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UNIFORM CONVERGENCE ON SPACES OF MULTIFUNCTIONS

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ABSTRACT. We study spaces of multifunctions with closed values, multifunctions with closed graphs, USCO multifunctions, minimal USCO multifunctions and the space of densely continuous forms as metric spaces, equipped with the topology of uniform convergence. We give conditions under which these metric spaces are complete.

In what follows let X, Y be Hausdorff topological spaces. Let Z be a topological space. The symbol B^c, $\overline{B}$ and $\operatorname{int} B$ will stand for the complement, closure and interior of $B \subset Z$, respectively. We denote by 2^Z the space of all closed subsets of Z, by $\mathrm{CL}(Z)$ the space of all nonempty closed subsets of Z and by $K(Z)$ the space of all nonempty compact sets in Z.

Let (Z, d) be a metric space. The open d-ball with center $z_0 \in Z$ and radius $\varepsilon > 0$ will be denoted by $S_\varepsilon[z_0]$ and the ε parallel body $\bigcup\limits_{a \in A} S_\varepsilon[a]$ for subset A of Z will be denoted by $S_\varepsilon[A]$.

The distance between a point z and a nonempty set A will be denoted by $d(z, A)$, where

$$d(z, A) = \inf\{d(z, a) : \ a \in A\}.$$

The diameter of a nonempty subset A of Z will be denoted by $\operatorname{diam} A$, where

$$\operatorname{diam} A = \sup\{d(z, y) : \ z \in A \ \text{and} \ y \in A\}.$$

The Hausdorff metric H_d on 2^Z is defined by

$$H_d(A, B) = \max\Big\{\sup\{d(a, B) : \ a \in A\}, \ \sup\{d(b, A) : \ b \in B\}\Big\}$$

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if A and B are nonempty. If $A \neq \emptyset$ take $H_d(A, \emptyset) = H_d(\emptyset, A) = \infty$. H_d defines an (extended-valued) metric on 2^Z. We will often use the following equality on $\mathrm{CL}(Z)$:

$$H_d(A, B) = \inf\bigl\{\varepsilon > 0 : \ A \subset S_\varepsilon[B] \text{ and } B \subset S_\varepsilon[A]\bigr\}.$$

Now let X be a topological space and (Y, d) be a metric space.

Denote by $F(X, 2^Y)$ the set of all multifunctions with closed values from X to Y. Let ϱ be the (extended-valued) metric on $F(X, 2^Y)$ defined by

$$\varrho(\Phi, \Psi) = \sup\bigl\{H_d(\Phi(x), \Psi(x)) : \ x \in X\bigr\}$$

for each $\Phi, \Psi \in F(X, 2^Y)$. The open sphere of radius ε around Φ with the centre Φ looks like:

$$\bigl\{\Psi \in F(X, 2^Y) : \ \varrho(\Phi, \Psi) < \varepsilon\bigr\}.$$

It is known ([Be]) that if d is a complete metric on Y, then H_d is a complete metric on 2^Y. Then the following holds:

Theorem 1. *Let X be a topological space and (Y, d) be a complete metric space. Then $(F(X, 2^Y), \varrho)$ is a complete metric space.*

By the active boundary of F at x_0 ($\operatorname{Frac} F(x_0)$) we mean

$$\operatorname{Frac} F(x_0) = \bigcap\bigl\{\ \overline{F(W) \setminus F(x_0)} : \ W \in \mathscr{B}(x_0)\bigr\}$$

where $\mathscr{B}(x_0)$ stands for a neighbourhood base at x_0 and $F(W) = \bigcup\bigl\{F(x) : x \in W\bigr\}$, see [Do].

Denote by $G(X, 2^Y)$ the set of all multifunctions with closed graphs i.e., if $\Phi \in G(X, 2^Y)$ the set $\bigl\{(x, y) : \ y \in \Phi(x)\bigr\}$ is a closed set in $X \times Y$.

Combining [Be, Lemma 6.1.15, 6.1.16] we obtain the following result.

Proposition 2. *Let X and Y be Hausdorff topological spaces. A multifunction F from X to Y has closed graph if and only if $F(x)$ is a closed set and $F(x)$ contains* $\operatorname{Frac} F(x)$ *for all $x \in X$.*

Theorem 3. *If (Y, d) is a metric space, then the following are equivalent.*

(a) *Y is locally compact.*

(b) *For every space X, $G(X, 2^Y)$ is a closed set in $(F(X, 2^Y), \varrho)$.*

Proof.

(a) $\implies$ (b) Let $\Phi \in F(X, 2^Y)$ be in the closure of $G(X, 2^Y)$ in $(F(X, 2^Y), \varrho)$. Let $\bigl\{\Phi_n : \ n \in \mathbb{Z}^+\bigr\}$ ($\mathbb{Z}^+ = \{1, 2, 3, \dots\}$) be a sequence in $G(X, 2^Y)$ convergent to Φ in $(F(X, 2^Y), \varrho)$. By Proposition 2 it is sufficient to prove that $\operatorname{Frac} \Phi(x) \subset \Phi(x)$ for all $x \in X$.

Suppose that this is not true. Then there exists $x_0 \in X$, $y_0 \in Y$ such that $y_0 \in \operatorname{Frac}\Phi(x_0)$ and $y_0 \notin \Phi(x_0)$. Let $\varepsilon > 0$ be such that

$$S_\varepsilon[\Phi(x_0)] \cap S_\varepsilon[y_0] = \emptyset \qquad \text{and} \qquad \overline{S_\varepsilon[y_0]} \quad \text{is compact.}$$

Since $y_0 \in \operatorname{Frac}\Phi(x_0)$, there is a net

$$\{x_W : W \in \mathscr{B}(x_0)\}, \qquad \text{where} \quad x_W \in W$$

(where $\mathscr{B}(x_0)$ stands for a neighbourhood base at x_0), in X which converges to x_0 and a net

$$\{y_W : W \in \mathscr{B}(x_0)\}, \qquad \text{where} \quad y_W \in S_{\frac{\varepsilon}{2}}[y_0] \cap (\Phi(x_W) \setminus \Phi(x_0)), \text{ in } Y.$$

Sequence $\{\Phi_n : n \in \mathbb{Z}^+\}$ converges to Φ in $(F(X, 2^Y), \varrho)$, then for $\frac{\varepsilon}{2}$ there exists n_0, such that $\Phi_{n_0} \in S_{\frac{\varepsilon}{2}}[\Phi]$.

Then there exists a net $\{z_W : W \in \mathscr{B}(x_0)\}$ in Y such that

$$d(z_W, y_W) < \frac{\varepsilon}{2} \qquad \text{and} \qquad z_W \in \Phi_{n_0}(x_W) \setminus \Phi_{n_0}(x_0).$$

The net $\{z_W : W \in \mathscr{B}(x_0)\}$ is a subset of compact set $\overline{S_\varepsilon[y_0]}$ and then has a cluster point z_0. Thus $z_0 \in \operatorname{Frac}\Phi_{n_0}(x_0)$ and since

$$\Phi_{n_0}(x_0) \subset S_{\frac{\varepsilon}{2}}[\Phi(x_0)],$$

we have that $z_0 \notin \Phi_{n_0}(x_0)$. Since $\Phi_{n_0} \in G(X, 2^Y)$, by Proposition 2 it is a contradiction.

(b) $\implies$ (a) follows from [HoMc, Theorem 8] □

Theorem 4. *If (Y, d) is a metric space, then the following are equivalent.*

(a) *Y is locally countably compact.*

(b) *For every first countable space X, $G(X, 2^Y)$ is a closed set in $(F(X, 2^Y), \varrho)$.*

Proof.

(a) $\implies$ (b) can be proved by modification of the proof of (a) $\implies$ (b) of the previous theorem.

(b) $\implies$ (a) follows from [HoMc, Theorem 8′]. □

Theorem 5. *Let X be a topological space and (Y, d) be a locally compact complete metric space. Then $(G(X, 2^Y), \varrho)$ a is complete metric space.*

Proof. By Theorem 1, $(F(X, 2^Y), \varrho)$ is a complete metric space and by Theorem 3, $(G(X, 2^Y), \varrho)$ is a closed set in $(F(X, 2^Y), \varrho)$, so $(G(X, 2^Y), \varrho)$ is a complete metric space. □

A multifunction $F\colon X \to Y$ is said to be upper semicontinuous (USC) at x_0 if for every open set O which contains $F(x_0)$ there exists a neighbourhood V of x_0 such that $F(V) \subset O$.

Denote by $U(X,Y)$ the space of upper semicontinuous nonempty compact-valued (USCO) multifunctions. It is known that every USCO multifunction has a closed graph so $U(X,Y) \subset G(X,2^Y)$, see [DL], [Ch].

Theorem 6. *Let X be a topological space and (Y,d) be a complete metric space. Then $(U(X,Y),\varrho)$ is a complete metric space.*

Proof. Since by Theorem 1, $F(X,2^Y)$ is a complete metric space, it is sufficient to show that $U(X,Y)$ is a closed set in $F(X,2^Y)$. Let $\Phi \in F(X,2^Y)$ be in the closure of $U(X,Y)$ in $(F(X,2^Y),\varrho)$. Let $\{\Phi_n : n \in \mathbb{Z}^+\}$ be a sequence in $U(X,Y)$ convergent to Φ in $(F(X,2^Y),\varrho)$. Since for a complete metric space (Y,d) the set $K(Y)$ is a closed set in $(2^Y,H_d)$, see [Be], $\Phi(x)$ is a nonempty compact set for every $x \in X$.

To prove that $\Phi \in U(X,Y)$, it is sufficient to show that Φ is USC at each $x \in X$. Suppose it is not true. Then for some $x_0 \in X$ there exists an open set O which contains $\Phi(x_0)$ and such that for every neighbourhood V of x_0 there exists $x_V \in V$ with $\Phi(x_V) \cap O^c \neq \emptyset$.

Since $\Phi(x_0)$ is a compact set, we have that $S_\varepsilon[\Phi(x_0)] \subset O$ for some $\varepsilon > 0$.

Consider $S_{\frac{\varepsilon}{4}}[\Phi]$. There is $n_0 \in \mathbb{Z}^+$ such that $\Phi_n \in S_{\frac{\varepsilon}{4}}[\Phi]$ for every $n \geq n_0$.

Let $n_1 > n_0$. Since Φ_{n_1} is USC at x_0, there exists $W \in \mathscr{B}(x_0)$ such that $\Phi_{n_1}(W) \subset S_{\frac{\varepsilon}{4}}[\Phi_{n_1}(x_0)]$.

Since $x_W \in W$ we have

$$\Phi_{n_1}(x_W) \subset S_{\frac{\varepsilon}{4}}[\Phi_{n_1}(x_0)]$$

and since $\Phi_{n_1} \in S_{\frac{\varepsilon}{4}}[\Phi]$ we have

$$\Phi_{n_1}(x_0) \subset S_{\frac{\varepsilon}{4}}[\Phi(x_0)],$$

then

$$\Phi_{n_1}(x_W) \subset S_{\frac{\varepsilon}{4}}[\Phi_{n_1}(x_0)] \subset S_{\frac{\varepsilon}{2}}[\Phi(x_0)].$$

Then from

$$\Phi(x_W) \cap (S_\varepsilon[\Phi(x_0)])^c \neq \emptyset \qquad \text{and} \qquad \Phi_{n_1}(x_W) \subset S_{\frac{\varepsilon}{2}}[\Phi(x_0)]$$

it follows that $H_d(\Phi(x_W),\Phi_{n_1}(x_W)) > \frac{\varepsilon}{4}$, a contradiction since $\Phi_{n_1} \in S_{\frac{\varepsilon}{4}}[\Phi]$. $\square$

A multifunction $\Phi \in U(X,Y)$ is said to be minimal USCO ([DL]) if it is USCO and does not contain properly any other USCO multifunction from $U(X,Y)$.

By an easy application of the Kuratowski-Zorn principle we can guarantee that every USCO multifunction from X to Y contains a minimal USCO multifunction from X to Y ([DL]).

Minimal multifunctions were also studied in [Ho1], [M1], [M2].

Denote by $M(X,Y)$ the space of all minimal USCO multifunctions.

Theorem 7. *Let X be a topological space and (Y,d) be a metric space. Then $M(X,Y)$ is a closed set in $(U(X,Y),\varrho)$.*

Proof. Let Φ be in the closure of $M(X,Y)$ in $(U(X,Y),\varrho)$. Let Ψ be a minimal USCO multifunction contained in Φ. We claim that $\Phi=\Psi$ (Φ, Ψ are identified with their graphs). Suppose that there is $(x_0,y_0)\in\Phi\setminus\Psi$. Then there is an open neighbourhood U of x_0 and $\varepsilon>0$ such that

$$\bigl(U\times S_\varepsilon[y_0]\bigr)\cap\Psi=\emptyset.$$

Let $\bigl\{\Phi_n:\ n\in\mathbb{Z}^+\bigr\}$ be a sequence in $M(X,Y)$ convergent to Φ in $(U(X,Y),\varrho)$.

Consider $S_{\frac{\varepsilon}{2}}[\Phi]$. Then there exists $n_0\in\mathbb{Z}^+$ such that $\varrho(\Phi_{n_0},\Phi)<\frac{\varepsilon}{2}$. Put

$$\Omega=\Phi_{n_0}\setminus\bigl(U\times S_{\frac{\varepsilon}{2}}[y_0]\bigr)\qquad\text{and}\qquad\Omega(x)=\bigl\{y:\ (x,y)\in\Omega\bigr\}.$$

We claim that $\Omega(x)\neq\emptyset$ for all $x\in X$.

It is sufficient to show that

$$\Omega(x)\neq\emptyset\qquad\text{for all}\quad x\in U.$$

Let $x\in U$, since Ψ is contained in Φ and $\Phi_{n_0}\in S_{\frac{\varepsilon}{2}}[\Phi]$, then there is $y\in\Psi(x)$ and $z\in\Phi_{n_0}(x)$ such that $d(y,z)<\frac{\varepsilon}{2}$. Then $z\notin S_{\frac{\varepsilon}{2}}[y_0]$. Thus we have that $\Omega(x)\neq\emptyset$.

Consider Ω as a multifunction from X to Y. By [DL], graph of Φ_{n_0} is a closed set, so graph of Ω is closed and then Ω is an USCO multifunction. $\Omega\subset\Phi_{n_0}$ and we claim that $\Omega\neq\Phi_{n_0}$. Since $\Phi_{n_0}\in S_{\frac{\varepsilon}{2}}[\Phi]$, there exists $y\in\Phi_{n_0}(x_0)$ such that $d(y_0,y)<\frac{\varepsilon}{2}$ so

$$\Omega(x_0)\neq\Phi_{n_0}(x_0).$$

Thus Φ_{n_0} is not a minimal USCO multifunction, a contradiction. $\square$

By using Theorem 6 and Theorem 7 we obtain the following theorem:

Theorem 8. *Let X be a topological space and (Y,d) be a complete metric space. Then $(M(X,Y),\varrho)$ is a complete metric space.*

Now we define a densely continuous form from X to Y ([HM]).

Denote by $\mathrm{DC}(X,Y)$ the set of functions from X to Y which are continuous at all points of some dense subset of X.

Let f be a function from X to Y. Define

$$C(f)=\bigl\{x\in X:\ f\text{ is continuous at }x\bigr\}.$$

Let

$$f\restriction C(f)=\bigl\{(x,y)\in X\times Y:\ x\in C(f),\ \ y=f(x)\bigr\}.$$

We define the set $D(X,Y)$ of densely continuous form by

$$D(X,Y)=\bigl\{\,\overline{f\restriction C(f)}:\ f\in\mathrm{DC}(X,Y)\bigr\}.$$

The densely continuous forms from X to Y may be considered as multifunctions ($D(X,Y) \subset F(X, 2^Y)$).

Let X be a topological space and (Y,d) be a metric space. If $\Phi \in D(X,Y)$ and $A \subset X$, we say that Φ is bounded on A, provided that the set $\Phi(A)$ is a bounded set of Y. Then we say that Φ is locally bounded if for all $x \in X$ there exists a neighbourhood $U(x)$ of x such that Φ is bounded on $U(x)$.

Now define $D^\star(X,Y)$ to be the set of members of $D(X,Y)$, that are locally bounded.

Let (Y,d) be a metric space. A metric space (Y,d) is called b-compact, if every bounded subset of Y has compact closure ([Ho2]).

Similar as in [Ho1] we show the following facts.

Let (Y,d) be a b-compact, then

$$D^\star(X,Y) \subset M(X,Y).$$

In fact, since (Y,d) is a b-compact, if $\Phi \in D^\star(X,Y)$, then for all $x \in X$, $\Phi(x)$ is a nonempty compact set. By a result of B e r g e [Ber, p. 112] any multifunction with closed graph which has a compact range is upper semicontinuous. Then

$$D^\star(X,Y) \subset U(X,Y).$$

Now by [DL, Theorem 4.7], if $\Phi \in D^\star(X,Y)$, then Φ is minimal USCO and

$$D^\star(X,Y) \subset M(X,Y).$$

If X is a Baire space and (Y,d) is a b-compact metric space, then

$$M(X,Y) \subset D^\star(X,Y).$$

In fact, if Φ is a USC multifunction with nonempty values, then by [Fo] there is a dense subset E of X such that Φ is lower semicontinuous at each $x \in E$. Then, if $\Phi \in M(X,Y)$, from the minimality of Φ is easy to show that, for each $x \in E$, $\Phi(x)$ must be single-valued. Then any selection of Φ is continuous in each $x \in X$ and by [DL], $\Phi \in D(X,Y)$. It is easy to show that, if (Y,d) is a b-compact, every USCO multifunction from X to Y is locally bounded.

As a result we have that if X is a Baire space and (Y,d) is a b-compact, then

$$M(X,Y) = D^\star(X,Y).$$

By using of Theorem 8, by above mentioned and from the fact, that every b-compact is complete, we have the following result:

Theorem 9. *Let X be a Baire space and (Y,d) be a b-compact space. Then $(D^\star(X,Y), \varrho)$ is a complete metric space.*

Proposition 10. ([Ho1]) *Let X, Y be topological spaces and Y be locally compact. If $\Phi \in D(X,Y)$, then there is an open dense set U in X such that Φ is upper semicontinuous at every point in U and for every $x \in U$ $\Phi(x)$ is a nonempty compact set.*

Theorem 11. *Let X be a Baire space and (Y,d) be a locally compact metric space. Then $D(X,Y)$ is a closed set in $(G(X,2^Y),\varrho)$.*

P r o o f. The proof uses some of the ideas of the proof of [Ho1, Theorem 4.3]. Let $\Phi \in G(X,2^Y)$ be in the closure of $D(X,Y)$ in $(G(X,2^Y),\varrho)$.

Put for each $n \in \mathbb{Z}^+$

$$B_n = \left\{x \in X:\ \operatorname{diam}\Phi(x) < \tfrac{1}{n}\right\} \cup \left\{x \in X:\ \Phi(x) = \emptyset\right\}.$$

We prove that $\operatorname{int} B_n$ is a dense set in X. Let $\left\{\Phi_k:\ k \in \mathbb{Z}^+\right\}$ be a sequence in $D(X,Y)$ convergent to Φ in $(G(X,2^Y),\varrho)$. Consider $S_{\frac{1}{4n}}[\Phi]$. Then there exists $k_n \in \mathbb{Z}^+$ such that

$$H_d(\Phi(x),\Phi_{k_n}(x)) < \frac{1}{4n} \qquad \text{for all} \quad x \in X. \tag{1}$$

Since $\Phi_{k_n} \in D(X,Y)$, there exists a dense set A_{k_n} in X such that $\Phi_{k_n}(x)$ is a singleton and Φ_{k_n} is USC at every $x \in A_{k_n}$ (see proof of [Ho1, Proposition 2.2]). We have $\operatorname{diam}\Phi(x) < \frac{1}{2n}$ for every $x \in A_{k_n}$. Thus $A_{k_n} \subset B_n$. Let $x_0 \in A_{k_n}$, Φ_{k_n} is USC at x_0, then there exists a neighbourhood V of x_0 such that

$$\Phi_{k_n}(V) \subset S_{\frac{1}{4n}}[\Phi_{k_n}(x_0)].$$

Let $x \in V$, then

$$\Phi(x) \subset S_{\frac{1}{4n}}[\Phi_{k_n}(x)] \subset S_{\frac{1}{4n}}[S_{\frac{1}{4n}}[\Phi_{k_n}(x_0)]] \subset S_{\frac{1}{2n}}[\Phi_{k_n}(x_0)].$$

Since $\Phi_{k_n}(x_0)$ is a singleton, we have that $\operatorname{diam}\Phi(x) < \frac{1}{n}$ or $\Phi(x) = \emptyset$ and then $V \subset B_n$. Thus $\operatorname{int} B_n$ is a dense set in X.

By Proposition 10, for all Φ_k there exists an open dense set U_k in X such that Φ_k is USC at every point in U_k and for every $x \in U_k$, $\Phi_k(x)$ is a nonempty compact set. Because of (1) and the properties of H_d we have that $\Phi(x)$ is nonempty for all $x \in U_{k_n}$. Put $D_n = \operatorname{int} B_n \cap U_{k_n}$. Then D_n is an open dense set in X.

Put $B = \bigcap\limits_{n \in \mathbb{Z}^+} D_n$. Then B is a dense set in X since X is a Baire space. For all $x \in B$ we have that $\Phi(x)$ is a singleton.

Let $x \in X$. If $\Phi(x) \neq \emptyset$, choose $s(x) \in \Phi(x)$. Let z be an arbitrary point in Y. We define a function $f\colon X \to Y$ as follows:

$$f(x) = \begin{cases} s(x) & \text{if } \Phi(x) \neq \emptyset, \\ z & \text{otherwise.} \end{cases}$$

We claim that f is continuous at every $x \in B$. Let $x_0 \in B$ and let $\varepsilon > 0$. For $\frac{\varepsilon}{3}$ there exists $n \in \mathbb{Z}^+$ such that $\frac{1}{4n} < \frac{\varepsilon}{3}$. Φ_{k_n} is USC at x_0, thus there is an open neighbourhood $U \subset U_{k_n}$ of x_0 such that

$$\emptyset \neq \Phi_{k_n}(x) \subset S_{\frac{\varepsilon}{3}}[\Phi_{k_n}(x_0)] \qquad \text{for every} \quad x \in U.$$

Since $\Phi_{k_n} \in S_{\frac{\varepsilon}{3}}[\Phi]$ and $\Phi_{k_n}(x) \neq \emptyset$ for every $x \in U$, we have that $\Phi(x) \neq \emptyset$ for every $x \in U$.
Let $x \in U$. We have

$$f(x) \in S_{\frac{\varepsilon}{3}}[\Phi_{k_n}(x)] \subset S_{\frac{\varepsilon}{3}}[S_{\frac{\varepsilon}{3}}[\Phi_{k_n}(x_0)]] \subset S_{\frac{\varepsilon}{3}}[S_{\frac{\varepsilon}{3}}[S_{\frac{\varepsilon}{3}}[f(x_0)]]] \subset S_\varepsilon[f(x_0)].$$

Now we prove that $\Phi = \overline{f \restriction C(f)}$. Suppose that it is not true. Then there is

$$(x_0, y_0) \in \Phi \setminus \overline{f \restriction C(f)}.$$

There is an open neighbourhood U of x_0 and $S_\varepsilon[y_0]$ for some $\varepsilon > 0$, such that

$$(U \times S_\varepsilon[y_0]) \cap \overline{f \restriction C(f)} = \emptyset.$$

There is $k \in \mathbb{Z}^+$ such that

$$\Phi_k \in S_{\frac{\varepsilon}{4}}[\Phi].$$

Then there exists $z \in \Phi_k(x_0)$ with $d(y_0, z) < \frac{\varepsilon}{4}$.

Let $g_k \in \mathrm{DC}(X, Y)$ be such that $\Phi_k = \overline{g_k \restriction C(g_k)}$. There is $x_1 \in U \cap U_k$ such that

$$d(g_k(x_1), y_0) < \frac{\varepsilon}{4}.$$

Since $\Phi_k(x_1) = g_k(x_1)$ and Φ_k is USC at x_1, for $\frac{\varepsilon}{4}$ there exists an open neighbourhood O of x_1 such that for all $x \in O$

$$\Phi_k(x) \subset S_{\frac{\varepsilon}{4}}[g_k(x_1)].$$

Let $a \in U \cap O \cap B$. We have $f(a) = \Phi(a)$ and

$$f(a) \in S_{\frac{\varepsilon}{4}}[\Phi_k(a)] \subset S_{\frac{\varepsilon}{2}}[g_k(x_1)] \subset S_\varepsilon[y_0],$$

a contradiction. □

The following example shows that the condition of Bairness of the space X in the above theorem is essential.

Example 1. Denote by $\mathbb{Q}$ the set of all rational numbers and by $\mathbb{R}$ the set of all real numbers. Let $X = \mathbb{Q}$, $Y = \mathbb{R}$ and consider both spaces with the usual metric d. Let h be a bijection from $\mathbb{Z}^+$ to $\mathbb{Q}$. For each $n \in \mathbb{Z}^+$ define the function $f_n \colon X \to Y$ as follows: If $n = 1$, then

$$f_1(x) = \begin{cases} \frac{1}{2} & x \geq h(1), \\ 0 & x < h(1). \end{cases}$$

Suppose we have defined $f_1, \dots, f_n$. We define f_{n+1} as follows: Let $\varepsilon_{n+1} > 0$ be such that $|h(n+1) - h(i)| > 2\varepsilon_{n+1}$ for all $i < n+1$. Then

$$f_{n+1}(x) = \begin{cases} f_n(x) + \frac{1}{2^{n+1}} & h(n+1) \leq x < h(n+1) + \varepsilon_{n+1}, \\ f_n(x) & \text{otherwise}, \end{cases}$$

It is easy to see that $C(f_n)$ is a dense set in X for every $n \in \mathbb{Z}^+$.
Put $F_n = \overline{f_n \upharpoonright C(f_n)}$ for every $n \in \mathbb{Z}^+$. $F_n \in D(X,Y)$ for every $n \in \mathbb{Z}^+$.

Now define the multifunction F from X to Y as follows: If $x = h(1)$, then

$$F(x) = \left\{0, \tfrac{1}{2}\right\}$$

If $x = h(n)$, where $n > 1$, then

$$F(x) = \left\{f_{n-1}(x),\ f_{n-1}(x) + \tfrac{1}{2^n}\right\}.$$

It is easy to see that the sequence $\left\{F_n :\ n \in \mathbb{Z}^+\right\}$ converges to F pointwise. Since for all $x \in X$

$$H_d(F_n(x), F_{n+1}(x)) \le \frac{1}{2^{n+1}},$$

$\left\{F_n :\ n \in \mathbb{Z}^+\right\}$ is Cauchy. By Theorem 5 there exists a multifunction $H \in G(X, 2^Y)$ such that $\left\{F_n :\ n \in \mathbb{Z}^+\right\}$ converges to H in $(G(X,2^Y), \varrho)$. Then $H(x) = F(x)$ for all $x \in X$. Since $F(x)$ consist of two elements for all $x \in X$, we have $F \notin D(X,Y)$.

The proof of the next theorem follows from Theorem 5 and Theorem 12.

Theorem 12. *Let X be a Baire space and (Y,d) be a locally compact complete metric space. Then $(D(X,Y), \varrho)$ is a complete metric space.*

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