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DIOPHANTINE EQUATIONS FOR MORGAN-VOYCE AND OTHER MODIFIED ORTHOGONAL POLYNOMIALS

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ABSTRACT. It is well-known that Morgan-Voyce polynomials $B_n(x)$ and $b_n(x)$ satisfy both a Sturm-Liouville equation of second order and a three-term recurrence equation ([SWAMY, M.: *Further properties of Morgan-Voyce polynomials*, Fibonacci Quart. **6** (1968), 167–175]). We study Diophantine equations involving these polynomials as well as other modified classical orthogonal polynomials with this property. Let $A, B, C \in \mathbb{Q}$ and $\{p_k(x)\}$ be a sequence of polynomials defined by

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x - c_0 \\ p_{n+1}(x) &= (x - c_n)p_n(x) - d_n p_{n-1}(x), \qquad n = 1, 2, \dots, \end{aligned}$$

with

$$(c_0, c_n, d_n) \in \left\{(A, A, B), (A + B, A, B^2), (A, Bn + A, \tfrac{1}{4}B^2n^2 + Cn)\right\}$$

with $A \neq 0$, $B > 0$ in the first, $B \neq 0$ in the second and $C > -\frac{1}{4}B^2$ in the third case. We show that the Diophantine equation

$$\mathscr{A} p_m(x) + \mathscr{B} p_n(y) = \mathscr{C}$$

with $m > n \geq 4$, $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbb{Q}$, $\mathscr{A}\mathscr{B} \neq 0$ has at most finitely many solutions in rational integers x, y.

1. Introduction

A topic of particular interest in number theory is the general Diophantine equation

$$\mathscr{A} p_m(x) + \mathscr{B} p_n(y) = \mathscr{C} \tag{1}$$

in rational integers (x, y) with $\{p_k(x)\}$ being an infinite family of polynomials

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with rational coefficients and $\mathscr{A},\mathscr{B},\mathscr{C}$ some fixed rational parameters satisfying $\mathscr{A}\mathscr{B}\neq 0$. Bilu–Tichy's criterion [1] enables one to decide quite algorithmically the problem whether there is a finite number of solutions in rational integers (x,y) of (1) for given $m,n\geq 2$ or not. Equation (1) sometimes arises while investigating combinatorial counting and enumeration problems, so that it is likely the case that there appears a three-term recurrence relating the polynomials

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x-c_0 \\ p_{n+1}(x) &= (x-c_n)p_n(x)-d_np_{n-1}(x), \qquad n=1,2,\dots, \end{aligned} \tag{2}$$

where c_n and d_n are parameters depending just on n. Finiteness for (1) has already been proved in case of $c_n=0$ and $d_n=$const in [3] by Dujella and Tichy, $c_n=0$ and $d_n=n^2$ in [5] by Kirschenhofer, Pethő and Tichy in some special cases of m, n. In [6], Kirschenhofer and Pfeiffer could give a direct combinatorial interpretation of some special case of (2) in terms of colored permutations. Moreover, in [7] these authors attacked the general case $c_n=0$ and arbitrary d_n. Results of finiteness of number of integral solutions (x,y) in case of $(\mathscr{A},\mathscr{B},\mathscr{C})=(1,-1,0)$ could be given under certain growth conditions of d_n. Other results have recently been obtained by the authors of this paper for $p_k(x)=\binom{x}{k}$ in [12] and $p_k(x)$ being the classical Jacobi, Laguerre or Hermite polynomials in [11]. In all these cases it has been shown that 'two interval monotonicity' of stationary points of the polynomials $p_k(x)$, i.e. informally speaking, the convexity of the local maxima, is a useful condition to get reductions in the above-mentioned criterion. In this paper we continue the investigation from [11], this time starting from a recurrence of type (2). We give necessary and sufficient conditions on c_n and d_n for $\{p_k(x)\}$ to satisfy second order Sturm-Liouville differential equations which automatically inherit the wanted 'two interval monotonicity' property. The general results will finally be applied to the classical Morgan-Voyce polynomials $B_n(x)$ and $b_n(x)$ which are known to satisfy both a recurrence of type (2) and a differential equation ([13]), i.e.

$$\begin{aligned} &B_0(x)=1,\quad B_1(x)=x+1,\quad B_{n+1}(x)=(x+2)B_n(x)-B_{n-1}(x),\quad \text{resp.}\\ &b_0(x)=1,\quad b_1(x)=x+2,\quad b_{n+1}(x)=(x+2)b_n(x)-b_{n-1}(x) \end{aligned}$$

$$\begin{aligned} x(x+4)B_n''(x)+3(x+2)B_n'(x)-n(n+2)B_n(x)&=0,\quad \text{resp.}\\ x(x+4)b_n''(x)+2(x+1)b_n'(x)-n(n+1)b_n(x)&=0. \end{aligned}$$

The Morgan-Voyce polynomials $B_n(x)$ and $b_n(x)$ can also be given explicitly by

$$B_n(x)=\sum_{j=0}^{n}\binom{n+j-1}{n-j}x^j,\qquad b_n(x)=\sum_{j=0}^{n}\binom{n+j}{n-j}x^j, \tag{3}$$

see [13]. They are also related to the well-known Fibonacci polynomials $F(x)$ by

$$B_n(x^2) = \frac{1}{x}F_{2n+2}(x), \qquad b_n(x^2) = F_{2n+1}(x).$$

2. Main result

Theorem 1. *Let $\{p_k(x)\}$ be a polynomial sequence satisfying (2). Assume one of the following conditions ($A, B, C \in \mathbb{Q}$):*

(1) *$c_0 = A$, $c_n = A$, $d_n = B$, with $A \neq 0$ and $B > 0$,*
(2) *$c_0 = A + B$, $c_n = A$, $d_n = B^2$, with $B \neq 0$,*
(3) *$c_0 = A$, $c_n = Bn + A$, $d_n = \frac{1}{4}B^2n^2 + Cn$, with $C > -\frac{1}{4}B^2$.*

Then the Diophantine equation

$$\mathscr{A} p_m(x) + \mathscr{B} p_n(y) = \mathscr{C}$$

with $m > n \geq 4$, $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbb{Q}$, $\mathscr{A}\mathscr{B} \neq 0$ has at most finitely many solutions in rational integers x, y.

To start with, we recall the connection between three-term recurrences and Sturm-Liouville differential equations as given in [4] and [9]:

Lemma 2. *The following conditions are equivalent:*

(i) *The second-order Sturm-Liouville differential equation ($n \geq 0$)*

$$\sigma(x)p_n''(x) + \tau(x)p_n'(x) - n((n-1)a + d)p_n(x) = 0, \tag{4}$$

with $\sigma(x) = ax^2 + bx + c \not\equiv 0$, $\tau = dx + e$, $a, b, c, d, e \in \mathbb{R}$, $d \neq -ta$ for all $t \in \mathbb{Z}_{\geq 0}$ has a (up to a factor depending on n) unique infinite polynomial family solution $\{p_n(x)\}$ of exact degree n.

(ii) *The family $\{p_n(x)\}$ satisfies a three-term recurrence of type (2) with*

$$c_0 = -\frac{e}{d}$$
$$c_n = -\frac{2nb((n-1)a + d) - e(2a - d)}{(2na + d)((2n-2)a + d)}$$
$$d_n = \frac{n((n-2)a + d)}{((2n-1)a + d)((2n-3)a + d)}\left(-c + \frac{((n-1)b + e)(((n-1)a + d)b - ae)}{((2n-2)a + d)^2}\right).$$

The properties of Lemma 2 are of course shared by all classical orthogonal polynomials (Jacobi, Laguerre, Hermite) but also by Bessel polynomials. On the other hand, one has by Favard's Theorem (see for instance [14]), that all polynomial families defined by a three-term recurrence of shape (2) are orthogonal with respect to some moment functional. If one demands orthogonality with respect to a positive definite moment functional (in order to use all known facts about zeros of orthogonal polynomials etc.), then one exactly gets just Jacobi, Laguerre and Hermite polynomials up to a linear transformation $x \mapsto \nu_1 x + \nu_2$ with $\nu_1 \nu_2 \in \mathbb{R}$ (see [10] and [2]). Hence, one can completely characterize all positive definite orthogonal solutions of (4) just by looking to the coefficients a, b, c, d, e (see [4]). Note that by 'characterization' we mean up to a factor depending only on n (see Lemma 2, first condition).

Lemma 3. *The only infinite chains of polynomial solutions of* (4) *which are orthogonal with respect to some positive definite moment functional are*

(i) *Hermite:* $a = 0$, $b = 0$, $c = 1$, *with* $d < 0$,

(ii) *Laguerre:* $a = 0$, $b = 1$, $c = -\alpha$, *with* $d\alpha + e > 0$,

(iii) *Jacobi:* $a = -1$, $b = \alpha + \beta$, $c = -\alpha\beta$, *with* $\alpha < \beta$ *and* $d < 0$, $d\alpha + e > 0$, $d\beta + e < 0$.

Recall that, for instance, by Jacobi we mean all classical Jacobi polynomials $P_n^{(\mu_1,\mu_2)}(x)$ with possibly shifted argument $x \mapsto \nu_1 x + \nu_2$ (for the notation see the Askey-scheme [8]). We now plug these values into condition (2) of Lemma 2 and by solving the equations with computer algebra system MAPLE to the rest of the variables we get a characterization by the three-term recurrence.

Corollary 4.

(i) *The parameters* $c_0 = A$, $c_n = A$, $d_n = B$ *with* $B > 0$ *give (shifted) Jacobi polynomials satisfying*

$$(x^2 - 2Ax + A^2 - 4B)p_n''(x) + 3(x - A)p_n'(x) - n(n+2)p_n(x) = 0.$$

(ii) *The parameters* $c_0 = A + B$, $c_n = A$, $d_n = B^2$ *with* $B \neq 0$ *give (shifted) Jacobi polynomials satisfying*

$$(x^2 - 2Ax + A^2 - 4B)p_n''(x) + 2(x - A - B)p_n'(x) - n(n+1)p_n(x) = 0.$$

(iii) *The parameters* $c_0 = A$, $c_n = Bn + A$, $d_n = \frac{1}{4}B^2 n^2 + Cn$ *with* $C > -\frac{1}{4}B^2$ *give* all *(shifted) Laguerre polynomials* $(B \neq 0)$ *satisfying*

$$\left(x + \frac{1}{2}B - A + \frac{2C}{B}\right)p_n''(x) - \frac{2}{B}(x - A)p_n'(x) + \frac{2}{B}np_n(x) = 0$$

and all *(shifted) Hermite polynomials* $(B = 0)$ *satisfying*

$$p_n''(x) - \frac{1}{C}(x - A)p_n'(x) + \frac{1}{C}np_n(x) = 0.$$

Note that all Hermite and Laguerre polynomials (with possible shifts) are totally characterized by the choices of the given c_n and d_n, while for Jacobi polynomials we just give simple choices. Nevertheless, the classical Morgan-Voyce polynomials $B_n(x)$ and $b_n(x)$ defined by $(c_0, c_n, d_n) = (-2, -2, 1)$ (which means $A = -2$ and $B = 1$ in the first case) resp. $(c_0, c_n, d_n) = (-1, -2, 1)$ (which means $A = -2$ and $B = 1$ in the second case) are in the given classes and we immediately see that they are simple transformations of Jacobi polynomials ([13]). The calculations to obtain Corollary 4 are straightforward with MAPLE, we just mention one crucial point. In the first Jacobi case after solving the system one also has to deal with the equation

$$(x^2 - 2Ax + A^2 - 4B)p_n''(x) + (x - A)p_n'(x) - n^2 p_n(x) = 0.$$

However, one easily sees that such equation implies $B = 0$, which is not allowed by $B > 0$.

Polynomial solutions of Sturm-Liouville equations are always two-interval monotone provided $\sigma'(x) - 2\tau(x) \not\equiv 0$ (see [11]). This means there are always two real intervals I_1 and I_2 on which the modulus of the values of the stationary points is strictly decreasing on I_1 and increasing on I_2. All polynomial families above are two-interval monotone except for the first case of the (shifted) Jacobi with $A = 0$ in Corollary 4. We add the condition $A \neq 0$ in our further investigations. We employ now the reduction of the Bilu-Tichy criterion from [11]:

Lemma 5. *Let $\{p_k(x)\}$ be a polynomial family with simple stationary points and have the two-interval monotonicity property. Then, if there exist no parameters $v_2, v_1, v_0, \mathscr{A}$ such that for some integers $m, n \geq 4$ it holds that*

$$p_m(x) = \mathscr{A} p_n(v_2 x^2 + v_1 x + v_0), \tag{5}$$

then the original Diophantine equation (1) *has at most finitely many solutions in rational integers x, y.*

The task is now, to come to a contradiction if one supposes a quadratic representation like (5). For this purpose we compare coefficients on both sides of (5). Let $p_m(x) = x^m + k_{m-1}^{(m)} x^{m-1} + k_{m-2}^{(m)} x^{m-2} + \cdots + k_0^{(0)}$. Then we have the following recursive relations

$$\begin{aligned}
k_{m-1}^{(m)} &= -c_0 - \sum_{j=1}^{m-1} c_j, \\
k_{m-2}^{(m)} &= -\sum_{j=1}^{m-1} \left(c_j k_{j-1}^{(j)} + d_j \right), \\
k_{m-t}^{(m)} &= -\sum_{j=t-1}^{m-1} \left(c_j k_{j-t+1}^{(j)} + d_j k_{j-t+2}^{(j-1)} \right), \qquad t \geq 3.
\end{aligned}$$

We now list the uppermost coefficient equations of (5). We do not give an 'Eq. 5' because we do not need it in the sequel.

- **Eq. 0**, $[x^m]$:

$$1 = \mathscr{A} v_2^n$$

- **Eq. 1**, $[x^{m-1}]$:

$$k_{m-1}^{(m)} = \mathscr{A}\left(n v_2^{n-1} v_1\right)$$

- **Eq. 2**, $[x^{m-2}]$:

$$k_{m-2}^{(m)} = \mathscr{A}\left(n v_2^{n-1} v_0 + \frac{n(n-1)}{2} v_2^{n-2} v_1^2 + k_{n-1}^{(n)} v_2^{n-1}\right)$$

- **Eq. 3**, $[x^{m-3}]$:

$$k_{m-3}^{(m)} = \mathscr{A}\left(n(n-1) v_2^{n-2} v_1 v_0 + \frac{n(n-1)(n-2)}{6} v_2^{n-3} v_1^3 + k_{n-1}^{(n)}(n-1) v_2^{n-2} v_1\right)$$

- **Eq. 4**, $[x^{m-4}]$:

$$\begin{aligned} k_{m-4}^{(m)} = \mathscr{A}\Bigg(&\frac{n(n-1)}{2} v_2^{n-2} v_0^2 + \frac{n(n-1)(n-2)}{2} v_2^{n-3} v_1^2 v_0 + \frac{n(n-1)(n-2)(n-3)}{24} v_2^{n-4} v_1^4 + \\ &+ k_{n-1}^{(n)}(n-1) v_2^{n-2} v_0 + k_{n-1}^{(n)} \frac{(n-1)(n-2)}{2} v_2^{n-3} v_1^2 + k_{n-2}^{(n)} v_2^{n-2}\Bigg) \end{aligned}$$

- **Eq. 6**, $[x^{m-6}]$:

$$\begin{aligned} k_{m-6}^{(m)} = \mathscr{A}\Bigg(&\frac{n(n-1)(n-2)}{6} v_2^{n-3} v_0^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{24} v_2^{n-5} v_1^4 v_0 \\ &+ \frac{n(n-1)(n-2)(n-3)}{4} v_2^{n-4} v_1^2 v_0^2 + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{720} v_2^{n-6} v_1^6 \\ &+ k_{n-1}^{(n)} \frac{(n-1)(n-2)}{2} v_2^{n-3} v_0^2 + k_{n-1}^{(n)} \frac{(n-1)(n-2)(n-3)}{2} v_2^{n-4} v_1^2 v_0 \\ &+ k_{n-1}^{(n)} \frac{(n-1)(n-2)(n-3)(n-4)}{24} v_2^{n-5} v_1^4 + k_{n-2}^{(n)}(n-2) v_2^{n-3} v_0 \\ &+ k_{n-2}^{(n)} \frac{(n-2)(n-3)}{2} v_2^{n-4} v_1^2 + k_{n-3}^{(n)} v_2^{n-3}\Bigg) \end{aligned}$$

We start with the Jacobi case, in which $c_0 = A \neq 0$.

$$\begin{aligned} k_{m-1}^{(m)} &= -Am \\ k_{m-2}^{(m)} &= \frac{1}{2}(m-1)(A^2 m - 2B) \\ k_{m-3}^{(m)} &= -\frac{1}{6}A(m-1)(m-2)(A^2 m - 6B) \\ k_{m-4}^{(m)} &= \frac{1}{24}(m-2)(m-3)(A^4 m^2 - 12A^2 Bm - A^4 m + 12A^2 B + 12B^2) \\ k_{m-6}^{(m)} &= \frac{1}{720}(m-3)(m-4)(m-5)(A^6 m^3 - 3A^6 m^2 - 30A^4 Bm^2 + 180A^2 mB^2 \\ &\quad + 2A^6 m + 90A^4 mB - 360A^2 B^2 - 60A^4 B - 120B^3) \end{aligned}$$

Combing the equations Eq. 0, 1, 2, 3 and 4 we get

$$v_2^2 = \frac{2n}{B(2n+1)}.$$

Finally, Eq. 6 gives $(2n+1)(n-2)B^3/n^2 = 0$, which is a contradiction. In the second Jacobi case $c_0 = A + B$ and $d_n = B^2$ we calculate

$$
\begin{aligned}
k_{m-1}^{(m)} &= -B - Am\\
k_{m-2}^{(m)} &= \frac{1}{2}(m-1)(A^2m + 2AB - 2B^2)\\
k_{m-3}^{(m)} &= -\frac{1}{6}(m-2)(A^3m^2 - A^3m + 3A^2mB - 6AmB^2 - 3A^2B + 6AB^2 - 6B^3)\\
k_{m-4}^{(m)} &= \frac{1}{24}(m-2)(m-3)(A^4m^2 - A^4m + 4A^3mB - 12A^2mB^2\\
&\quad - 4A^3B - 12B + 12A^2B^2 - 24AB^3)\\
k_{m-6}^{(m)} &= \frac{1}{720}(m-3)(m-4)(m-5)(A^6m^3 - 30A^4B^2m^2 - 3A^6m^2 + 6A^5Bm^2 + 90A^4mB^2\\
&\quad - 120A^3mB^3 + 2A^6m + 180A^2mB^4 - 18A^5mB - 360A^2B^4 - 60A^4B^2\\
&\quad + 240A^3B^3 - 120B^6 + 12A^5B + 360AB^5)
\end{aligned}
$$

This time Eq. 0–3 yield the contradiction $B^3(2m+1)(m-2) = 0$. Next, in the Laguerre-Hermite case we have the following upper coefficients:

$$
\begin{aligned}
k_{n-1}^{(n)} &= -\frac{1}{2}n(Bn - B + 2A)\\
k_{n-2}^{(n)} &= \frac{1}{8}n(n-1)(B^2n^2 + 4ABn - 3B^2n - 4AB + 4A^2 - 4C + B^2)\\
k_{n-3}^{(n)} &= -\frac{1}{48}n(n-1)(n-2)(B^3n^3 - 6B^3n^2 + 6AB^2n^2 + 12BA^2n + 8B^3n - 12BCn - 18AB^2n\\
&\quad + 8A^3 + 6AB^2 - B^3 - 24AC + 20BC - 12BA^2)\\
k_{n-4}^{(n)} &= \frac{1}{384}n(n-1)(n-2)(n-3)(B^4n^4 - 10B^4n^3 + 8AB^3n_4^3 8AB^3n^2 - 24B^2Cn^2 + 29B^4n^2\\
&\quad + 24B^2A^2n^2 + 64AB^3n - 96BACn - 24B^4n + 104B^2Cn - 72B^2A^2n + 32BA^3n - 96A^2C\\
&\quad + 48C^2 - 32BA^3 + 24B^2A^2 + 16A^4 + B^4 - 80B^2C - 8AB^3 + 160BAC)\\
k_{n-6}^{(n)} &= \frac{1}{46080}n(n-5)(n-1)(n-2)(n-3)(n-4)(-12AB^5 - 160B^3A^3 + 240B^2A^4 + 4720B^2C^2\\
&\quad - 192BA^5 - 1636B^4C - 960A^4C + 2880A^2C^2 + 3200BA^3C - 6720BAC^2 + B^6 + 64A^6\\
&\quad - 960C^3 + 4032B^3AC - 4800B^2A^2C + 60B^4A^2 + 12AB^5n^5 + 1068AB^5n + 240B^2A^4n^2\\
&\quad - 4080B^2C^2n - 1740AB^5n^2 + 814B^6n^2 - 21B^6n^5 - 415B^6n + B^6n^6 - 1440B^4A^2n\\
&\quad + 4376B^4Cn - 720B^2A^4n - 1440B^2A^2Cn^2 - 1920BA^3Cn + 2880BAC^2n + 3840B^3ACn^2\\
&\quad - 8160B^3ACn + 6240B^2A^2Cn - 480B^3ACn^3 + 1280B^3A^3n + 192BA^5n - 545B^6n^3\\
&\quad + 160B^6n^4 - 960B^3A^3n^2 - 3060B^4Cn^2 + 720B^2C^2n^2 + 900AB^5n^3 + 160B^3A^3n^3\\
&\quad + 760B^4Cn^3 - 600B^4A^2n^3 + 60B^4A^2n^4 - 180AB^5n^4 - 60B^4Cn^4 + 1740B^4A^2n^2)
\end{aligned}
$$

In this case Eq. 0–3 and MAPLE give

$$Bn(2n-1)(n-1)(B^2n + 2C) = 0,$$

which does not vanish with exception of $B = 0$ (Hermite case). In the latter cases Eq. 4 gives $1/v_2^2 = 2C(2n-1)$ and finally Eq. 6 the wanted contradiction $(2n-1)(n-2)(n-1)nC^3 = 0$.

As an immediate consequence of Theorem 1 we have:

Corollary 6. *Let $\{p_k(x)\}$ be the Morgan-Voyce polynomials $\{B_k(x)\}$ and $\{b_k(x)\}$ be defined by relation (3). Then the Diophantine equation (1) has at most finitely many solutions for $m, n \geq 4$.*

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