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ON SOME NEW TYPE GENERALIZED DIFFERENCE SEQUENCE SPACES

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ABSTRACT. In this paper we introduce a new type of difference operator Δ_m^n for fixed $m, n \in \mathbb{N}$. We define the sequence spaces $\ell_\infty(\Delta_m^n)$, $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ and study some topological properties of these spaces. We obtain some inclusion relations involving these sequence spaces. These notions generalize many earlier existing notions on difference sequence spaces.

1. Introduction

Throughout the paper w, ℓ_∞, c, and c_o denote the spaces of all, bounded, convergent and null sequences $x = (x_k)$ with complex terms respectively, normed by $\|x\| = \sup\limits_k |x_k|$.

The zero sequence is denoted by $\theta = (0, 0, 0, \dots)$.

Kizmaz [4] defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_o(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_k) \in w : \ (\Delta x_k) \in Z\},$$

for $Z = \ell_\infty, c$ and c_o, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. The above spaces are Banach spaces, normed by

$$\|x\|_\Delta = |x_1| + \sup_k |\Delta x_k|.$$

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The notion was further generalized by Et and Colak [2] as follows:

$$Z(\Delta^n) = \{x = (x_k) \in w : \ (\Delta^n x_k) \in Z\},$$

for $Z = \ell_\infty, c$ and c_o, where $\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$. They showed that the above spaces are Banach spaces, normed by

$$\|(x_k)\|_{\Delta^n} = \sum_{i=1}^{n} |x_i| + \sup_k |\Delta^n x_k|.$$

Recently the idea was generalized by Tripathy and Esi [7] as follows:

Let $m \geq 0$, be a fixed integer, then

$$Z(\Delta_m) = \{x = (x_k) \in w : \ (\Delta_m x_k) \in Z\},$$

for $Z = \ell_\infty, c$ and c_o, where $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$ and $\Delta_0 x_k = x_k$ for all $k \in \mathbb{N}$. They showed that the above spaces are Banach spaces, normed by

$$\|(x_k)\|_{\Delta_m} = \sum_{i=1}^{m} |x_i| + \sup_k \|\Delta_m x_k\|.$$

The idea of Kizmaz [4] was applied for introducing different type of difference sequence spaces and for studying their different algebraic and topological properties by Tripathy ([5], [6]) and many others.

2. Definitions and preliminaries

A sequence space E said to be *solid* (or *normal*) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

A sequence space E is said to be *monotone* if it contains the canonical preimages of all its step spaces.

A sequence space E is said to be *convergence free* if $(y_k) \in E$ whenever $(x_k) \in E$ and $y_k = 0$ whenever $x_k = 0$.

A sequence space E is said to be a *sequence algebra* if $(x_k \cdot y_k) \in E$ whenever $(x_k) \in E$ and $(y_k) \in E$.

A sequence space E is said to be *symmetric* if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$, where π is a permutation on $\mathbb{N}$.

Let $m, n \geq 0$ be fixed integers, then we introduce the following new type of generalized difference sequence spaces

$$Z(\Delta_m^n) = \big\{x = (x_k) \in w : \ \Delta_m^n x = (\Delta_m^n x_k) \in Z\big\},$$

for $Z = \ell_\infty, c$ and c_o, where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbb{N}$. This generalized difference notion has the following binomial representation:

$$\Delta_m^n x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{k+m\nu} \qquad \text{for all} \quad k \in \mathbb{N}.$$

For $n = 1$, these spaces reduce to the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$ introduced and studied by Tripathy and Esi [7].

For $m = 1$, these represent the spaces $\ell_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$ introduced and studied by Et and Colak [2].

For $m = 1$ and $n = 1$, these spaces represent the spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [4].

3. Main results

In this section we state and prove the results of this article. The proof of the following two results are routine verifications.

Proposition 1. *The classes of sequences $\ell_\infty(\Delta_m^n)$, $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are normed linear spaces, normed by*

$$\|x\|_{\Delta_m^n} = \sum_{i=1}^{r} |x_i| + \sup_k |\Delta_m^n x_k|, \tag{1}$$

where $r = mn$ for $m \geq 1$, $n \geq 1$; $r = n$ for $m = 1$ and $r = m$ for $n = 1$.

Proposition 2.

2.1. $c_0(\Delta_m^n) \subset c(\Delta_m^n) \subset \ell_\infty(\Delta_m^n)$ *and the inclusions are proper.*

2.2. $Z(\Delta_m^i) \subset Z(\Delta_m^n)$ *for $Z = c, c_0$ and ℓ_∞, for $0 \leq i < n$ and the inclusions are strict.*

Theorem 3. *The sequence spaces $\ell_\infty(\Delta_m^n)$, $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are Banach spaces, under the norm* (1).

P r o o f. Let (x^s) be a Cauchy sequence in $\ell_\infty(\Delta_m^n)$, where $x^s = (x_i^s) = (x_1^s, x_2^s, x_3^s, \dots) \in \ell_\infty(\Delta_m^n)$ for each $s \in \mathbb{N}$. Then

$$\|x^s - s^t\|_{\Delta_m^n} = \sum_{i=1}^{r} |x_i^s - x_i^t| + \sup_k |\Delta_m^n(x_k^s - x_k^t)| \to 0 \qquad \text{as} \quad s, t \to \infty,$$

where $r = mn$ for $m \geq 1$, $n \geq 1$; $r = n$ for $m = 1$ and $r = m$ for $n = 1$.

Hence we obtain

$$|x_k^s - x_k^t| \to 0 \qquad \text{as} \quad s, t \to \infty,$$

for each $k \in \mathbb{N}$.

Therefore $(x_k^s) = (x_1^s, x_2^s, x_3^s, \dots)$ is a Cauchy sequence in $\mathbb{C}$, the set of complex numbers. Since $\mathbb{C}$ is complete, it is convergent, then

$$\lim_{s \to \infty} x_k^s = x_k$$

say, for each $k \in \mathbb{N}$. Since (x^s) is a Cauchy sequence, for each $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that

$$\|x^s - s^t\|_{\Delta_m^n} < \varepsilon$$

for all $s, t \geq n_0$. Hence

$$\sum_{i=1}^{m} |x_i^s - x_i^t| < \varepsilon$$

and

$$|\Delta_m^n(x_k^s - x_k^t)| = \left| \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (x_{k+m\nu}^s - x_{k+m\nu}^t) \right| < \varepsilon$$

for all $k \in \mathbb{N}$ and for all $s, t \geq n_0$.

On taking limit as $t \to \infty$, in the above two inequalities, we have

$$\lim_{t \to \infty} \sum_{i=1}^{m} |x_i^s - x_i^t| = \sum_{i=1}^{m} |x_i^s - x_i| < \varepsilon$$

and

$$\lim_{t \to \infty} |\Delta_m^n(x_i^s - x_i^t)| = |\Delta_m^n(x_i^s - x_i)| < \varepsilon$$

for all $s \geq n_0$. This implies that $\|x^s - x\|_{\Delta_m^n} < 2\varepsilon$ for all $s \geq n_0$, that is $x^s \to x$, as $s \to \infty$, where $x = (x_k)$. Also, since

$$\begin{aligned}|\Delta_m^n x_k| &= \left|\sum_{\nu=0}^{n}(-1)^\nu \binom{n}{\nu}(x_{k+m\nu})\right| = \left|\sum_{\nu=0}^{n}(-1)^\nu \binom{n}{\nu}(x_{k+m\nu} - x_{k+m\nu}^{n_0} + x_{k+m\nu}^{n_0})\right| \\ &\leq \left|\sum_{\nu=0}^{n}(-1)^\nu \binom{n}{\nu}(x_{k+m\nu}^{n_0} - x_{k+m\nu})\right| + \left|\sum_{\nu=0}^{n}(-1)^\nu \binom{n}{\nu})(x_{k+m\nu}^{n_0})\right| \\ &\leq \|x^{n_0} - x\|_{\Delta_m^n} + \|\Delta_m^n x_k^{n_0}\| = O(1).\end{aligned}$$

Hence $x \in \ell_\infty(\Delta_m^n)$. Therefore $\ell_\infty(\Delta_m^n)$ is a Banach space.

Similarly it can be shown that the spaces $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are also Banach spaces.

Since $\ell_\infty(\Delta_m^n)$, $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are Banach spaces with continuous coordinates, that is

$$\|x^s - x\|_{\Delta_m^n} \to 0 \implies |x_k^s - x_k| \to 0 \qquad \text{as} \quad s \to \infty,$$

for each $k \in \mathbb{N}$. □

We now state the following result:

Proposition 4. *The spaces $\ell_\infty(\Delta_m^n)$, $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are BK-spaces.*

Proposition 5. *The spaces $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are nowhere dense subsets of $\ell_\infty(\Delta_m^n)$.*

Proof. From Proposition 2.1 it follows that the inclusions $c(\Delta_m^n) \subset \ell_\infty(\Delta_m^n)$ and $c_0(\Delta_m^n) \subset \ell_\infty(\Delta_m^n)$ are strict. Further from Theorem 3, it follows that the spaces $c_0(\Delta_m^n)$ and $c(\Delta_m^n)$ are closed. Hence the spaces $c_0(\Delta_m^n)$ and $c(\Delta_m^n)$ are nowhere dense subsets of $\ell_\infty(\Delta_m^n)$. □

Theorem 6.

6.1. *The spaces $\ell_\infty(\Delta_m^n)$, $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are not solid spaces in general. For $m = n = 0$, the spaces ℓ_∞ and c_0 are solid.*

6.2. *The space $c_0(\Delta)$ is symmetric.*

6.3. *The spaces $\ell_\infty(\Delta_m^n)$, $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are not symmetric in general.*

6.4. *The spaces $\ell_\infty(\Delta_m^n)$, $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are not convergence free.*

6.5. *The spaces $\ell_\infty(\Delta_m^n)$, $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are not monotone in general.*

Proof.

6.1: That the spaces ℓ_∞ and c_0 are solid is well known. To show that the spaces $\ell_\infty(\Delta_m^n)$ and $c(\Delta_m^n)$ are not solid in general, let $m = n = 2$. Consider the sequence (x_k) defined by $x_1 = 1$ and $x_{k+1} = x_k + k + 2$ for all $k \in \mathbb{N}$. Then (x_k) belongs to $\ell_\infty(\Delta_2^2)$ and $c(\Delta_2^2)$ both. Consider the sequence of scalars (α_k) defined by $\alpha_k = 1$ for $k = 3i$, for $i \in \mathbb{N}$ and $\alpha_k = 0$, otherwise. Then $(\alpha_k x_k)$ neither belongs to $c(\Delta_2^2)$ nor to $\ell_\infty(\Delta_2^2)$. Hence the spaces $\ell_\infty(\Delta_m^n)$ and $c(\Delta_m^n)$ are not solid in general.

To show that the space $c_0(\Delta_m^n)$ is not solid in general, let $m = n = 2$. Consider the sequence (x_k) defined by $x_k = 1$ for all $k \in \mathbb{N}$ and the sequence (α_k) defined as above. Then $(x_k) \in c_0(\Delta_2^2)$, but $(\alpha_k x_k) \notin c_0(\Delta_2^2)$. Hence $\ell_\infty(\Delta_m^n)$ is not solid.

6.2: The proof is known.

6.3: To show that the spaces $c(\Delta_m^n)$ and $\ell_\infty(\Delta_m^n)$ are not symmetric in general, let $m = n = 2$ and consider the sequence (x_k) defined by $x_1 = 1$ and $x_{k+1} = x_k + k + 2$ for all $k \in \mathbb{N}$. Consider the rearranged sequence (y_k) of (x_k) defined as

$$y_k = \begin{cases} x_k, & \text{if } k = 3n-2,\ n \in \mathbb{N}, \\ x_{k+1}, & \text{if } k \text{ is even and } k \neq 3n-2,\ n \in \mathbb{N}, \\ x_{k-1}, & \text{if } k \text{ is odd and } k \neq 3n-2,\ n \in \mathbb{N}. \end{cases}$$

Then (y_k) neither belongs to $c(\Delta_2^2)$ nor to $\ell_\infty(\Delta_2^2)$.

Hence the spaces $c(\Delta_m^n)$ and $\ell_\infty(\Delta_m^n)$ and are not symmetric in general.

Next to show that the space $c_0(\Delta_m^n)$ is not symmetric in general, let $m = n = 2$ and consider the sequence (x_k) defined by $x_k = 1$ if k is odd and $x_k = 2$ if k is even, for all $k \in \mathbb{N}$. Consider its rearrangement defined by

$$y_k = \begin{cases} 2, & \text{if } k = i^2,\ i \in \mathbb{N}\ , \\ 1, & \text{otherwise.} \end{cases}$$

Then $(x_k) \in c_0(\Delta_2^2)$, but $(y_k) \notin c_0(\Delta_2^2)$.

Hence the space $c_0(\Delta_m^n)$ is not symmetric in general.

6.4: Let $m = n = 3$ and consider the sequence (x_k) defined by $x_k = 1$, for all $k \in \mathbb{N}$. Then $(x_k) \in c_0(\Delta_3^3) \subset c(\Delta_3^3) \subset \ell_\infty(\Delta_3^3)$. Now consider the sequence (y_k) defined by $y_k = k^2$, for all $k \in \mathbb{N}$, then $(y_k) \notin \ell_\infty(\Delta_3^3)$. Hence the spaces $c_0(\Delta_3^3)$, $c(\Delta_3^3)$ and $\ell_\infty(\Delta_3^3)$ are not convergence free.

6.5: First we show that the spaces $\ell_\infty(\Delta_m^n)$ and $c(\Delta_m^n)$ are not monotone in general. Let $m = 3$ and $n = 2$. Consider the sequence $x = (x_k)$ defined by $x_1 = 1$ and $x_{k+1} = x_k + k + 1$, for all $k \in \mathbb{N}$. Then $(x_k) \in c(\Delta_3^2)$ and $\ell_\infty(\Delta_3^2)$. Now consider the sequence (y_k) in its pre-image space defined by $y_k = 1$, for k odd and $y_k = 0$, for k even. Then (y_k) neither belongs to $c(\Delta_3^2)$ nor to $\ell_\infty(\Delta_3^2)$. Hence the spaces $c(\Delta_3^2)$ and $\ell_\infty(\Delta_3^2)$ are not monotone.

Next we show that the space $c_0(\Delta_m^n)$ is not monotone in general. Let $m = 3$ and $n = 2$. Consider the sequence $x = (x_k)$ defined by $x_k = 2$, for all $k \in \mathbb{N}$. Then $(x_k) \in c_0(\Delta_3^2)$. Now consider the sequence (y_k) in its pre-image space, defined as above. Then $(y_k) \notin c_0(\Delta_3^2)$. Hence the spaces $c_0(\Delta_3^2)$ is not monotone. $\square$

The proof of the following result is easy, so omitted.

PROPOSITION 7.

(i) *$Z(\Delta) \subset Z(\Delta_m^n)$, for $Z = \ell_\infty$, c and c_0.*

(ii) *$c(\Delta_m^n) \subset c_0(\Delta_m^n)$.*

(iii) *If m is even, then $c(\Delta_m^n) \subset c_0(\Delta_m^n)$.*

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REFERENCES

[1] COOKE, R. G.: *Infinite Matrices and Sequence Spaces*, MacMillan, London, 1950.

[2] ET, M.—COLAK, R.: *On some generalized difference spaces*, Soochow J. Math. **21** (1995), 377–386.

[3] KAMTHAN, P. K.—GUPTA, M.: *Sequence Spaces and Series*, Marcel Dekker Inc., New York, 1981.

[4] KIZMAZ, H.: *On certain sequence spaces*, Canad. Math. Bull. **24** (1981), 169–176.

[5] TRIPATHY, B. C.: *A class of difference sequence sequences related to the p-normed space ℓ^p*, Demonstratio Math. **36** (2003), 867–872.

[6] TRIPATHY, B. C.: *On some class of difference paranormed sequence spaces associated with multiplier sequences*, Int. J. Math. Math. Sci. **2** (2003), 159–166.

[7] TRIPATHY, B. C.—ESI, A.: *A new type of difference sequence spaces*, Internat. J. Sci. Tech. **1** (2006), 11–14.

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