon/2 \qquad \text{if} \quad s < s_0.$$

$$\left|U(s_0, s) - U(s_0, s_0) - (GAP)\int_{s_0}^{s} U\right| < \epsilon/2 \qquad \text{if} \quad s_0 < s.$$

If $s < s_0$, then

$$\begin{aligned} |U(s_0, s) - U(s_0, s_0)| &= |U(s_0, s_0) - U(s_0, s)| \\ &\le \left|U(s_0, s_0) - U(s_0, s) - (GAP)\int_s^{s_0} U\right| + \left|(GAP)\int_s^{s_0} U\right| \\ &< \epsilon/2 + |\phi(s_0) - \phi(s)| < \epsilon/2 + \epsilon/2 = \epsilon \qquad \text{by (3.2).} \end{aligned}$$

Similar, if $s_0 < s$.

Hence $U(s_0, \cdot)$ is continuous at the point s_0. □

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Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road, Kolkata-700019
INDIA
E-mail: gangulydk@yahoo.co.in
ranumukherjee05@yahoo.co.in