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THE DESIGN OF THE CENTURY

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ABSTRACT. We construct a 2-chromatic Steiner system $S(2,4,100)$ in which every block contains three points of one colour and one point of the other colour. The existence of such a design has been open for over 25 years.

1. The background

A *Steiner system* $S(t,k,v)$ is an ordered pair $(V,\mathscr{B})$ where V is a set of cardinality v, the *base set*, and $\mathscr{B}$ is a collection of k-subsets of V, the *blocks*, which collectively have the property that every t-element subset of V is contained in precisely one block. Elements of V are called *points*. In this paper we are principally concerned with the case in which $t = 2$ and $k = 4$. Steiner systems $S(2,4,v)$ exist if and only if $v \equiv 1$ or $4 \pmod{12}$, [4]; such values of v are called *admissible*. Given a Steiner system $S(2,4,v)$, we may ask whether it is possible to colour each point of the base set V with one of two colours, say red or blue, so that no block is monochromatic. A Steiner system $S(2,4,v)$ having this property is said to be 2*-chromatic* or to have a *blocking set*. It was shown in [5] that 2-chromatic $S(2,4,v)$s exist for all admissible v with the possible exception of three values, $v = 37$, 40 and 73. Existence for these three values was established in [3]. Perhaps we should also remark here that it is known that for all $v \geq 25$ there exists a Steiner system $S(2,4,v)$ which is not 2-chromatic, [8].

In a 2-chromatic $S(2,4,v)$ let c and $v - c$ be the cardinalities of the red and blue colour classes, respectively. If b_1, b_2 and b_3 are the numbers of blocks with

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colour patterns $RRRB$, $RRBB$ and $RBBB$, respectively, then by counting pairs we have:

$$\begin{aligned} 3b_1 + b_2 &= \frac{c\,(c-1)}{2}, \\ b_2 + 3b_3 &= \frac{(v-c)(v-c-1)}{2}, \\ 3b_1 + 4b_2 + 3b_3 &= c\,(v-c). \end{aligned}$$

Solving the equations for b_2 gives $b_2 = (4vc - 4c^2 + v - v^2)/4$, which is non-negative for

$$\frac{v-\sqrt{v}}{2} \le c \le \frac{v+\sqrt{v}}{2}.$$

Furthermore, in the extreme cases where $\{c, v-c\} = \big\{(v-\sqrt{v}\,)/2, (v+\sqrt{v}\,)/2\big\}$ it follows that $b_2 = 0$; i.e. every block contains three points of one colour and one of the other colour. Moreover, the monochromatic triples of each colour appearing in the blocks form Steiner systems $S\big(2, 3, (v-\sqrt{v}\,)/2\big)$ and $S(2, 3, (v+\sqrt{v}\,)/2)$. An $S(2, 3, w)$ is usually called a *Steiner triple system* and denoted by STS(w); they exist if and only if $w \equiv 1$ or 3 (mod 6), [6]. A modern account of Kirkman's work is given in [1]. From the preceding discussion, it is easy to deduce that a 2-chromatic $S(2, 4, v)$ having all blocks containing three points of one colour and one of the other colour can exist only if v is of the form $(12s+2)^2$ or $(12s+10)^2$, $s \ge 0$.

The smallest non-trivial case is therefore $v = 100$, and has become known as "the Design of the Century". Its existence, and possible construction, has been a problem in Design Theory for over 25 years. An early reference is [7]. In this paper we construct the design. We make no claim for uniqueness and, indeed, we think it highly unlikely.

2. The method

The cardinalities of the two colour classes are 55, the red points, and 45, the blue points. Denote the former by $A_0, A_1, \ldots, A_{54}$ and the latter by $\infty, B_0, B_1, \ldots, B_{43}$. We will seek an $S(2, 4, 100)$ having an automorphism σ of order 11 defined by

$$\sigma : A_i \mapsto A_{i+5 \pmod{55}}, \quad B_j \mapsto B_{j+4 \pmod{44}}, \quad \infty \mapsto \infty.$$

Our method is based on a simple backtrack algorithm with four distinct stages.

Stage 1. Select systems STS(55) and STS(45), both having automorphism σ, on the red and blue points respectively. The latter is an example of a 4-rotational STS(v); such systems exist for $v \equiv 1, 9, 13$ or 21 (mod 24), [2].

Stage 2. The blue system has 30 orbits under the automorphism. We partition these into five classes of six orbits, and label each class with a different point from the set $\{A_0, A_1, A_2, A_3, A_4\}$. Within each class we then assign the label to one block of each of the six orbits in such a way that the blocks to which the label is assigned form a partial parallel class; i.e. the blocks are pairwise disjoint. The assignment of red points to the other blocks of blue points is completely determined by σ. It is clear that this assignment ensures that there are no repeated pairs of a blue point with a red point.

Stage 3. The red system has 45 orbits under the automorphism. We next deal with the blue point ∞. In the course of performing stage 2 of the algorithm the point ∞ will have been paired with two of the five subsets $\{A_{i+j} : j = 0, 5, 10, \ldots, 50\}$, $i = 0, 1, 2, 3, 4$. We assign ∞ to all blocks of a single orbit whose red points cover the remaining three subsets.

Stage 4. This leaves 44 orbits of the red system. As in stage 2 we partition these into four classes of 11 orbits and label each class with a different point from the set $\{B_0, B_1, B_2, B_3\}$. Within each class, we then assign the label, say X, to one block of each of the 11 orbits in such a way that the blocks to which X is assigned form a partial parallel class, say $\mathscr{P}$. We attempt to do this while satisfying the further constraint that none of the 22 red points with which X has already been paired in stage 2 occur in $\mathscr{P}$. This latter is, of course, a very severe constraint. Again, the assignment of the blue points to the other blocks of red points is completely determined by σ.

Finally, we make a brief remark about our implementation of the algorithm. Stages 3 and 4 execute very quickly on a modern computer system and we always ran the backtracking to completion. However, for each particular choice of systems STS(55) and STS(45), we did not run the backtracking of stage 2 to completion, preferring instead to return to stage 1 after a certain period of time and select new systems.

3. The design

Listed below are 75 blocks which, under the mapping σ, give "the Design of the Century". As described in the last section, the construction of the design involved significant computing. However, it is perfectly feasible, although perhaps a little tedious, to check the design by hand, and the dedicated reader is invited to do this.

$B0$	$B1$	$B9$	$A0$	$B4$	$B6$	$B23$	$A0$	$B8$	$B11$	$B13$	$A0$
$B12$	$B20$	$B10$	$A0$	$B36$	$B5$	$B7$	$A0$	$B2$	$B3$	$B38$	$A0$
$B0$	$B4$	$B33$	$A1$	$B40$	$B3$	$B6$	$A1$	$B8$	$B24$	$B5$	$A1$
$B16$	$B7$	$B11$	$A1$	$B1$	$B2$	$B21$	$A1$	$B9$	$B13$	$B42$	$A1$
$B0$	$B6$	$B24$	$A2$	$B8$	$B19$	$B40$	$A2$	$B4$	$B31$	$B3$	$A2$
$B9$	$B14$	$B43$	$A2$	$B1$	$B7$	$B29$	$A2$	$B5$	$B26$	$B2$	$A2$
$B0$	$B14$	$B21$	$A3$	$B4$	$B35$	$B41$	$A3$	$B1$	$B10$	$B31$	$A3$
$B5$	$B23$	∞	$A3$	$B2$	$B7$	$B18$	$A3$	$B22$	$B34$	$B3$	$A3$
$B28$	$B1$	$B14$	$A4$	$B4$	$B22$	$B26$	$A4$	$B0$	$B38$	∞	$A4$
$B29$	$B39$	$B2$	$A4$	$B37$	$B5$	$B19$	$A4$	$B3$	$B11$	$B35$	$A4$
$A25$	$A29$	$A19$	$B0$	$A20$	$A32$	$A5$	$B0$	$A35$	$A48$	$A18$	$B0$
$A15$	$A39$	$A11$	$B0$	$A41$	$A43$	$A8$	$B0$	$A21$	$A42$	$A13$	$B0$
$A31$	$A14$	$A28$	$B0$	$A16$	$A9$	$A12$	$B0$	$A17$	$A23$	$A49$	$B0$
$A37$	$A44$	$A33$	$B0$	$A22$	$A34$	$A38$	$B0$				
$A35$	$A36$	$A13$	$B1$	$A10$	$A17$	$A18$	$B1$	$A25$	$A44$	$A11$	$B1$
$A20$	$A43$	$A19$	$B1$	$A5$	$A37$	$A39$	$B1$	$A15$	$A6$	$A12$	$B1$
$A30$	$A23$	$A28$	$B1$	$A26$	$A29$	$A34$	$B1$	$A21$	$A38$	$A48$	$B1$
$A27$	$A42$	$A9$	$B1$	$A32$	$A49$	$A7$	$B1$				
$A5$	$A7$	$A21$	$B2$	$A20$	$A25$	$A46$	$B2$	$A35$	$A41$	$A11$	$B2$
$A40$	$A54$	$A15$	$B2$	$A30$	$A47$	$A19$	$B2$	$A36$	$A37$	$A23$	$B2$
$A26$	$A39$	$A24$	$B2$	$A16$	$A32$	$A53$	$B2$	$A31$	$A12$	$A22$	$B2$
$A17$	$A8$	$A9$	$B2$	$A13$	$A28$	$A49$	$B2$				
$A40$	$A43$	$A32$	$B3$	$A35$	$A44$	$A15$	$B3$	$A10$	$A20$	$A38$	$B3$
$A5$	$A27$	$A47$	$B3$	$A25$	$A6$	$A13$	$B3$	$A21$	$A26$	$A36$	$B3$
$A31$	$A42$	$A11$	$B3$	$A16$	$A28$	$A34$	$B3$	$A41$	$A9$	$A29$	$B3$
$A12$	$A17$	$A48$	$B3$	$A8$	$A24$	$A54$	$B3$	$A0$	$A11$	$A37$	∞

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