VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-007-0047-y
Math. Slovaca **57** (2007), No. 6, 571–588

CONFIDENCE REGIONS IN A MULTIVARIATE REGRESSION MODEL WITH CONSTRAINTS II

Lubomír Kubáček

(*Communicated by Gejza Wimmer*)

ABSTRACT. The multivariate model, where not only parameters of the mean value of the observation matrix, but also some other parameters occur in constraints, is considered in the paper. Some basic inference is presented under the condition that the covariance matrix is either unknown, or partially unknown, or known.

1. Introduction

The multivariate regression model, where not only parameters of the mean value matrix but also other parameters are involved in the constraints (constraints of the type II) is dealt with.

As a motivation example the following problem can serve. A group of points on the Earth surface is a basis for an investigation of the recent crustal movement. At different times $t_1, \dots, t_m$, positions of points are estimated, e.g. by navigation satellites (GPS). After m measurements, $i = 1, \dots, m$, estimated coordinates of the investigated points are at our disposal. If they are on the same Earth block, they must satisfy some constraints because their relative position are not changed. If they are not on the same block, then in the constraints some new parameters occur. These new parameters characterize unknown shifts among different blocks.

The aim of the paper is to contribute to the theory of such models.

2000 Mathematics Subject Classification: Primary 62J05.
Keywords: multivariate model, confidence region, constraints.
Supported by the grant of the Council of Czech Republic MSM 6 198 959 214.

2. Notation and auxiliary statements

The investigated model will be denoted as

$$\underline{\boldsymbol{Y}} \sim_{nm} (\mathbf{X}\mathbf{B}_1, \boldsymbol{\Sigma} \otimes \mathbf{I}), \quad \mathbf{G}_1\mathbf{B}_1 + \mathbf{G}_2\mathbf{B}_2 + \mathbf{G}_0 = \mathbf{O}. \tag{1}$$

Here $\underline{\boldsymbol{Y}}$ is an $n \times m$ random matrix with the mean value $E(\underline{\boldsymbol{Y}}) = \mathbf{X}\mathbf{B}_1$, $\mathbf{X}$ is an $n \times k_1$ known matrix (design matrix), $\mathbf{B}_1$ is an $k_1 \times m$ matrix of the unknown parameters (coordinates of the investigated points), $\mathbf{G}_1, \mathbf{G}_2$ and $\mathbf{G}_0$ are given matrices of the type $q \times k_1$, $q \times k_2$ and $q \times m$, respectively, and $\mathbf{B}_2$ is a $q \times k_2$ matrix of unknown parameters occurring in the constraints only. The rows of the random matrix $\underline{\boldsymbol{Y}}$ are independent and their common covariance matrix is $\boldsymbol{\Sigma}$.

The model is regular if the rank $r(\mathbf{X})$ of the matrix $\mathbf{X}$ is $r(\mathbf{X}_{n,k_1}) = k_1 < n$, $r(\mathbf{G}_1, \mathbf{G}_2) = q < k_1 + k_2$ and $r(\mathbf{G}_2) = k_2 < q$. The matrix $\boldsymbol{\Sigma}$ is positive definite (p.d.).

The model can be rewritten as

$$\begin{gathered}\mathrm{vec}(\underline{\boldsymbol{Y}}) \sim_{nm} [(\mathbf{I} \otimes \mathbf{X})\mathrm{vec}(\mathbf{B}_1), \boldsymbol{\Sigma} \otimes \mathbf{I}],\\ (\mathbf{I} \otimes \mathbf{G}_1)\mathrm{vec}(\mathbf{B}_1) + (\mathbf{I} \otimes \mathbf{G}_2)\mathrm{vec}(\mathbf{B}_2) + \mathrm{vec}(\mathbf{G}_0) = \boldsymbol{0}.\end{gathered} \tag{2}$$

(Here $\mathrm{vec}(\mathbf{A}) = \mathrm{vec}(\boldsymbol{a}_1, \dots, \boldsymbol{a}_m) = (\boldsymbol{a}_1', \dots, \boldsymbol{a}_m')'$.)

The notation can be compared with the notation of the univariate regression model with constraints of the type II

$$\boldsymbol{Y}^* \sim_n (\mathbf{X}^*\boldsymbol{\beta}_1, \boldsymbol{\Sigma}^*), \quad \mathbf{G}_1^*\boldsymbol{\beta}_1 + \mathbf{G}_2^*\boldsymbol{\beta}_2 + \boldsymbol{g}_0 = \boldsymbol{0},$$

$r(\mathbf{X}^*_{n,k_1}) = k_1 < n$, $r(\mathbf{G}^*_{1,(q,k_1)}, \mathbf{G}^*_{2,(q,k_2)}) = q < k_1 + k_2$, $r(\mathbf{G}^*_{2,(q,k_2)}) = k_2 < q$. Thus it is useful to state some results from the theory of univariate regular linear models with constraints of the type II.

In the following text the notation from [6] is used, i.e. if $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{A}^-$ is an $n \times m$ matrix with the property $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$. If $\mathbf{N}$ is an $n \times n$ positive semidefinite matrix, then $\mathbf{A}^-_{m(N)}$ is an $n \times m$ matrix with the property $\forall\{\boldsymbol{y} \in \mathscr{M} = \{\mathbf{A}\boldsymbol{u} :\ \boldsymbol{u} \in R^n\}\}\, \mathbf{A}\mathbf{A}^-_{m(N)}\boldsymbol{y} = \boldsymbol{y}$ & $\forall\{\boldsymbol{y} \in \mathscr{M}(\mathbf{A})\}$ $\forall\{\boldsymbol{x} :\ \mathbf{A}\boldsymbol{x} = \boldsymbol{y}\}\, \|\mathbf{A}^-_{m(N)}\boldsymbol{y}\| = \sqrt{\boldsymbol{y}'\left(\mathbf{A}^-_{m(N)}\right)'\mathbf{N}\mathbf{A}^-_{m(N)}\boldsymbol{y}} \leq \|\boldsymbol{x}\|_N$. The symbol $\mathbf{A}^+$ means the $n \times m$ matrix such that $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, $\mathbf{A}\mathbf{A}^+ = (\mathbf{A}\mathbf{A}^+)'$, $(\mathbf{A}^+\mathbf{A})' = \mathbf{A}^+\mathbf{A}$. Further $\mathbf{P}_A = \mathbf{A}\mathbf{A}^+ = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}$, $\mathbf{M}_A = \mathbf{I} - \mathbf{P}_A$.

Lemma 2.1. *Let* $\mathbf{C}^* = (\mathbf{X}^*)'(\boldsymbol{\Sigma}^*)^{-1}\mathbf{X}^*$. *Then*

$$\left[\begin{pmatrix} (\mathbf{X}^*)', & (\mathbf{G}^*)_1' \\ \mathbf{O}, & (\mathbf{G}^*)_2' \end{pmatrix}^-_{m\begin{pmatrix} \Sigma^*, & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix}}\right]' = \begin{pmatrix} \boxed{11}, & \boxed{12} \\ \boxed{21}, & \boxed{22} \end{pmatrix},$$

$$\begin{aligned}
\boxed{11} &= \Big\{\mathbf{I} - (\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'[\mathbf{M}_{G_2^*}\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'\mathbf{M}_{G_2^*}]^+\mathbf{G}_1^*\Big\}\\
&\quad \times(\mathbf{C}^*)^{-1}(\mathbf{X}^*)'(\mathbf{\Sigma}^*)^{-1},\\
\boxed{12} &= (\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'[\mathbf{M}_{G_2^*}\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'\mathbf{M}_{G_2^*}]^+,\\
\boxed{21} &= -\Big\{[(\mathbf{G}_2^*)']^-_{m[G_1^*(C^*)^{-1}(G^*)_1']}\Big\}'\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{X}^*)'(\mathbf{\Sigma}^*)^{-1},\\
\boxed{22} &= \Big\{[(\mathbf{G}_2^*)']^-_{m[G_1^*(C^*)^{-1}(G_1^*)']}\Big\}'.
\end{aligned}$$

P r o o f. Two equalities must be valid (cf. [6]).

(i)

$$\begin{pmatrix} (\mathbf{X}^*)', & (\mathbf{G}_1^*)' \\ \mathbf{O}, & (\mathbf{G}_2^*)' \end{pmatrix}\begin{pmatrix} (\mathbf{X}^*)', & (\mathbf{G}_1^*)' \\ \mathbf{O}, & (\mathbf{G}_2^*)' \end{pmatrix}^-_{m\begin{pmatrix} \Sigma^*, & 0 \\ 0, & 0\end{pmatrix}}\begin{pmatrix} (\mathbf{X}^*)', & (\mathbf{G}_1^*)' \\ \mathbf{O}, & (\mathbf{G}_2^*)' \end{pmatrix} = \begin{pmatrix} (\mathbf{X}^*)', & (\mathbf{G}_1^*)' \\ \mathbf{O}, & (\mathbf{G}_2^*)' \end{pmatrix}$$

and

(ii)

$$\begin{pmatrix} \mathbf{\Sigma}^*, & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix}\begin{pmatrix} (\mathbf{X}^*)', & (\mathbf{G}_1^*)' \\ \mathbf{O}, & (\mathbf{G}_2^*)' \end{pmatrix}^-_{m\begin{pmatrix} \Sigma^*, & 0 \\ 0, & 0\end{pmatrix}}\begin{pmatrix} (\mathbf{X}^*)', & (\mathbf{G}_1^*)' \\ \mathbf{O}, & (\mathbf{G}_2^*)' \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{X}^*, & \mathbf{O} \\ \mathbf{G}_1^*, & \mathbf{G}_2^* \end{pmatrix}\left[\begin{pmatrix} (\mathbf{X}^*)', & (\mathbf{G}_1^*)' \\ \mathbf{O}, & (\mathbf{G}_2^*)' \end{pmatrix}^-_{m\begin{pmatrix} \Sigma^*, & 0 \\ 0, & 0\end{pmatrix}}\right]'\begin{pmatrix} \mathbf{\Sigma}^*, & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix}.$$

The equality (i) is easy to be proved.

The relationships

$$\begin{aligned}
&\mathbf{X}^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' - \mathbf{X}^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'[\mathbf{M}_{G_2^*}\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'\mathbf{M}_{G_2^*}]^+\\
&\times\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' - \mathbf{X}^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'[(\mathbf{G}_2^*)']^-_{m[G_1^*(C^*)^{-1}(G_1^*)']}(\mathbf{G}_2^*)'\\
&= \mathbf{X}^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' - \mathbf{X}^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'\\
&+\mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1} - [\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}\\
&\times\mathbf{G}_2^*\Big\{(\mathbf{G}_2^*)'[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}\mathbf{G}_2^*\Big\}^{-1}(\mathbf{G}_2^*)'[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'\\
&+\mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}\Big\}[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)'] - \mathbf{X}^*(\mathbf{C}^*)^{-1}(\mathbf{G}_2^*)'[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}\\
&\times(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}\mathbf{G}_2^*\Big\{(\mathbf{G}_2^*)'[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}\mathbf{G}_2^*\Big\}^{-1} = \mathbf{O},
\end{aligned}$$

$$[\mathbf{M}_{G_2^*}\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'\mathbf{M}_{G_2^*}]^+\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'$$
$$= [\mathbf{M}_{G_2^*}\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'\mathbf{M}_{G_2^*}]^+[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)'],$$
$$[\mathbf{M}_{G_2^*}\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'\mathbf{M}_{G_2^*}]^+ = [\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}$$
$$-[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}\mathbf{G}_2^*\Big\{(\mathbf{G}_2^*)'[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'$$
$$+\mathbf{G}_2^*(\mathbf{G}^*)_2']^{-1}\mathbf{G}_2^*\Big\}^{-1}[(\mathbf{G}_2^*)'\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}$$

and

$$\begin{aligned}
&[(\mathbf{G}_2^*)']^-_{m[G_1^*(C^*)^{-1}(G_1^*)']}\\
&= [\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}\mathbf{G}_2^*\Big\{(\mathbf{G}_2^*)'[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'\\
&\quad +\mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}\mathbf{G}_2^*\Big\}^{-1}
\end{aligned}$$

must be taken into account in order to prove (ii). □

Corollary 2.2. *In the regular univariate linear model with constraints II the BLUEs of the parameters $\boldsymbol\beta_1$ and $\boldsymbol\beta_2$ can be expressed with respect to* Lemma 2.1 (cf. also, e.g. [2])

$$\begin{aligned}
\hat{\hat{\boldsymbol\beta}}_1 &= \boxed{11}\,\boldsymbol{Y}^* + \boxed{12}(-\boldsymbol{g}_0)\\
&= \hat{\boldsymbol\beta}_1 - (\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'[\mathbf{M}_{G_2^*}\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'\mathbf{M}_{G_2^*}]^+(\mathbf{G}_1^*\hat{\boldsymbol\beta}_1 + \boldsymbol{g}_0),\\
\hat{\boldsymbol\beta}_1 &= (\mathbf{C}^*)^{-1}(\mathbf{X}^*)'(\boldsymbol\Sigma^*)^{-1}\boldsymbol{Y}^*,\\
\hat{\hat{\boldsymbol\beta}}_2 &= \boxed{21}\,\boldsymbol{Y}^* + \boxed{22}(-\boldsymbol{g}_0) = -\Big\{[(\mathbf{G}_2^*)']^-_{m[G_1^*(C^*)^{-1}(G_1^*)']}\Big\}'\\
&\quad\times(\mathbf{G}_1^*\hat{\boldsymbol\beta}_1 + \boldsymbol{g}_0),\\
\mathrm{Var}(\hat{\hat{\boldsymbol\beta}}_1) &= (\mathbf{C}^*)^{-1} - (\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'[\mathbf{M}_{G_2^*}\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'\mathbf{M}_{G_2^*}]^+\mathbf{G}_1^*(\mathbf{C}^*)^{-1},\\
\mathrm{Var}(\hat{\hat{\boldsymbol\beta}}_2) &= \Big\{(\mathbf{G}_2^*)'[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}\mathbf{G}_2^*\Big\}^{-1} - \mathbf{I},\\
\mathrm{cov}(\hat{\hat{\boldsymbol\beta}}_1,\hat{\hat{\boldsymbol\beta}}_2) &= -(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)'[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}\mathbf{G}_2^*\\
&\quad\times\Big\{(\mathbf{G}_2^*)'[\mathbf{G}_1^*(\mathbf{C}^*)^{-1}(\mathbf{G}_1^*)' + \mathbf{G}_2^*(\mathbf{G}_2^*)']^{-1}\mathbf{G}_2^*\Big\}^{-1}.
\end{aligned}$$

Corollary 2.3. *Using the model* (2) *in relationships from* Corollary 2.2 *we have in the model* (2) *and* (1), *respectively,*

$$\mathrm{vec}(\widehat{\widehat{\mathbf{B}}}_1) = \mathrm{vec}(\widehat{\mathbf{B}}_1) - \left[\boldsymbol\Sigma\otimes(\mathbf{X}'\mathbf{X})^{-1}\right](\mathbf{I}\otimes\mathbf{G}_1')$$
$$\times\left\{\mathbf{M}_{(I\otimes G_2)}(\mathbf{I}\otimes\mathbf{G}_1)\left[\boldsymbol\Sigma\otimes(\mathbf{X}'\mathbf{X})^{-1}\right](\mathbf{I}\otimes\mathbf{G}_1')\mathbf{M}_{(I\otimes G_2)}\right\}^+\mathrm{vec}(\mathbf{G}_1\widehat{\mathbf{B}}_1 + \mathbf{G}_0).$$

Since $\mathbf{M}_{(I\otimes G_2)} = \mathbf{I}\otimes\mathbf{M}_{G_2}$, *we can write*

$$\left\{\mathbf{M}_{(I\otimes G_2)}(\mathbf{I}\otimes\mathbf{G}_1)\left[\boldsymbol{\Sigma}\otimes(\mathbf{X}'\mathbf{X})^{-1}\right](\mathbf{I}\otimes\mathbf{G}_1')\mathbf{M}_{(I\otimes G_2)}\right\}^+$$
$$= \left\{\boldsymbol{\Sigma}\otimes\left[\mathbf{M}_{G_2}\mathbf{G}_1(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1'\mathbf{M}_{G_2}\right]\right\}^+ = \boldsymbol{\Sigma}^{-1}\otimes\left[\mathbf{M}_{G_2}\mathbf{G}_1(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1'\mathbf{M}_{G_2}\right]^+.$$

Thus

$$\begin{aligned}
\mathrm{vec}(\widehat{\widehat{\mathbf{B}}}_1) &= \mathrm{vec}(\widehat{\mathbf{B}}_1) - \left(\mathbf{I}\otimes\left\{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1'\left[\mathbf{M}_{G_2}\mathbf{G}_1(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1'\mathbf{M}_{G_2}\right]^+\right\}\right)\\
&\qquad\times\mathrm{vec}(\mathbf{G}_1\widehat{\mathbf{B}}_1+\mathbf{G}_0)\\
&= \mathrm{vec}\left\{\widehat{\mathbf{B}}_1 - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1'\left[\mathbf{M}_{G_2}\mathbf{G}_1(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1'\mathbf{M}_{G_2}\right]^+(\mathbf{G}_1\widehat{\mathbf{B}}_1+\mathbf{G}_0)\right\},\\
\mathrm{vec}(\widehat{\mathbf{B}}_1) &= \left\{\mathbf{I}\otimes\left[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\right]\right\}\mathrm{vec}(\underline{\mathbf{Y}}) = \mathrm{vec}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}}].
\end{aligned}$$

Analogously

$$\begin{aligned}
\mathrm{vec}(\widehat{\widehat{\mathbf{B}}}_2) &= -\left(\mathbf{I}\otimes\mathbf{G}_2'\right)^{-}_{m\{(I\otimes G_1)[\Sigma\otimes(X'X)^{-1}](I\otimes G_1')\}}\Big]'\,\mathrm{vec}(\mathbf{G}_1\widehat{\mathbf{B}}_1+\mathbf{G}_0)\\
&= -\left\{\mathbf{I}\otimes\left[(\mathbf{G}_2')^{-}_{m[G_1(X'X)^{-1}G_1']}\right]'\right\}\mathrm{vec}(\mathbf{G}_1\widehat{\mathbf{B}}_1+\mathbf{G}_0)\\
&= \mathrm{vec}\left\{-\left[(\mathbf{G}_2')^{-}_{m[G_1(X'X)^{-1}G_1']}\right]'(\mathbf{G}_1\widehat{\mathbf{B}}_1+\mathbf{G}_0)\right\},\\
\mathrm{Var}[\mathrm{vec}(\widehat{\widehat{\mathbf{B}}}_1)] &= \boldsymbol{\Sigma}\otimes\left\{\mathbf{M}_{G_1'M_{G_2}}(\mathbf{X}'\mathbf{X})\mathbf{M}_{G_1'M_{G_2}}\right\}^+,\\
\mathrm{Var}[\mathrm{vec}(\widehat{\widehat{\mathbf{B}}}_2)] &= \boldsymbol{\Sigma}\otimes\left(\left\{\mathbf{G}_2'\left[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1+\mathbf{G}_2\mathbf{G}_2'\right]^{-1}\mathbf{G}_2\right\}^{-1}-\mathbf{I}\right),\\
\mathrm{cov}[\mathrm{vec}(\widehat{\widehat{\mathbf{B}}}_1),\mathrm{vec}(\widehat{\widehat{\mathbf{B}}}_2)] &= -\boldsymbol{\Sigma}\otimes\left[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1'(\mathbf{G}_2')^{-}_{m[G_1(X'X)^{-1}G_1']}\right].
\end{aligned}$$

3. Estimation of parameters of Σ

The matrix $\boldsymbol{\Sigma}$ can be either known, or known partially, or it is fully unknown. Three typical situations occur. The matrix is of the form $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$, where $\mathbf{V}$ is $m\times m$ p.d. known matrix and $\sigma^2\in(0,\infty)$ is an unknown parameter. The other form is $\boldsymbol{\Sigma} = \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i$, where $\mathbf{V}_1,\dots,\mathbf{V}_p$, are given $m\times m$ symmetric matrices and $\boldsymbol{\vartheta} = (\vartheta_1,\dots,\vartheta_p)'$ is an unknown vector parameter, $\boldsymbol{\vartheta}\in\underline{\vartheta}\subset R^p$ (p-dimensional Euclidean space), $\underline{\vartheta}$ an open set. The last possibility is that $\boldsymbol{\Sigma}$ is fully unknown.

Let the residual matrix $\underline{\boldsymbol{Y}} - \mathbf{X}\widehat{\widehat{\mathbf{B}}}_1$ be denoted as $\underline{\boldsymbol{v}}_{II}$. With respect to Corollary 2.3

$$\underline{\boldsymbol{v}}_{II} = \underline{\boldsymbol{v}} + \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1'[\mathbf{M}_{G_2}\mathbf{G}_1(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1'\mathbf{M}_{G_2}]^{+}(\mathbf{G}_1\widehat{\mathbf{B}}_1 + \mathbf{G}_0),$$

where $\underline{\boldsymbol{v}} = \underline{\boldsymbol{Y}} - \mathbf{X}\widehat{\mathbf{B}}_1 = \underline{\boldsymbol{Y}} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\boldsymbol{Y}} = \mathbf{M}_X\underline{\boldsymbol{Y}}$ and $\widehat{\mathbf{B}}_1$ is the BLUE in the model $\operatorname{vec}(\underline{\boldsymbol{Y}}) \sim_{nm} [(\mathbf{I}\otimes\mathbf{X})\operatorname{vec}(\mathbf{B}_1), \boldsymbol{\Sigma}\otimes\mathbf{I}]$ without constraints.

Obviously

$$\operatorname{vec}(\underline{\boldsymbol{v}}_{II}) \sim_{nm} [\boldsymbol{0}, \boldsymbol{\Sigma}\otimes(\mathbf{M}_X + \mathbf{P}_{X(X'X)^{-1}G_1'M_{G_2}})]$$

(the matrices $\underline{\boldsymbol{v}}$ and $\widehat{\mathbf{B}}_1$ are uncorrelated).

Lemma 3.1. *Let $n\times m$ matrix $\mathbf{U}$ be normally distributed, i.e.*

$$\operatorname{vec}(\mathbf{U}) \sim N_{nm}(\boldsymbol{0}, \boldsymbol{\Sigma}\otimes\mathbf{W}),$$

where $m\times m$ matrix $\boldsymbol{\Sigma}$ is p.d.. Then $\mathbf{U}'\mathbf{W}^{-}\mathbf{U} \sim W_m(r(\mathbf{W}), \boldsymbol{\Sigma})$ (Wishart distribution).

Proof. Let $\mathbf{W} = \mathbf{J}\mathbf{J}'$, where $\mathbf{J}$ is $n\times r(\mathbf{W})$ matrix, $r(\mathbf{J}) = r(\mathbf{W})$. Let $\mathbf{K}$ be $n\times r(\mathbf{W})$ matrix with the property $r(\mathbf{K}) = r(\mathbf{W})$, $\mathbf{K}'\mathbf{J} = \mathbf{I}$. Then

$$(\mathbf{I}\otimes\mathbf{K}')\operatorname{vec}(\mathbf{U}) \sim N_{r(W_m)}(\boldsymbol{0}, \boldsymbol{\Sigma}\otimes\mathbf{I}),$$

i.e. the rows of the matrix $\mathbf{K}'\mathbf{U}$ are independent and they have the common covariance matrix $\boldsymbol{\Sigma}$. Therefore

$$\mathbf{U}'\mathbf{K}\mathbf{K}'\mathbf{U} = \mathbf{U}'\mathbf{W}^{+}\mathbf{U} \sim W_m(r(\mathbf{W}), \boldsymbol{\Sigma}).$$

Since $\mathscr{M}(\mathbf{U}) \subset \mathscr{M}(\mathbf{W})$ with probability 1, $\mathbf{U}'\mathbf{W}^{+}\mathbf{U} = \mathbf{U}'\mathbf{W}^{-}\mathbf{U}$ for any g-inverse of the matrix $\mathbf{W}$. □

Corollary 3.2. *It is valid that*

$$\underline{\boldsymbol{v}}_{II}'\underline{\boldsymbol{v}}_{II} \sim W_m(n + q - (k_1 + k_2), \boldsymbol{\Sigma}).$$

Proof. Since $\mathbf{M}_X + \mathbf{P}_{X(X'X)^{-1}G_1'M_{G_2}}$ is an idempotent matrix, thus the identity matrix is also its g-inverse and

$$\begin{aligned} r(\mathbf{M}_X + \mathbf{P}_{X(X'X)^{-1}G_1'M_{G_2}}) &= \operatorname{Tr}(\mathbf{M}_X + \mathbf{P}_{X(X'X)^{-1}G_1'M_{G_2}}) \\ &= \operatorname{Tr}(\mathbf{M}_X) + r(\mathbf{G}_1'\mathbf{M}_{G_2}) = n - k_1 + q - k_2. \end{aligned}$$

The last equality is implied by the relationship

$$r\begin{pmatrix} \mathbf{G}_1' \\ \mathbf{G}_2' \end{pmatrix} = q = r(\mathbf{G}_1'\mathbf{M}_{G_2}) + r(\mathbf{G}_2) = r(\mathbf{G}_1'\mathbf{M}_{G_2}) + k_2.$$

□

Lemma 3.3. *If $\underline{\mathbf{Y}}$ is normally distributed and $\mathbf{\Sigma} = \sigma^2\mathbf{V}$, i.e.*

$$\mathrm{vec}(\underline{\mathbf{v}}_{II}) \sim N_{nm}\Big\{\mathbf{0}, \sigma^2\Big[\mathbf{V} \otimes (\mathbf{M}_X + \mathbf{P}_{X(X'X)^{-1}G_1'M_{G_2}})\Big]\Big\},$$

then the estimator of σ^2, which is unbiased and with the minimum variance, is

$$\hat{\sigma}^2_{II} = \mathrm{Tr}(\underline{\mathbf{v}}'_{II}\underline{\mathbf{v}}_{II}\mathbf{V}^{-1})/[m(n+q-k_1-k_2)] \sim \sigma^2\frac{\chi^2_{m(n+q-k_1-k_2)}}{m(n+q-k_1-k_2)}.$$

P r o o f. Since $\mathbf{V}^{-1} \otimes \mathbf{I}$ is a g-inverse of the matrix

$$\mathbf{V} \otimes (\mathbf{M}_X + \mathbf{P}_{X(X'X)^{-1}G_1'M_{G_2}}),$$

thus the expression in the statement is only a transcription of the well known formula from the theory of linear univariate models

$$\hat{\sigma}^2_{II} = \big[\mathrm{vec}(\underline{\mathbf{v}}_{II})\big]'(\mathbf{V}^{-1} \otimes \mathbf{I})\,\mathrm{vec}(\underline{\mathbf{v}}_{II})/[m(n+q-k_1-k_2)].$$

□

If $\underline{\mathbf{Y}}$ is not normally distributed, then $\hat{\sigma}^2_{II}$ is still at least unbiased estimator of σ^2.

Theorem 3.4. *Let $\mathbf{\Sigma} = \sum_{I=1}^{p} \vartheta_i\mathbf{V}_i$, $\mathbf{V}_1, \dots, \mathbf{V}_p$, known $n \times n$ symmetric matrices, $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_p)' \in \underline{\vartheta}$ (open set) and let $\boldsymbol{\vartheta}_0$ be an approximate value of the vector $\boldsymbol{\vartheta}$. Then the $\boldsymbol{\vartheta}_0$-MINQUE of an estimable function $\mathbf{h}'\boldsymbol{\vartheta}, \boldsymbol{\vartheta} \in \underline{\vartheta}$, is*

$$\widehat{\mathbf{h}'\boldsymbol{\vartheta}} = \sum_{I=1}^{p} \lambda_i\,\mathrm{Tr}(\underline{\mathbf{v}}'_{II}\underline{\mathbf{v}}_{II}\mathbf{\Sigma}_0^{-1}\mathbf{V}_i\mathbf{\Sigma}_0^{-1})/(n+q-k_1-k_2), \quad \mathbf{S}_{\Sigma_0^{-1}}\boldsymbol{\lambda} = \mathbf{h},$$

where

$$\mathbf{\Sigma}_0 = \sum_{i=1}^{p} \vartheta_i^{(0)}\mathbf{V}_i, \quad \{\mathbf{S}_{\Sigma_0^{-1}}\}_{i,j} = \mathrm{Tr}(\mathbf{\Sigma}_0^{-1}\mathbf{V}_i\mathbf{\Sigma}_0^{-1}\mathbf{V}_j), \qquad i,j = 1,\dots,p.$$

If the matrix $\mathbf{S}_{\Sigma_0^{-1}}$ is regular, then

$$\widehat{\boldsymbol{\vartheta}} = \frac{1}{n+q-k_1-k_2}\mathbf{S}^{-1}_{\Sigma_0^{-1}}\begin{pmatrix} \mathrm{Tr}(\underline{\mathbf{v}}'_{II}\underline{\mathbf{v}}_{II}\mathbf{\Sigma}_0^{-1}\mathbf{V}_1\mathbf{\Sigma}_0^{-1}) \\ \vdots \\ \mathrm{Tr}(\underline{\mathbf{v}}'_{II}\underline{\mathbf{v}}_{II}\mathbf{\Sigma}_0^{-1}\mathbf{V}_p\mathbf{\Sigma}_0^{-1}) \end{pmatrix}.$$

In the case of normality of the observation vector $\underline{\mathbf{Y}}$

$$\mathrm{Var}_{\vartheta_0}(\widehat{\boldsymbol{\vartheta}}) = \frac{2}{n_1+q-k_1-k_2}\mathbf{S}^{-1}_{\Sigma_0^{-1}}.$$

P r o o f. In the univariate regular linear model with constraints II, i.e. $\boldsymbol{Y}^* \sim_n \left(\mathbf{X}^*\boldsymbol{\beta}, \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i\right)$, $\boldsymbol{g}_0 + \mathbf{G}_1^*\boldsymbol{\beta}_1 + \mathbf{G}_2^*\boldsymbol{\beta}_2 = \boldsymbol{0}$, it is valid

$$\widehat{\boldsymbol{h}'\boldsymbol{\vartheta}} = \sum_{i=1}^{p} \lambda_i \boldsymbol{v}'_{II}(\boldsymbol{\Sigma}_0^*)^{-1}\mathbf{V}_i(\boldsymbol{\Sigma}_0^*)^{-1}\boldsymbol{v}_{II}, \quad \mathbf{S}_{(M_{X^*M_{(G_1^*)'M_{G_2^*}}}\Sigma_0^* M_{X^*M_{(G_1^*)'M_{G_2^*}}})^+}\boldsymbol{\lambda} = \boldsymbol{h}.$$

If the matrix $\mathbf{S}_{(M_{X^*M_{(G_1^*)'M_{G_2^*}}}\Sigma_0^* M_{X^*M_{(G_1^*)'M_{G_2^*}}})^+}$ is regular, then

$$\widehat{\boldsymbol{\vartheta}} = \mathbf{S}^{-1}_{(M_{X^*M_{(G_1^*)'M_{G_2^*}}}\Sigma_0^* M_{X^*M_{(G_1^*)'M_{G_2^*}}})^+}\begin{pmatrix} \boldsymbol{v}'_{II}(\boldsymbol{\Sigma}^*)_0^{-1}\mathbf{V}_1(\boldsymbol{\Sigma}^*)_0^{-1}\boldsymbol{v}_{II} \\ \vdots \\ \boldsymbol{v}'_{II}(\boldsymbol{\Sigma}^*)_0^{-1}\mathbf{V}_p(\boldsymbol{\Sigma}^*)_0^{-1}\boldsymbol{v}_{II} \end{pmatrix}.$$

In the multivariate model it means

$$\begin{aligned}\widehat{\boldsymbol{h}'\boldsymbol{\vartheta}} &= \sum_{i=1}^{p} \lambda_i [\mathrm{vec}(\underline{\boldsymbol{v}}_{II})]'(\boldsymbol{\Sigma}_0^{-1}\otimes \mathbf{I})(\mathbf{V}_i\otimes \mathbf{I})(\boldsymbol{\Sigma}_0^{-1}\otimes \mathbf{I})\,\mathrm{vec}(\underline{\boldsymbol{v}}_{II}) \\ &= \sum_{i=1}^{p} \lambda_i \,\mathrm{Tr}(\underline{\boldsymbol{v}}'_{II}\underline{\boldsymbol{v}}_{II}\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}), \quad \mathbf{S}_*\boldsymbol{\lambda} = \boldsymbol{h},\end{aligned}$$

where

$$* = \left[\mathbf{M}_{(I\otimes X)M_{(I\otimes G_1')M_{(I\otimes G_2)}}}(\boldsymbol{\Sigma}_0\otimes \mathbf{I})\mathbf{M}_{(I\otimes X)M_{(I\otimes G_1')M_{(I\otimes G_2)}}}\right]^+.$$

Since

$$\begin{aligned}\mathbf{M}_{(I\otimes X)M_{(I\otimes G_1')M_{(I\otimes G_2)}}} &= \mathbf{M}_{(I\otimes X)M_{(I\otimes G_1')(I\otimes M_{G_2})}} = \mathbf{M}_{(I\otimes X)(I\otimes M_{G_1'M_{G_2}})} \\ &= \mathbf{M}_{[I\otimes(XM_{G_1'M_{G_2}})]} = \mathbf{I}\otimes \mathbf{M}_{XM_{G_1'M_{G_2}}} \implies \\ * &= \left[\left(\mathbf{I}\otimes \mathbf{M}_{XM_{G_1'M_{G_2}}}\right)(\boldsymbol{\Sigma}_0\otimes \mathbf{I})\left(\mathbf{I}\otimes \mathbf{M}_{XM_{G_1'M_{G_2}}}\right)\right]^+ \\ &= \boldsymbol{\Sigma}_0^{-1}\otimes \mathbf{M}_{XM_{G_1'M_{G_2}}} \implies \\ \{\mathbf{S}_*\}_{i,j} &= \mathrm{Tr}\left[\left(\boldsymbol{\Sigma}_0^{-1}\otimes \mathbf{M}_{XM_{G_1'M_{G_2}}}\right)(\mathbf{V}_i\otimes \mathbf{I})\left(\boldsymbol{\Sigma}_0^{-1}\otimes \mathbf{M}_{XM_{G_1'M_{G_2}}}\right)(\mathbf{V}_j\otimes \mathbf{I})\right] \\ &= \mathrm{Tr}(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j)\,\mathrm{Tr}\left(\mathbf{M}_{XM_{G_1'M_{G_2}}}\right) = \left[n - r\left(\mathbf{M}_{G_1'M_{G_2}}\right)\right] \\ &\quad\times \mathrm{Tr}(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j) = \{n - [k_1 - r(\mathbf{G}_1'\mathbf{M}_{G_2})]\}\,\mathrm{Tr}(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j) \\ &= [n + q - (k_1 + k_2)]\,\mathrm{Tr}(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j) \implies \\ \mathbf{S}_* &= [n + q - (k_1 + k_2)]\mathbf{S}_{\Sigma_0^{-1}}.\end{aligned}$$

Now it is obvious how to finish the proof. □

More about MINQUE cf. [7].

4. Confidence regions

The normality of the observation matrix $\underline{\boldsymbol{Y}}$ is assumed in this section.

An estimator

$$\Phi(\mathbf{B}_1, \mathbf{B}_2) = \mathrm{Tr}(\mathbf{A}_1\mathbf{B}_1) + \mathrm{Tr}(\mathbf{A}_2\mathbf{B}_2)$$

($\mathbf{A}_1$ is a given $m \times k_1$ matrix and $\mathbf{A}_2$ is a given $m \times k_2$ matrix) is also normally distributed and

$$\mathrm{Tr}\left[(\mathbf{A}_1, \mathbf{A}_2)\begin{pmatrix} \widehat{\widehat{\mathbf{B}_1}} \\ \widehat{\widehat{\mathbf{B}_2}} \end{pmatrix}\right] \sim$$

$$\sim N_1\left(\mathrm{Tr}\left[(\mathbf{A}_1, \mathbf{A}_2)\begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix}\right], \mathrm{Var}\left\{\mathrm{Tr}\left[(\mathbf{A}_1, \mathbf{A}_2)\begin{pmatrix} \widehat{\widehat{\mathbf{B}_1}} \\ \widehat{\widehat{\mathbf{B}_2}} \end{pmatrix}\right]\right\}\right),$$

$$\mathrm{Var}\left\{\mathrm{Tr}\left[(\mathbf{A}_1, \mathbf{A}_2)\begin{pmatrix} \widehat{\widehat{\mathbf{B}_1}} \\ \widehat{\widehat{\mathbf{B}_2}} \end{pmatrix}\right]\right\} = \mathrm{Tr}\left[\mathbf{A}_1\left(\mathbf{M}_{G_1' M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1' M_{G_2}}\right)^{+}\mathbf{A}_1'\boldsymbol{\Sigma}\right]$$

$$+\,\mathrm{Tr}\left[\mathbf{A}_2\left(\left\{\mathbf{G}_1'\left[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1' + \mathbf{G}_2\mathbf{G}_2'\right]^{-1}\mathbf{G}_2\right\}^{-1} - \mathbf{I}\right)\mathbf{A}_2'\boldsymbol{\Sigma}\right]$$

$$-2\,\mathrm{Tr}\left[\mathbf{A}_1(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1'(\mathbf{G}_2')^{-}_{m[G_1(X'X)^{-1}G_1']}\mathbf{A}_2'\boldsymbol{\Sigma}\right]$$

(cf. Corollary 2.3).

If $\boldsymbol{\Sigma}$ is given, the determination of the $(1-\alpha)$-confidence interval for $\Phi(\cdot, \cdot\cdot)$ is elementary.

If $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$, where $\mathbf{V}$ is a given p.d. $m \times m$ matrix and σ^2 is an unknown parameter, it is sufficient to take into account Lemma 3.3, i.e. the $(1-\alpha)$-confidence interval is $[d, u]$

$$\begin{aligned} d &= \mathrm{Tr}\left[(\mathbf{A}_1, \mathbf{A}_2)\begin{pmatrix} \widehat{\widehat{\mathbf{B}_1}} \\ \widehat{\widehat{\mathbf{B}_2}} \end{pmatrix}\right] - t_{m(n+q-k_1-k_2)}\left(1 - \frac{\alpha}{2}\right)\hat{\sigma}_{II}\sqrt{Q}, \\ u &= \mathrm{Tr}\left[(\mathbf{A}_1, \mathbf{A}_2)\begin{pmatrix} \widehat{\widehat{\mathbf{B}_1}} \\ \widehat{\widehat{\mathbf{B}_2}} \end{pmatrix}\right] + t_{m(n+q-k_1-k_2)}\left(1 - \frac{\alpha}{2}\right)\hat{\sigma}_{II}\sqrt{Q}, \end{aligned}$$

where $t_{m(n+q-k_1-k_2)}(1-\frac{\alpha}{2})$ is $(1-\frac{\alpha}{2})$-quantile of the Student distribution with $m(n+q-k_1-k_2)$ degrees of freedom, $\hat{\sigma}_{II}$ is given by Lemma 3.3 and

$$Q = \operatorname{Tr}\left[\mathbf{A}_1\left(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}}\right)^{+}\mathbf{A}_1'\mathbf{V}\right] +$$
$$+\operatorname{Tr}\left[\mathbf{A}_1\left(\left\{\mathbf{G}_1'\left[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1' + \mathbf{G}_2\mathbf{G}_2'\right]^{-1}\mathbf{G}_2\right\}^{-1} - \mathbf{I}\right)\mathbf{A}_2'\mathbf{V}\right]$$
$$-2\operatorname{Tr}\left[\mathbf{A}_1(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}_1'(\mathbf{G}_2')^{-}_{m[G_1(X'X)^{-1}G_1']}\mathbf{A}_2'\mathbf{V}\right].$$

If the matrix $\boldsymbol{\Sigma}$ is a priori unknown, its estimator $\underline{\mathbf{v}}_{II}'\underline{\mathbf{v}}_{II}/(n+q-k_1-k_2)$ from Corollary 3.2 is at our disposal only. Thus the problem arises whether this estimator can be used instead of the actual matrix $\boldsymbol{\Sigma}$. The following statement can help in a decision. For the sake of simplicity a procedure is demonstrated on the function $\Phi(\mathbf{B}_1) = \operatorname{Tr}(\mathbf{A}\mathbf{B}_1)$, where $\mathbf{A}$ is a given $m \times k_1$ matrix, only. (For more general situation cf. also [3].)

Theorem 4.1. *Let $\varepsilon > 0$ be such small positive real number that decreasing the confidence level $1-\alpha$ to $1-\alpha-\varepsilon$ can be tolerated. Let*

$$\mathbf{U} = \mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^{+}\mathbf{A}'.$$

If

$$\frac{t^2}{n+q-k_1-k_2}\left\{\boldsymbol{\Sigma} * \boldsymbol{\Sigma} + \operatorname{diag}(\boldsymbol{\Sigma})[\operatorname{diag}(\boldsymbol{\Sigma})]'\right\} \prec \frac{\kappa \operatorname{Tr}(\mathbf{U}\boldsymbol{\Sigma})}{\sqrt{\operatorname{Tr}(\mathbf{U}^2)}}\mathbf{1}\mathbf{1}', \tag{3}$$

then the estimator of $\boldsymbol{\Sigma}$, i.e.

$$\widehat{\boldsymbol{\Sigma}} = \frac{\mathbf{W}}{n+q-k_1-k_2} = \underline{\mathbf{v}}_{II}\underline{\mathbf{v}}_{II}'/(n+q-k_1-k_2)$$

can be used for a determination of confidence region and the confidence level is at least $1-\alpha-\varepsilon$, where $1-\alpha$ is a level under the known matrix $\boldsymbol{\Sigma}$.

Here $\kappa = 1 - \chi_1^2(1-\alpha-\varepsilon)/\chi_1^2(1-\alpha)$, $*$ means the Hadamard product of matrices, i.e. $\{\mathbf{K} * \mathbf{L}\}_{i,j} = \{\mathbf{K}\}_{i,j}\{\mathbf{L}\}_{i,j}$, t is sufficiently large real number with the property that

$$\sigma_{i,j} \in \left[\frac{w_{i,j}}{f} - t\sqrt{(\sigma_{i,i}\sigma_{j,j} + \sigma_{i,j}^2)/f}, \frac{w_{i,j}}{f} + t\sqrt{(\sigma_{i,i}\sigma_{j,j} + \sigma_{i,j}^2)/f}\right]$$

occurs with sufficiently large probability, $f = n+q-k_1-k_2$, $w_{i,j} = \{\mathbf{W}\}_{i,j}$ and $\prec$ means such ordering matrices, that $\mathbf{K} \prec \mathbf{L}$ means $\{\mathbf{K}\}_{i,j} \le \{\mathbf{L}\}_{i,j}$ for all i and j. Symbol $\operatorname{diag}(\boldsymbol{\Sigma})$ means the vector of diagonal entries of the matrix $\boldsymbol{\Sigma}$.

P r o o f. Since $\mathbf{W} = \underline{\boldsymbol{v}}'_{II}\underline{\boldsymbol{v}}_{II} \sim W_m(f, \boldsymbol{\Sigma}), f = n + q - k_1 - k_2$, it holds $w_{i,j} \sim_1 [f\sigma_{i,j}, f(\sigma_{i,i}\sigma_{j,j} + \sigma^2_{i,j})]$. Obviously $\{\boldsymbol{\Sigma} * \boldsymbol{\Sigma} + \mathrm{diag}(\boldsymbol{\Sigma})[\mathrm{diag}(\boldsymbol{\Sigma})]'\}_{i,j} = \sigma_{i,i}\sigma_{j,j} + \sigma^2_{i,j}$. If t is sufficiently large, then regarding the Chebyshev inequality

$$P\left\{\left|\frac{w_{i,j}}{f} - \sigma_{i,j}\right| < t\sqrt{\frac{\sigma_{i,i}\sigma_{j,j} + \sigma^2_{i,j}}{f}}\right\} \geq 1 - \frac{1}{t_2} \approx 1.$$

When $\boldsymbol{\Sigma}$ is given, the $(1-\alpha)$-confidence interval for the function $\mathrm{Tr}(\mathbf{AB}_1)$ is

$$\left[\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) - \sqrt{\chi^2_1(1-\alpha)\,\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma})}, \mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) + \sqrt{\chi^2_1(1-\alpha)\,\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma})}\right].$$

Small changes of the entries of the matrix $\boldsymbol{\Sigma}$ imply small changes of the boundaries of the interval. Let the matrix of changes $\delta\boldsymbol{\Sigma}$ satisfy the inequality

$$|\,\mathrm{Tr}(\mathbf{U}\delta\boldsymbol{\Sigma})| \leq \left[1 - \frac{\chi^2_1(1-\alpha-\varepsilon)}{\chi^2_1(1-\alpha)}\right]\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma}) = \kappa\,\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma}), \tag{4}$$

where $\varepsilon > 0$ is a sufficiently small number chosen in advance. Then

$$\begin{aligned}
&P\Big\{\mathrm{Tr}(\mathbf{AB}_1) \in \Big[\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) - \sqrt{\chi^2_1(1-\alpha)[\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma}) + |\,\mathrm{Tr}(\mathbf{U}\delta\boldsymbol{\Sigma})|]};\\
&\qquad \mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) + \sqrt{\chi^2_1(1-\alpha)[\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma}) + |\,\mathrm{Tr}(\mathbf{U}\delta\boldsymbol{\Sigma})|]}\Big]\Big\}\\
&\geq P\Big\{\mathrm{Tr}(\mathbf{AB}_1) \in \Big[\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) - \sqrt{\chi^2_1(1-\alpha)[\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma}) - |\,\mathrm{Tr}(\mathbf{U}\delta\boldsymbol{\Sigma})|]};\\
&\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) + \sqrt{\chi^2_1(1-\alpha)[\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma}) - |\,\mathrm{Tr}(\mathbf{U}\delta\boldsymbol{\Sigma})|]}\Big]\Big\} \geq P\Big\{\mathrm{Tr}(\mathbf{AB}_1)\\
&\in \left[\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) - \sqrt{\chi^2_1(1-\alpha)\left[\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma}) - \left(1 - \frac{\chi^2_1(1-\alpha-\varepsilon)}{\chi^2_1(1-\alpha)}\right)\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma})\right]};\right.\\
&\left.\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) + \sqrt{\chi^2_1(1-\alpha)\left[\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma}) - \left(1 - \frac{\chi^2_1(1-\alpha-\varepsilon)}{\chi^2_1(1-\alpha)}\right)\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma})\right]}\right]\Big\}\\
&= P\Big\{\mathrm{Tr}(\mathbf{AB}_1) \in \Big[\mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) - \sqrt{\chi^2_1(1-\alpha)\,\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma})}; \mathrm{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1)\\
&\qquad + \sqrt{\chi^2_1(1-\alpha)\,\mathrm{Tr}(\mathbf{U}\boldsymbol{\Sigma})}\Big]\Big\} = 1 - \alpha - \varepsilon.
\end{aligned}$$

Let $c > 0, \delta\mathbf{\Sigma} = c\mathbf{U}$ and at the same time $|\operatorname{Tr}(\mathbf{U}\delta\mathbf{\Sigma})| \leq \kappa\operatorname{Tr}(\mathbf{U\Sigma})$. Then $c = \kappa\operatorname{Tr}(\mathbf{U\Sigma})/\operatorname{Tr}(\mathbf{U}^2)$. If

$$\|\delta\mathbf{\Sigma}\| \leq \kappa\frac{\operatorname{Tr}(\mathbf{U\Sigma})}{\operatorname{Tr}(\mathbf{U}^2)}\|\mathbf{U}\| = \kappa\frac{\operatorname{Tr}(\mathbf{U\Sigma})}{\|\mathbf{U}\|^2}\|\mathbf{U}\| = \kappa\frac{\operatorname{Tr}(\mathbf{U\Sigma})}{\sqrt{\operatorname{Tr}(\mathbf{U}^2)}},$$

then $|\operatorname{Tr}(\mathbf{U}\delta\mathbf{\Sigma})| \leq \|\delta\mathbf{\Sigma}\|\|\mathbf{U}\| \leq \kappa\operatorname{Tr}(\mathbf{U\Sigma})$ (cf. (4)).

If t is sufficiently large, then $|\hat\sigma_{i,j} - \sigma_{i,j}|$ is smaller than $t\sqrt{(\sigma_{i,i}\sigma_{j,j} + \sigma_{i,j}^2)/f}$ and thus the condition (3) implies the validity of the statement. □

If $\mathbf{\Sigma} = \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i$, then sometimes, under some condition, the confidence region can be determined in a similar way as in Theorem 4.1.

Theorem 4.2. *Let* $\mathbf{\Sigma} = \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i, \boldsymbol{u}_A = [\operatorname{Tr}(\mathbf{UV}_1\mathbf{U}),\dots,\operatorname{Tr}(\mathbf{UV}_p\mathbf{U})]'$ *and the other notations be the same as in* Theorem 4.1. *Then*

$$\left\{\boldsymbol{\vartheta}:\ \forall\{i=1,\dots,p\}|\vartheta_i - \hat\vartheta_i| \leq \sqrt{\tfrac{p}{\gamma}}\sqrt{\left\{\frac{2\mathbf{S}^{-1}_{\Sigma_0^{-1}}}{f}\right\}_{i,i}}\right\}$$
$$\subset\left\{\boldsymbol{\vartheta}:\ \boldsymbol{\vartheta} = \boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta}, \tfrac{f}{2}\delta\boldsymbol{\vartheta}'\mathbf{S}_{\Sigma_0^{-1}}\delta\boldsymbol{\vartheta} \leq \tfrac{f}{2}\frac{\kappa^2[\operatorname{Tr}(\mathbf{U\Sigma}_0)]^2}{\boldsymbol{u}_A'\mathbf{S}^{-1}_{\Sigma_0^{-1}}\boldsymbol{u}_A}\right\}$$
$$\implies P\Bigg\{\operatorname{Tr}(\mathbf{AB}_1) \in \Bigg[\operatorname{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) - \sqrt{\chi_1^2(1-\alpha)\operatorname{Tr}(\mathbf{U}\widehat{\mathbf{\Sigma}})},$$
$$\operatorname{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) + \sqrt{\chi_1^2(1-\alpha)\operatorname{Tr}(\mathbf{U}\widehat{\mathbf{\Sigma}})}\Bigg]\Bigg\} \geq 1 - \alpha - \varepsilon.$$

Here $\widehat{\mathbf{\Sigma}} = \sum_{i=1}^{p}\hat\vartheta_i\mathbf{V}_i, \widehat{\boldsymbol{\vartheta}}$ *is given by* Theorem 3.4, $\gamma > 0$ *is sufficiently small number such that* $1-\gamma$ *can be considered as a practical certainty.*

P r o o f. Analogously as in Theorem 4.1 let $\delta\mathbf{\Sigma} = \sum_{i=1}^{p}\mathbf{V}_i\delta\vartheta_i$ satisfy the inequality $\operatorname{Tr}(\mathbf{U}\delta\mathbf{\Sigma})| = |\boldsymbol{u}_A'\delta\boldsymbol{\vartheta}| \leq \kappa\operatorname{Tr}(\mathbf{U\Sigma})$. A sufficient condition for the validity of this inequality is

$$\frac{f}{2}\delta\boldsymbol{\vartheta}'\mathbf{S}_{\Sigma_0^{-1}}\delta\boldsymbol{\vartheta} \leq \frac{f}{2}\frac{\kappa^2[\operatorname{Tr}(\mathbf{U\Sigma})]^2}{\boldsymbol{u}_A'\mathbf{S}^{-1}_{\Sigma_0^{-1}}\boldsymbol{u}_A}.$$

If $|\boldsymbol{u}'_A \delta\boldsymbol{\vartheta}| \leq \kappa \operatorname{Tr}(\mathbf{U\Sigma})$, then

$$P\Big\{\operatorname{Tr}(\mathbf{AB}_1) \in \Big[\operatorname{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) - \sqrt{\chi^2_1(1-\alpha)\operatorname{Tr}[\mathbf{U}(\boldsymbol{\Sigma}+\delta\boldsymbol{\Sigma})]},$$
$$\operatorname{Tr}(\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1) + \sqrt{\chi^2_1(1-\alpha)\operatorname{Tr}[\mathbf{U}(\boldsymbol{\Sigma}+\delta\boldsymbol{\Sigma})]}\Big]\Big\} \geq 1-\alpha-\varepsilon.$$

With respect to the Bonferroni rule (cf. [1, p. 492]) and the Chebyshev inequality

$$P\left\{\forall\{i=1,\dots,p\}|\vartheta^*_1 - \hat{\vartheta}_i| \leq \sqrt{\tfrac{p}{\gamma}}\sqrt{\left\{\tfrac{2}{f}\mathbf{S}^{-1}_{\Sigma_0^{-1}}\right\}_{i,i}}\right\} \approx 1-\gamma$$

and thus the statement is proved. □

Remark 4.3. A verification of the inequality

$$\frac{p}{\gamma} \ll \frac{f}{2}\kappa^2 \frac{[\operatorname{Tr}(\mathbf{U\Sigma})]^2}{\boldsymbol{u}'_A \mathbf{S}^{-1}_{\Sigma_0^{-1}} \boldsymbol{u}_A} \tag{5}$$

can serve as a basis for a preliminary decision, whether $\widehat{\boldsymbol{\Sigma}}$ can be used instead of the $\boldsymbol{\Sigma}$ in a construction of the confidence interval for the function $\operatorname{Tr}(\mathbf{AB}_1)$. It follows from the following consideration.

Even the set

$$\left\{\delta\boldsymbol{\vartheta}:\ \tfrac{f}{2}\delta\boldsymbol{\vartheta}'\mathbf{S}_{\Sigma_0^{-1}}\delta\boldsymbol{\vartheta} \leq \frac{p}{\gamma}\right\} \tag{6}$$

is included into the set

$$\left\{\delta\boldsymbol{\vartheta}:\ \forall\{i=1,\dots,p\}|\delta\vartheta_i| \leq \sqrt{\tfrac{p}{\gamma}}\sqrt{\left\{\tfrac{2}{f}\mathbf{S}^{-1}_{\Sigma_0^{-1}}\right\}_{i,i}}\right\}, \tag{7}$$

also the set (6) covers the actual value $\boldsymbol{\vartheta}^*$ of the parameter $\boldsymbol{\vartheta}$ for sufficiently large p/γ with practical certainty. Thus (5) implies the validity of $|\boldsymbol{u}'_A\delta\boldsymbol{\vartheta}| \leq \kappa \operatorname{Tr}(\mathbf{U\Sigma})$ with certainty.

In [4] the problem how large should be the value t $(= \frac{p}{\gamma})$ in some situations is solved. It was found out that $t = 3$ can be sufficiently large. Thus it seems that the condition (5) can be sometimes too rigorous in practice.

Until now scalar functions of the parameter matrix $\begin{pmatrix}\mathbf{B}_1\\ \mathbf{B}_2\end{pmatrix}$ was considered. A special vector function of the parameter $\mathbf{B}_1$ is $\Phi(\mathbf{B}_1) = (\boldsymbol{b}'\mathbf{B}_1\mathbf{A}')'$, where $\boldsymbol{b}$ is a k_1- dimensional given vector and $\mathbf{A}$ is a given $s\times m$ matrix, where $r(\mathbf{A}) = s \leq m$.

Theorem 4.4. *The $(1-\alpha)$-confidence ellipsoid for the function $\Phi(\mathbf{B}_1)$ is*

$$\mathscr{E}_\Phi(1-\alpha) = \Big\{\boldsymbol{u}:\ \boldsymbol{u}\in R^s,\ \tfrac{n+q-(k_1+k_2)-s+1}{s}(\boldsymbol{u}-\mathbf{A}\widehat{\widehat{\mathbf{B}}}{}'_1\boldsymbol{b})'$$
$$\times\frac{(\mathbf{A}\underline{\boldsymbol{v}}'_{II}\underline{\boldsymbol{v}}_{II}\mathbf{A}')^{-1}}{\boldsymbol{b}'(\mathbf{M}_{G'_1M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G'_1M_{G_2}})^{+}\boldsymbol{b}}(\boldsymbol{u}-\mathbf{A}\widehat{\widehat{\mathbf{B}}}{}'_1\boldsymbol{b}) \le F_{s,n+q-(k_1+k_2)-s+1}(1-\alpha)\Big\}.$$

P r o o f. Since

$$(\mathbf{A}\otimes\boldsymbol{b}')\operatorname{vec}(\widehat{\widehat{\mathbf{B}}}_1) \sim N_s\Big[(\mathbf{A}\otimes\boldsymbol{b}')\operatorname{vec}(\mathbf{B}_1),\ \boldsymbol{b}'\Big(\mathbf{M}_{G'_1M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G'_1M_{G_2}}\Big)^{+}\boldsymbol{b}\mathbf{A}\Sigma\mathbf{A}'\Big],$$
$$\boldsymbol{b}'\Big(\mathbf{M}_{G'_1M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G'_1M_{G_2}}\Big)^{+}\boldsymbol{b}\mathbf{A}\underline{\boldsymbol{v}}'_{II}\underline{\boldsymbol{v}}_{II}\mathbf{A}'$$
$$\sim W_s\Big[n+q-k_1-k_2,\ \boldsymbol{b}'\Big(\mathbf{M}_{G'_1M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G'_1M_{G_2}}\Big)^{+}\boldsymbol{b}\mathbf{A}\Sigma\mathbf{A}'\Big]$$

and the vector $(\mathbf{A}\otimes\boldsymbol{b}')\operatorname{vec}(\widehat{\widehat{\mathbf{B}}}{}'_1)$ and the matrix

$$\boldsymbol{b}'\Big(\mathbf{M}_{G'_1M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G'_1M_{G_2}}\Big)^{+}\boldsymbol{b}\mathbf{A}\underline{\boldsymbol{v}}'_{II}\underline{\boldsymbol{v}}_{II}\mathbf{A}'$$

are stochastically independent, the Hotelling theorem (cf. [5]) can be used, i.e.

$$[(\mathbf{A}\otimes\boldsymbol{b}')\operatorname{vec}(\mathbf{B}_1)-(\mathbf{A}\otimes\boldsymbol{b}')\operatorname{vec}(\widehat{\widehat{\mathbf{B}}}_1)]'\Big[\boldsymbol{b}'\Big(\mathbf{M}_{G'_1M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G'_1M_{G_2}}\Big)^{+}\boldsymbol{b}$$
$$\times\mathbf{A}\underline{\boldsymbol{v}}'_{II}\underline{\boldsymbol{v}}_{II}\mathbf{A}'\Big]^{-1}[(\mathbf{A}\otimes\boldsymbol{b}')\operatorname{vec}(\mathbf{B}_1)-(\mathbf{A}\otimes\boldsymbol{b}')\operatorname{vec}(\widehat{\widehat{\mathbf{B}}}_1)] \sim \frac{\chi^2_s}{\chi^2_{n_q-k_1-k_2-s+1}},$$

where χ^2_s and $\chi^2_{h+q-k_1-k_2-s+1}$ are stochastically independent. The relationship for the $(1-\alpha)$-confidence ellipsoid can be now easily obtained. □

Another special vector function of $\mathbf{B}_1$ is $\mathbf{A}\mathbf{B}_1\boldsymbol{b}$, where $\mathbf{A}$ is a $s\times k_1$ given matrix of the rank $r(\mathbf{A}) = s \le k_1$ and $\boldsymbol{b}$ is a given m-dimensional given vector.

Theorem 4.5. *Let $\mathscr{M}(\mathbf{A}') \subset \mathscr{M}(\mathbf{M}_{G'_1M_{G_2}})$ and $r(\mathbf{A}_{s,k}) = s \le k_1$, i.e. $\{\mathbf{A}\}_{i,\cdot}\mathbf{B}_1\boldsymbol{b}$, $i=1,\dots,s$, are unbiasedly estimable functions with nonzero dispersions of their estimators. Then $(1-\alpha)$-confidence ellipsoid for the function $\Phi(\mathbf{B}_1) = \mathbf{A}\mathbf{B}_1\boldsymbol{b}$ is*

$$\mathscr{E}_\Phi(1-\alpha) = \Big\{\boldsymbol{u}:\ \boldsymbol{u}\in R^s,\ (\boldsymbol{u}-\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1\boldsymbol{b})'\Big[\mathbf{A}\Big(\mathbf{M}_{G'_1M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G'_1M_{G_2}}\Big)^{+}\mathbf{A}'\Big]^{-1}$$
$$\times(\boldsymbol{u}-\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1\boldsymbol{b})\Big/\Big(s\frac{\boldsymbol{b}'\underline{\boldsymbol{v}}'_{II}\underline{\boldsymbol{v}}_{II}\boldsymbol{b}}{n+q-k_1-k_2}\Big) \le F_{s,n+q-k_1-k_2}(1-\alpha)\Big\}.$$

P r o o f. Since

$$\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1\boldsymbol{b} \sim N_s\left[\mathbf{A}\mathbf{B}_1\boldsymbol{b}, \boldsymbol{b}'\Sigma\boldsymbol{b}\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^+\mathbf{A}'\right],$$

$$\widehat{\boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{b}} = \boldsymbol{b}'\underline{\boldsymbol{v}}'_{II}\underline{\boldsymbol{v}}_{II}\boldsymbol{b}\Big/[n+q-(k_1+k_2)] \sim \boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{b}\frac{\chi^2_{n+q-k_1-k_2}}{n+q-k_1-k_2},$$

the random vector $\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1\boldsymbol{b}$ and the random variable $\boldsymbol{b}'\widehat{\boldsymbol{\Sigma}}\boldsymbol{b}$ are stochastically independent and $r\left[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^+\mathbf{A}'\right] = s$, the statement is obvious. □

Let the $(1-\alpha)$-confidence region $\mathscr{E}_\Phi(1-\alpha)$ must be determined for the matrix function $\Phi(\mathbf{B}_1) = \mathbf{A}\mathbf{B}_1\mathbf{D}$, where $r(\mathbf{A}_{r,k_1}) = r \leq k_1, r(\mathbf{D}_{m,s}) = s \leq m, \mathscr{M}(\mathbf{A}') \subset \mathscr{M}(\mathbf{M}_{G_1'M_{G_2}})$ and it is assumed that $\boldsymbol{\Sigma}$ is known. The confidence region can be obtained in a standard way as a set

$$\mathscr{E}_\Phi(1-\alpha) = \Big\{\mathbf{U}_{r,s}:\ \operatorname{Tr}\Big\{(\mathbf{U}'-\mathbf{D}'\widehat{\widehat{\mathbf{B}}}{}_1'\mathbf{A}')\Big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^+\mathbf{A}'\Big]^{-1} \times(\mathbf{U}-\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1\mathbf{D})(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1} \leq \chi^2_{rs}(1-\alpha)\Big\}\Big\}.$$

In an analogous way as in the preceding part of this section, the expression for $\mathscr{E}_\Phi(1-\alpha)$ can be investigated when $\boldsymbol{\Sigma}$ is either unknown, or partially unknown. The case $\boldsymbol{\Sigma}$ is unknown will be investigated only.

Lemma 4.6. *Let*

$$\mathbf{Q} = (\mathbf{D}'\mathbf{B}_1'\mathbf{A}' - \mathbf{D}'\widehat{\widehat{\mathbf{B}}}{}_1'\mathbf{A}')\Big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^+\mathbf{A}'\Big]^{-1}(\mathbf{A}\mathbf{B}_1\mathbf{D} - \mathbf{A}\widehat{\widehat{\mathbf{B}}}_1\mathbf{D})$$

and

$$\eta(\delta\boldsymbol{\Sigma}) = \operatorname{Tr}\Big\{\mathbf{Q}[\mathbf{D}'(\boldsymbol{\Sigma}+\delta\boldsymbol{\Sigma})\mathbf{D}]^{-1}\Big\} - \operatorname{Tr}[\mathbf{Q}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}].$$

Then

$$\eta(\delta\boldsymbol{\Sigma}) = -\operatorname{Tr}[\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{Q}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}]$$
$$\sim_1 \Big(-r\operatorname{Tr}[\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}], 2r\operatorname{Tr}[(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}\mathbf{D}]\Big).$$

P r o o f. Since

$$\left\{\frac{\partial\operatorname{Tr}[\mathbf{Q}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}]}{\partial\boldsymbol{\Sigma}}\right\}_{i,j} = -\operatorname{Tr}\left[\mathbf{Q}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\frac{\partial\boldsymbol{\Sigma}}{\partial\sigma_{i,j}}\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\right]$$
$$= \begin{cases} -2\boldsymbol{e}_j'\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{Q}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{e}_i, & i\neq j,\\ -\boldsymbol{e}_i'\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{Q}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{e}_i, & i=j,\end{cases}$$

the random variable $\eta(\delta\mathbf{\Sigma})$, for sufficiently small $\delta\mathbf{\Sigma}$, can be expressed as

$$\eta(\delta\mathbf{\Sigma}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial \operatorname{Tr}[\mathbf{Q}(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}]}{\partial \sigma_{i,j}} \delta\sigma_{i,j}$$
$$= -\sum_{i=1}^{n} \operatorname{Tr}[\mathbf{D}(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{Q}(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{e}_i\boldsymbol{e}_i']\delta\sigma_{i,i}$$
$$-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \operatorname{Tr}[\mathbf{D}(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{Q}(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{D}'(\boldsymbol{e}_i\boldsymbol{e}_j' + \boldsymbol{e}_j\boldsymbol{e}_i')]\delta\sigma_{i,j}$$
$$= -\operatorname{Tr}[\mathbf{D}(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{Q}(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\mathbf{\Sigma}].$$

Let $(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\mathbf{\Sigma}\mathbf{D}(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1} = \mathbf{U}$. Then

$$E\Big\{-\operatorname{Tr}[\mathbf{D}(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{Q}(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\mathbf{\Sigma}]\Big\} = -E[\operatorname{Tr}(\mathbf{Q}\mathbf{U})]$$
$$= -E\bigg(\operatorname{Tr}\Big\{\mathbf{D}'(\mathbf{B}_1 - \widehat{\widehat{\mathbf{B}}}_1)'\mathbf{A}'\Big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^{+}\mathbf{A}'\Big]^{-1}\mathbf{A}(\mathbf{B}_1 - \widehat{\widehat{\mathbf{B}}}_1)\mathbf{D}\mathbf{U}\Big\}\bigg)$$
$$= -E\bigg[\big[\operatorname{vec}(\mathbf{B}_1 - \widehat{\widehat{\mathbf{B}}}_1)\big]'\bigg((\mathbf{D}\mathbf{U}\mathbf{D}') \otimes \Big\{\mathbf{A}'\Big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^{+}\mathbf{A}'\Big]^{-1}\mathbf{A}\Big\}\bigg)$$
$$\times \operatorname{vec}(\mathbf{B}_1 - \widehat{\widehat{\mathbf{B}}}_1)\bigg]$$
$$= -\operatorname{Tr}\bigg[\bigg((\mathbf{D}\mathbf{U}\mathbf{D}') \otimes \Big\{\mathbf{A}'\Big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^{+}\mathbf{A}'\Big]^{-1}\mathbf{A}\Big\}\bigg)$$
$$\times\Big[\mathbf{\Sigma} \otimes (\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^{+}\Big]\bigg]$$
$$= -\operatorname{Tr}(\mathbf{D}\mathbf{U}\mathbf{D}'\mathbf{\Sigma})\operatorname{Tr}\Big\{\mathbf{A}'\big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^{+}\mathbf{A}'\big]^{-1}$$
$$\times\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^{+}\Big\} = -\operatorname{Tr}\big[\mathbf{D}(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\mathbf{\Sigma}\mathbf{D}(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\mathbf{\Sigma}\big]$$
$$\times\operatorname{Tr}\Big\{\big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^{+}\mathbf{A}'\big]^{-1}\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^{+}\mathbf{A}'\Big\}$$
$$= -r\operatorname{Tr}[(\mathbf{D}'\mathbf{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\mathbf{\Sigma}\mathbf{D}].$$

Further

$$\operatorname{Var}[\operatorname{Tr}(\mathbf{U}\mathbf{Q})] = \operatorname{Var}\Big\{(\mathbf{B}_1 - \widehat{\widehat{\mathbf{B}}})'\mathbf{A}'\big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^{+}\mathbf{A}'\big]^{-1}$$
$$\times\mathbf{A}(\mathbf{B}_1 - \widehat{\widehat{\mathbf{B}}})\mathbf{D}\mathbf{U}\mathbf{D}'\Big\} = \operatorname{Var}\bigg[\big[\operatorname{vec}(\mathbf{B}_1 - \widehat{\widehat{\mathbf{B}}}_1)\big]'\bigg((\mathbf{D}\mathbf{U}\mathbf{D}') \otimes \Big\{\mathbf{A}'\big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'$$
$$\times\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^{+}\mathbf{A}'\big]^{-1}\mathbf{A}\Big\}\bigg)\operatorname{vec}(\mathbf{B}_1 - \widehat{\widehat{\mathbf{B}}}_1)\bigg]$$

$$= 2\,\mathrm{Tr}\Big\{\Big[\Big((\mathbf{DUD}') \otimes \Big\{\mathbf{A}'\big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^+\mathbf{A}'\big]^{-1}\mathbf{A}\Big\}\Big)$$
$$\times[\boldsymbol{\Sigma} \otimes (\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^+]\Big]^2\Big\} = \mathrm{Tr}(\mathbf{DUD}'\boldsymbol{\Sigma}\mathbf{DUD}'\boldsymbol{\Sigma})$$
$$\times\,\mathrm{Tr}\Big\{\mathbf{A}'\big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^+\mathbf{A}'\big]^{-1}\mathbf{A}$$
$$\times(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^+\mathbf{A}'\big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^+\mathbf{A}'\big]^{-1}$$
$$\times\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^+\Big\} = 2\,\mathrm{Tr}\big[\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}\delta\boldsymbol{\Sigma}\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}$$
$$\times\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}\big]\,\mathrm{Tr}(\mathbf{I}_{r,r})$$
$$= 2r\,\mathrm{Tr}\big[(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}\mathbf{D}\big].$$

□

Theorem 4.7. *Let $t > 0$ be a given sufficiently large number and let $\delta\boldsymbol{\Sigma}$ satisfy the inequality*

$$-r\,\mathrm{Tr}\big[(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}\mathbf{D}\big] + t\sqrt{2r\,\mathrm{Tr}\big[(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}\mathbf{D}\big]}$$
$$< \chi^2_{rs}(1-\alpha) - \chi^2_{rs}(1-\alpha-\varepsilon) \tag{8}$$

for a given $\varepsilon > 0$. Then

$$P\Big\{\mathbf{AB}_1\mathbf{D} \in \Big\{\mathbf{U}_{r,s}:\ \mathrm{Tr}\{(\mathbf{U}-\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1\mathbf{D})'\big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^+\mathbf{A}'\big]^{-1}$$
$$\times(\mathbf{U}-\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1\mathbf{D})\} \le \chi^2_{rs}(1-\alpha)\Big\}\Big\} \ge 1-\alpha-\varepsilon.$$

P r o o f. With respect to Lemma 4.6

$$\mathrm{Tr}\{(\mathbf{U}-\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1\mathbf{D})'\big[\mathbf{A}(\mathbf{M}_{G_1'M_{G_2}}\mathbf{X}'\mathbf{X}\mathbf{M}_{G_1'M_{G_2}})^+\mathbf{A}'\big]^{-1}$$
$$\times(\mathbf{U}-\mathbf{A}\widehat{\widehat{\mathbf{B}}}_1\mathbf{D})[\mathbf{D}'(\boldsymbol{\Sigma}+\delta\boldsymbol{\Sigma})\mathbf{D}]^{-1}\}$$
$$= \chi^2_{rs} - \mathrm{Tr}\big[\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{Q}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}\big] = \chi^2_{rs} + \eta(\delta\boldsymbol{\Sigma}).$$

If $\delta\boldsymbol{\Sigma}$ satisfy inequality (8), then $\eta(\delta\boldsymbol{\Sigma})$ is with sufficiently high probability smaller than $\chi^2_{rs}(1-\alpha) - \chi^2_{rs}(1-\alpha-\varepsilon) = \delta$, what is implied by the Chebyshev inequality. Thus

$$P\{\chi^2_{rs} + \eta(\delta\boldsymbol{\Sigma}) \le \chi^2_{rs}(1-\alpha)\} \approx P\{\chi^2_{rs} \le \chi^2_{rs}(1-\alpha) - \delta\}$$
$$= P\{\chi^2_{rs} \le \chi^2_{rs}(1-\alpha-\varepsilon)\} = 1-\alpha-\varepsilon.$$

□

Remark 4.8. For a first orientation let $\delta\boldsymbol{\Sigma} = c\boldsymbol{\Sigma}$, $c > 0$. Then

$$-r\,\mathrm{Tr}\big[(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}\mathbf{D}\big] + t\sqrt{2r\,\mathrm{Tr}\big[(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}\mathbf{D}(\mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})^{-1}\mathbf{D}'\delta\boldsymbol{\Sigma}\mathbf{D}\big]}$$
$$= -rsc + t\sqrt{2rsc^2} \le \delta \implies c < \frac{\delta}{-rs + t\sqrt{2rs}}, \quad |\delta\sigma_{i,j}| \le c|\sigma_{i,j}|,$$

what means that $|\delta\sigma_{i,j}| \le c|\sigma_{i,j}|$ implies a smaller destroy of the confidence level than $\varepsilon > 0$. Since

$$\hat{\sigma}_{i,j} \sim_1 \left(\sigma_{i,j}, \frac{1}{n+q-k_1-k_2}(\sigma_{i,i}\sigma_{j,j} + \sigma_{i,j}^2)\right),$$

we must have at our disposal the degrees of freedom $n + q - k_1 - k_2$ for the Wishart matrix large enough that at least

$$\sqrt{\frac{\sigma_{i,i}\sigma_{j,j} + \sigma_{i,j}^2}{n+q-k_1-k_2}} < \frac{c|\sigma_{i,j}|}{3}.$$

In such a case the estimator $\widehat{\boldsymbol{\Sigma}} = \frac{1}{n+q-k_1-k_2}\underline{\mathbf{v}}'_{II}\underline{\mathbf{v}}_{II}$ can be used in a determination of the $(1-\alpha)$-confidence region for the function $\Phi(\mathbf{B}_1) = \mathbf{A}\mathbf{B}_1\mathbf{D}$, instead of the actual matrix $\boldsymbol{\Sigma}$.

Acknowledgement. Author would like to thank to the reviewer for his externally thorough reading the manuscript. He corrected many misprints and errors.

REFERENCES

[1] HUMAK, K. M. S.: *Statistische Methoden der Modellbildung. Band I*, Akademie-Verlag, Berlin, 1977.

[2] KUBÁČEK, L.—KUBÁČKOVÁ, L.—VOLAUFOVÁ, J.: *Statistical Models with Linear Structures*, Veda, Bratislava, 1995.

[3] KUBÁČEK, L.—KUBÁČKOVÁ, L.: *Nonsensitiveness regions in universal models*, Math. Slovaca **50** (2000), 219–240.

[4] LEŠANSKÁ, E.: *Optimization of the size of nonsensitiveness regions*, Appl. Math. **47** (2002), 9–23.

[5] RAO, C. R.: *Linear Statistical Inference and Its Applications*, J. Wiley & Sons, New York-London-Sydney, 1965.

[6] RAO, C. R.—MITRA, S. K.: *Generalized Inverse of Matrices and Its Applications*, John Wiley & Sons, New York-London-Sydney-Toronto, 1971.

[7] RAO, C. R.—KLEFFE, J.: *Estimation of Variance Components and Applications*, North-Holland, Amsterdam, 1988.

Received 13. 10. 2005
Revised 1. 6. 2006

Katedra matematické analýzy
a aplikované matematiky
PF Univerzita Palackého
Tomkova 40
CZ–779 00 Olomouc
ČESKÁ REPUBLIKA
E-mail: kubacekl@risc.upol.cz