

NON-HOMEOMORPHIC DENSITY TOPOLOGIES

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ABSTRACT. In [ANISZCZYK, B.—FRANKIEWICZ, R.: *Non homeomorphic density topologies*, Bull. Polish Acad. Sci. Math. **34** (1986), 211–213] the authors constructed 2^c non-homeomorphic density topologies tied to the concept of measure. In [WROŃSKI, S.: *The number of non-homeomorphic I -density topologies*, Institute of Mathematics of Polish Academy of Sciences. Preprint 482, December 1990, XXXIV Semester in Banach Center, Theory of Real Functions] an analogous construction has been given for topologies tied to the concept of category. Here we present a generalization containing both the results mentioned above. This generalization based only on the topological ideas is independent of the concept of measure or category.

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Let $\langle \mathbf{1}, T \rangle$ be a fixed topological space. The symbols: Δ_T , L_T , I and C will stand for the family of nowhere dense sets, the algebra of sets with nowhere dense boundary, the interior operator and the closure operator of the space $\langle \mathbf{1}, T \rangle$.

By a density topology for $\langle \mathbf{1}, T \rangle$ we shall mean any topology D on the set $\mathbf{1}$ such that:

$$T \subset D \subset L_T, \\ \Delta_T = \Delta_D, \quad L_T = L_D.$$

Compare with [1] and [6].

Let \varkappa denote a positive cardinal. By a \varkappa -partition we shall mean a function $f: \varkappa \rightarrow L_T/\Delta_T$ satisfying the following conditions:

- (i) $(\forall \gamma < \varkappa)(f(\gamma) \neq 0)$
- (ii) $\sup_{\gamma < \varkappa} f(\gamma) = 1$
- (iii) $(\forall \gamma_1, \gamma_2 < \varkappa)(\gamma_1 \neq \gamma_2 \Rightarrow f(\gamma_1) \wedge f(\gamma_2) = 0).$

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The set of all κ -partitions will be denoted by $\text{Par}(\kappa)$. For $A \in L_T$ we denote by $[A]$ the respective member of L_T/Δ_T .

We say that a cardinal κ is acceptable if the following condition holds:

- (A) $(\forall x \in \mathbf{1}) (\exists f \in \text{Par}(\kappa)) (\forall A \in L_T)$
 $(x \in IA \implies (\forall \gamma < \kappa) ([A] \wedge f(\gamma) \neq 0)).$

The main theorem of our paper is the following:

THEOREM. (Main) *Let $\langle \mathbf{1}, T \rangle$ be a regular topological space. If an infinite cardinal κ is acceptable and such that $\text{card}(L_T/\Delta_T) = 2^\kappa$, and $\text{card}(\mathbf{1}) \leq 2^\kappa$, then there exist 2^{2^κ} non-homeomorphic density topologies for the space $\langle \mathbf{1}, T \rangle$. Moreover, none of those topologies has a non-trivial autohomeomorphism.*

We shall frequently use the following well-known fact:

LEMMA 1. (Preliminary) *If A, B are clopen and $A \cap C = B \cap C$ where C is a dense set, then $A = B$.*

Let \mathbb{U} be the set of all ultrafilters of the algebra L_T/Δ_T and let $i: L_T/\Delta_T \rightarrow P(\mathbb{U})$ be the mapping given by $i([A]) = \{j \in \mathbb{U} : [A] \in j\}$, for every $A \in L_T$. ($P(\mathbb{U})$ denotes the family of all subsets of the set \mathbb{U} .) Let \mathcal{I} be the topology on \mathbb{U} determined by the basis $\{i([A])\}_{A \in L_T \setminus \Delta_T}$. It is known that the topological space $\langle \mathbb{U}, \mathcal{I} \rangle$ is compact and extremally disconnected and moreover, its field of clopen subsets is just the image of L_T/Δ_T under the mapping i , called frequently the Stone-isomorphism.

Using properties of interior operator one can easily prove the following:

LEMMA 2. *For every $x \in \mathbf{1}$ the set $F_x = \{[A] \in L_T/\Delta_T : x \in IA\}$ is a proper filter of the algebra L_T/Δ_T .*

For $x \in \mathbf{1}$ we put $\mathbb{F}_x = \{j \in \mathbb{U} : F_x \subset j\}$.

LEMMA 3. *If $A \in L_T \setminus \Delta_T$, then:*

- (a) $IA \neq \emptyset$
 (b) $(\exists x \in A) (\mathbb{F}_x \subset i([A]))$.

Proof. It is easy to check that every boundary set with a nowhere dense boundary is itself nowhere dense. This fact and the assumption that $A \in L_T \setminus \Delta_T$ yields (a) and the existence of $x \in A$, such that $[A] \in F_x$. We will show that $\mathbb{F}_x \subset i([A])$. Indeed, if $j \in \mathbb{F}_x$, then $F_x \subset j$, and therefore $[A] \in j$. This means that $j \in i([A])$, by the definition of the Stone isomorphism i . \square

Let Sel denote the set of all functions $S: \mathbf{1} \rightarrow \mathbb{U}$ such that $(\forall x \in \mathbf{1}) (S(x) \in \mathbb{F}_x)$. The elements of the set Sel are called the selectors.

LEMMA 4. *For an arbitrary $S \in \text{Sel}$ the set $S(\mathbf{1})$ is a dense subset of the space $\langle \mathbb{U}, \mathcal{I} \rangle$. If $\langle \mathbf{1}, T \rangle$ is a Hausdorff space then the mapping S is one-to-one.*

Proof. To prove the first part let us assume that U is a base set of the space $\langle \mathbb{U}, \mathcal{I} \rangle$. Then $U = i([A])$, for some $A \in L_T \setminus \Delta_T$. By Lemma 3(b), there exists $x \in A$ such that $\mathbb{F}_x \subset i([A])$ and consequently $S(x) \in i([A]) \cap S(\mathbf{1}) = U \cap S(\mathbf{1})$. By the choice of U , this yields the density of $S(\mathbf{1})$.

To prove that S is one-to-one we will show that $\mathbb{F}_x \cap \mathbb{F}_y = \emptyset$ whenever $x \neq y$. Assume to the contrary that $j \in \mathbb{F}_x \cap \mathbb{F}_y$, for some $j \in \mathbb{U}$. Since $x \neq y$ and $\langle \mathbf{1}, T \rangle$ is a Hausdorff space, there exist sets $A, B \in T \subset L_T$ such that $x \in A$, $y \in B$, $A \cap B = \emptyset$. In such a situation $x \in IA$ and $y \in IB \subset I(-A)$ which means that $[A] \in F_x$ and $[-A] \in F_y$. Since $F_x, F_y \subset j$, we have $[A], [-A] \in j$ which is impossible because j is a proper filter. \square

For an arbitrary $S \in \text{Sel}$, let $\varphi_S : L_T \rightarrow P(\mathbf{1})$ be a mapping given by $\varphi_S(A) = S^{-1}(i([A]))$, for every $A \in L_T$.

THEOREM 1. *The following conditions hold for every $A, B \in L_T$:*

- (1) $IA \subset \varphi_S(A) \subset CA$
- (2) $A \equiv B \iff \varphi_S(A) = \varphi_S(B)$
- (3) $\varphi_S(A \cup B) = \varphi_S(A) \cup \varphi_S(B)$
- (4) $\varphi_S(A \cap B) = \varphi_S(A) \cap \varphi_S(B)$
- (5) $\varphi_S(-A) = -\varphi_S(A)$
- (6) $[A] \leq [B] \implies \varphi_S(A) \subset \varphi_S(B)$
- (7) $\varphi_S(A) = \emptyset \iff A \in \Delta_T$
- (8) $A \div \varphi_S(A) \in \Delta_T$.

Proof. To prove the first inclusion of (1) assume that $x \in IA$. This implies that $[A] \in F_x \subset S(x)$ and therefore $S(x) \in i([A])$ and finally $x \in S^{-1}(i([A])) = \varphi_S(A)$.

To prove (2) assume first that $A \equiv B$. Then $i([A]) = i([B])$ and consequently $\varphi_S(A) = S^{-1}(i([A])) = S^{-1}(i([B])) = \varphi_S(B)$.

Conversely, assume that $\varphi_S(A) = \varphi_S(B)$. This means that $S^{-1}(i([A])) = S^{-1}(i([B]))$ and consequently $S(S^{-1}(i([A]))) = S(S^{-1}(i([B])))$. This implies that $i([A]) \cap S(\mathbf{1}) = i([B]) \cap S(\mathbf{1})$ where $i([A])$ and $i([B])$ are clopen and $S(\mathbf{1})$ is dense (Lemma 4). Finally one gets that $A \equiv B$, by Preliminary Lemma. The assertions (3), (4) and (5) follow directly from the property of inverse images while the second inclusion of (1) follows from the first inclusion, by (5).

To prove (6) observe that if $[A] \leq [B]$, then $\varphi_S(A) = \varphi_S(A \cup B)$, by the definition of φ_S . Now, by (3), one gets $\varphi_S(A) = \varphi_S(A) \cup \varphi_S(B)$ and therefore $\varphi_S(A) \subset \varphi_S(B)$.

To prove (7) assume first that $\varphi_S(A) = \emptyset$. Now, by the definition of φ_S it follows that $i([A]) \cap S(\mathbf{1}) = \emptyset \cap S(\mathbf{1})$ where $i([A])$, \emptyset are clopen while $S(\mathbf{1})$ is dense. Thus, by Preliminary Lemma, $i([A]) = \emptyset$ which means that $A \in \Delta_T$.

Conversely, assume that $A \in \Delta_T$. Then $[A] = 0$, and therefore $i([A]) = \emptyset$. Finally $\varphi_S(A) = S^{-1}(i([A])) = S^{-1}(\emptyset) = \emptyset$.

To prove (8) observe first that

$$\begin{aligned} A \setminus \varphi_S(A) &= (IA \cup (A \setminus IA)) \setminus \varphi_S(A) \stackrel{(1)}{\subset} (\varphi_S(A) \cup (A \setminus IA)) \setminus \varphi_S(A) \\ &= (A \setminus IA) \setminus \varphi_S(A). \end{aligned}$$

But $A \setminus IA \in \Delta_T$ because $A \in L_T$. Thus $A \setminus \varphi_S(A) \in \Delta_T$. This implies that $\varphi_S(A) \setminus A = (-A) \setminus (-\varphi_S(A)) \stackrel{(5)}{=} (-A) \setminus \varphi_S(-A) \in \Delta_T$ and for this reason $A \div \varphi_S(A) = (A \setminus \varphi_S(A)) \cup (\varphi_S(A) \setminus A) \in \Delta_T$. \square

COROLLARY 1.

- (a) $\varphi_S(L_T) \subset L_T$
- (b) $\varphi_S \circ \varphi_S = \varphi_S$
- (c) $\varphi_S(L_T \setminus \Delta_T) \subset L_T \setminus \Delta_T$.

Proof. Since $\varphi_S(A) = (A \setminus (A \setminus \varphi_S(A))) \cup (\varphi_S(A) \setminus A)$ for every $A \in L_T$, then $\varphi_S(A) \in L_T$, by Theorem 1(8). This proves (a), and (b) follows from (a) by virtue of Theorem 1(8),(2).

To prove (c) assume that $A \in L_T \setminus \Delta_T$. By (a) it will be sufficient to prove that $\varphi_S(A) \notin \Delta_T$. Assume on the contrary. Then $\varphi_S(A) = \emptyset$, by Theorem 1(7) and (b). Again, by Theorem 1(7), it follows that $A \in \Delta_T$ which is a contradiction. \square

COROLLARY 2. *If $A \in L_T \setminus \Delta_T$, then $I\varphi_S(A) \neq \emptyset$.*

Proof. By Corollary 1(c) it follows that $\varphi_S(A) \in L_T \setminus \Delta_T$, which implies that $I\varphi_S(A) \neq \emptyset$, by Lemma 3(a). \square

It is easy to check that the family $\{\varphi_S(A)\}_{A \in L_T \setminus \Delta_T}$ is a topological basis. The topology determined by this basis on the set $\mathbf{1}$ will be further denoted by T_S .

LEMMA 5. *If $\langle \mathbf{1}, T \rangle$ is a Hausdorff space then the spaces $\langle \mathbf{1}, T_S \rangle$ and $\langle S(\mathbf{1}), \{S(\mathbf{1}) \cap U\}_{U \in \mathcal{I}} \rangle$ are homeomorphic.*

Proof. From Lemma 4 it follows that the required homeomorphism is just the mapping $S: \mathbf{1} \rightarrow S(\mathbf{1})$. \square

LEMMA 6. *$T_S \subset L_T$, if $\langle \mathbf{1}, T \rangle$ is a regular space then $T \subset T_S$.*

Proof. First let us note that the following assertion is true:

$$(*) \quad (\forall \mathcal{H} \subset L_T) (\exists G \in L_T) \left(G \subset \bigcup \mathcal{H} \wedge [G] = \sup_{H \in \mathcal{H}} [H] \right).$$

To prove the inclusion $T_S \subset L_T$ assume that $A \in T_S$. Then $A = \bigcup \varphi_S(\mathcal{F})$, for some family $\mathcal{F} \subset L_T$. By Corollary 1(a), $\varphi_S(\mathcal{F}) \subset L_T$ and thus, applying the assertion (*) to the family $\varphi_S(\mathcal{F})$, we get that there exists $B \in L_T$ such that $B \subset \bigcup \varphi_S(\mathcal{F}) = A$ and $[B] = \sup_{H \in \mathcal{F}} [\varphi_S(H)]$. By the definition of supremum, $(\forall H \in \mathcal{F}) ([\varphi_S(H)] \leq [B])$. This, together with Theorem 1(6) implies that $\varphi_S(\varphi_S(H)) \subset \varphi_S(B)$, for every $H \in \mathcal{F}$. Applying Corollary 1(b) we get that for every $H \in \mathcal{F}$, $\varphi_S(H) \subset \varphi_S(B)$, which yields that $A = \bigcup \varphi_S(\mathcal{F}) \subset \varphi_S(B)$. Thus, $B \subset A \subset \varphi_S(B)$ and finally $A \in L_T$ because $B \div \varphi_S(B) \in \Delta_T$, by virtue of Theorem 1(8).

To prove the inclusion $T \subset T_S$ assume that $A \in T$. We shall prove that every $x \in A$ has a neighbourhood which is a subset of A belonging to the basis of the space $\langle \mathbf{1}, T_S \rangle$. Take any $x \in A$. Since $\langle \mathbf{1}, T \rangle$ is a regular space, by virtue of the assumption, then there exists $U \in T \setminus \{\emptyset\} \subset L_T \setminus \Delta_T$ such that $x \in U \subset CU \subset A$. On the other hand, by Theorem 1(1), $U = IU \subset \varphi_S(U) \subset CU$ and combining all this we get that $x \in \varphi_S(U) \subset A$, as required. \square

COROLLARY 3. *If $\langle \mathbf{1}, T \rangle$ is a regular space, then $\Delta_T = \Delta_{T_S}$, $L_T = L_{T_S}$.*

Proof. From Lemma 6 and Corollary 2 we infer that the families $T \setminus \{\emptyset\}$ and $T_S \setminus \{\emptyset\}$ are cofinal. This, by virtue of [2, Proposition 1.2(i), Proposition 1.3], implies both equalities to be proved. \square

COROLLARY 4. *If $\langle \mathbf{1}, T \rangle$ is a regular space then for every selector S , the family T_S is a density topology for $\langle \mathbf{1}, T \rangle$.*

Proof. It is an immediate consequence of Lemma 6 and Corollary 3. \square

For every $x \in \mathbf{1}$, $f \in \text{Par}(\kappa)$ and $j \in \mathbb{F}_x$ let $\nabla(j, f) = \{P \subset \kappa : \sup f(P) \in j\}$. Moreover, let $\beta\kappa$ stand for the set of all ultrafilters of the algebra of all subsets of κ .

LEMMA 7. $\nabla(j, f) \in \beta\kappa$.

Proof. Clearly $\nabla(j, f)$ must be an ultrafilter because so is j . \square

For every $x \in \mathbf{1}$, $j \in \mathbb{F}_x$ let $B(j) = \{\nabla(j, f) : f \in \text{Par}(\kappa)\}$.

LEMMA 8. *If κ is an infinite cardinal and $\text{card}(L_T/\Delta_T) = 2^\kappa$, then $\text{card}(B(j)) \leq 2^\kappa$.*

Proof. From the definition of the set $B(j)$ it follows that its cardinality cannot exceed the number of mappings of κ into L_T/Δ_T . Thus, $\text{card}(B(j)) \leq \text{card}(L_T/\Delta_T)^\kappa = (2^\kappa)^\kappa = 2^\kappa$ because κ is an infinite cardinal by virtue of the assumption. \square

Let $\text{Im} = \{S(\mathbf{1}) : S \in \text{Sel}\}$.

LEMMA 9. *Let $Z_1, Z_2 \in \text{Im}$ and let $h: Z_1 \rightarrow Z_2$ be a homeomorphism of subspaces Z_1, Z_2 of the space $\langle \mathbb{U}, \mathcal{I} \rangle$. Then for every $f \in \text{Par}(\varkappa)$ there exists $g \in \text{Par}(\varkappa)$ satisfying the following condition:*

$$(B) \quad (\forall P \subset \varkappa) (h(Z_1 \cap i(\sup f(P))) = Z_2 \cap i(\sup g(P))).$$

Proof. Since for every $\gamma < \varkappa$ the set $Z_1 \cap i(f(\gamma))$ is clopen in the space Z_1 and the mapping h is a homeomorphism $h(Z_1 \cap i(f(\gamma)))$ is clopen in the space Z_2 . The algebra L_T/Δ_T is complete, so by Lemma 4 it follows that there exists a mapping $g: \varkappa \rightarrow L_T/\Delta_T$ such that $h(Z_1 \cap i(f(\gamma))) = Z_2 \cap i(g(\gamma))$ for every $\gamma < \varkappa$. We will prove that $g \in \text{Par}(\varkappa)$ i.e. it obeys the conditions (i), (ii) and (iii) of the definition of \varkappa -partition and moreover, it obeys the condition (B).

To prove (iii) suppose that $\gamma_1, \gamma_2 < \varkappa$ and $\gamma_1 \neq \gamma_2$. Then

$$\begin{aligned} i(g(\gamma_1) \wedge g(\gamma_2)) \cap Z_2 &= i(g(\gamma_1)) \cap Z_2 \cap i(g(\gamma_2)) \cap Z_2 \\ &= h(Z_1 \cap i(f(\gamma_1))) \cap h(Z_1 \cap i(f(\gamma_2))) \\ &= h(Z_1 \cap i(f(\gamma_1)) \cap i(f(\gamma_2))) \\ &= h(Z_1 \cap i(f(\gamma_1) \wedge f(\gamma_2))) \\ &= h(Z_1 \cap i(0)) = h(Z_1 \cap \emptyset) = h(\emptyset) = \emptyset. \end{aligned}$$

Now, by Lemma 4 and Preliminary Lemma it follows that $g(\gamma_1) \wedge g(\gamma_2) = 0$.

To prove (i) suppose to the contrary, that $g(\gamma_0) = 0$, for some $\gamma_0 < \varkappa$. Then $Z_2 \cap i(g(\gamma_0)) = Z_2 \cap i(0) = Z_2 \cap \emptyset = \emptyset$, which implies that $h(Z_1 \cap i(f(\gamma_0))) = \emptyset$. Clearly $Z_1 \cap i(f(\gamma_0)) = \emptyset = Z_1 \cap \emptyset$ because h is a homeomorphism. Applying again Lemma 4 and Preliminary Lemma one gets that $f(\gamma_0) = 0$ which is impossible since $f \in \text{Par}(\varkappa)$, by virtue of the assumption.

Starting the proof of (B) let us denote the closure operators of the space $\langle \mathbb{U}, \mathcal{I} \rangle$ and subspaces $\langle Z_1, \{Z_1 \cap U\}_{U \in \mathcal{I}} \rangle$, $\langle Z_2, \{Z_2 \cap U\}_{U \in \mathcal{I}} \rangle$ by $C_{\mathcal{I}}$, C_{Z_1} , C_{Z_2} , respectively. By Lemma 4, the sets Z_1, Z_2 are dense in the space $\langle \mathbb{U}, \mathcal{I} \rangle$ and for every $P \subset \varkappa$ the sets $\bigcup i(f(P))$, $\bigcup i(g(P))$ are open in this space. Therefore $C_{\mathcal{I}}(\bigcup i(f(P))) = C_{\mathcal{I}}(Z_1 \cap \bigcup i(f(P)))$ and $C_{\mathcal{I}}(\bigcup i(g(P))) = C_{\mathcal{I}}(Z_2 \cap \bigcup i(g(P)))$. Thus we get:

$$\begin{aligned} h(Z_1 \cap i(\sup f(P))) &= h(Z_1 \cap \sup i(f(P))) = h\left(Z_1 \cap C_{\mathcal{I}}\left(\bigcup i(f(P))\right)\right) \\ &= h\left(Z_1 \cap C_{\mathcal{I}}\left(Z_1 \cap \bigcup i(f(P))\right)\right) = h\left(C_{Z_1}\left(Z_1 \cap \bigcup i(f(P))\right)\right) \\ &= C_{Z_2}\left(h\left(Z_1 \cap \bigcup i(f(P))\right)\right) = C_{Z_2}\left(\bigcup_{\gamma \in P} h(Z_1 \cap i(f(\gamma)))\right) \\ &= C_{Z_2}\left(\bigcup_{\gamma \in P} (Z_2 \cap i(g(\gamma)))\right) = C_{Z_2}\left(Z_2 \cap \bigcup i(g(P))\right) \end{aligned}$$

$$\begin{aligned}
 &= Z_2 \cap C_{\mathcal{I}} \left(Z_2 \cap \bigcup i(g(P)) \right) = Z_2 \cap C_{\mathcal{I}} \left(\bigcup i(g(P)) \right) \\
 &= Z_2 \cap \sup i(g(P)) = Z_2 \cap i(\sup g(P)).
 \end{aligned}$$

Proving (ii) we will apply (B) with $P = \varkappa$. Then we have $h \left(Z_1 \cap i \left(\sup_{\gamma < \varkappa} f(\gamma) \right) \right)$
 $= Z_2 \cap i \left(\sup_{\gamma < \varkappa} g(\gamma) \right)$. But on the other hand $h \left(Z_1 \cap i \left(\sup_{\gamma < \varkappa} f(\gamma) \right) \right) \stackrel{(ii)}{=} h(Z_1 \cap i(1)) = h(Z_1 \cap \mathbb{U}) = h(Z_1) = Z_2 = Z_2 \cap \mathbb{U} = Z_2 \cap i(1)$. Comparing the right sides of the above series of equalities, we infer that $Z_2 \cap i \left(\sup_{\gamma < \varkappa} g(\gamma) \right) = Z_2 \cap i(1)$. Now, applying Lemma 4 and Preliminary Lemma, we get that $\sup_{\gamma < \varkappa} g(\gamma) = 1$. \square

LEMMA 10. *Given $Z \in \text{Im}$ and a mapping $h: Z \rightarrow \mathbb{U}$ such that $h(Z) \in \text{Im}$. If h is a homeomorphism of subspaces Z and $h(Z)$ of the space $\langle \mathbb{U}, \mathcal{I} \rangle$, then $B(j) = B(h(j))$, for every $j \in Z$.*

Proof. First we will prove that $B(j) \subset B(h(j))$. Take any member of $B(j)$, it has a form $\nabla(j, f)$ for some $f \in \text{Par}(\varkappa)$. By Lemma 9, it follows that there exists $g \in \text{Par}(\varkappa)$ such that for every $P \subset \varkappa$:

$$(B') \quad h(Z \cap i(\sup f(P))) = h(Z) \cap i(\sup g(P)).$$

Now let us consider the ultrafilter $\nabla(h(j), g) \in B(h(j))$, we will prove that $\nabla(j, f) \subset \nabla(h(j), g)$. Indeed, if $P \in \nabla(j, f)$, then $\sup f(P) \in j$. Since $j \in Z$, then $j \in Z \cap i(\sup f(P))$. Applying (B') we get that $h(j) \in h(Z) \cap i(\sup g(P))$ and therefore $\sup g(P) \in h(j)$ or in other words $P \in \nabla(h(j), g)$. Since $\nabla(j, f), \nabla(h(j), g) \in \beta\varkappa$, then the inclusion just proved implies that $\nabla(j, f) = \nabla(h(j), g)$ and consequently $\nabla(j, f) \in B(h(j))$. The inclusion $B(h(j)) \subset B(j)$ can be proved in a similar manner. \square

COROLLARY 5. *Let $\langle \mathbf{1}, T \rangle$ be a Hausdorff space. Let $S, S_1, S_2 \in \text{Sel}$, $S(\mathbf{1}) = Z$, $S_1(\mathbf{1}) = Z_1$, $S_2(\mathbf{1}) = Z_2$. Then the following conditions hold:*

- (a) *If for every distinct $j, k \in Z$, $B(j) \neq B(k)$ then the space $\langle \mathbf{1}, T_S \rangle$ has only trivial autohomeomorphism.*
- (b) *If for every $j \in Z_1$, $k \in Z_2$ we have $B(j) \neq B(k)$ then spaces $\langle \mathbf{1}, T_{S_1} \rangle$, $\langle \mathbf{1}, T_{S_2} \rangle$ are not homeomorphic.*

Proof. To prove (a) assume that a non-trivial autohomeomorphism of $\langle \mathbf{1}, T_S \rangle$ exists. Then, by Lemma 5, a non-trivial autohomeomorphism h exists also in the space $\langle Z, \{Z \cap U\}_{U \in \mathcal{I}} \rangle$. Let $s \in Z$ be such that $h(s) \neq s$. Then, by Lemma 10, $B(s) = B(h(s))$ and putting $j = s$, $k = h(s)$ we get that $j \neq k$ and

$B(j) = B(k)$, as required. The condition (b) can be proved in an analogous way. \square

For every $q \in \beta\kappa$ and $f \in \text{Par}(\kappa)$ we put

$$F(q, f) = \{a \in L_T / \Delta_T : (\exists P \in q)(a \geq \sup f(P))\}.$$

LEMMA 11. $F(q, f)$ is a proper filter of the algebra L_T / Δ_T .

Proof. $F(q, f)$ is a proper filter because so is q . \square

LEMMA 12. If κ is acceptable then for every $q \in \beta\kappa$ and $x \in \mathbf{1}$ there exists an ultrafilter $j \in \mathbb{F}_x$ such that $q \in B(j)$.

Proof. Take any $q \in \beta\kappa$ and $x \in \mathbf{1}$. Let $f \in \text{Par}(\kappa)$ be a partition satisfying the following condition:

$$(A') \quad (\forall A \in L_T)(x \in IA \implies (\forall \gamma < \kappa)([A] \wedge f(\gamma) \neq 0)).$$

By Lemma 11, $F(q, f)$ is a proper filter of L_T / Δ_T . Let F be the filter generated by $F(q, f) \cup F_x$. Then $F = \{a \in L_T / \Delta_T : (\exists b \in F(q, f))(\exists g \in F_x)(a \geq b \wedge g)\}$. We shall prove that the filter F is proper. Suppose to the contrary, that for some $b \in F(q, f)$ and $g \in F_x$, $b \wedge g = 0$. Since $b \in F(q, f)$, there exists $P \in q$ such that $b \geq \sup f(P)$. Since $g \in F_x$, there exists $G \in L_T$ such that $x \in IG$ and $g = [G]$. By (A'), one gets that $g \wedge f(\gamma) \neq 0$, for every $\gamma < \kappa$. Since the set P , as an element of an ultrafilter, must be nonempty, we have $g \wedge \sup f(P) \neq 0$. From $g \wedge b \geq g \wedge \sup f(P)$ we get $g \wedge b \neq 0$, a contradiction. Now, let j be an ultrafilter of L_T / Δ_T containing F . Since $F_x \subset F \subset j$, then $j \in \mathbb{F}_x$. To make sure that $q \in B(j)$ take any $P \in q$. Then, by the definition of $F(q, f)$, it follows that $\sup f(P) \in F(q, f) \subset j$ which means that $P \in \nabla(j, f)$. We have proved that $q \subset \nabla(j, f)$. Since both q and $\nabla(j, f)$ are ultrafilters then $q = \nabla(j, f)$ and consequently $q \in B(j)$. \square

Now we are ready to prove the main result of this paper. First, by Zermelo's well-ordering theorem, we equip the set $\mathbf{1}$ with a well-order and arrange its elements into a transfinite sequence $(x_\gamma)_{\gamma < \text{card}(\mathbf{1})}$. Next, we define a sequence of selectors $(S_\gamma)_{\gamma < 2^{2^\kappa}}$ where the following conditions are satisfied:

1. $(\forall \gamma < 2^{2^\kappa})(\forall j, k \in S_\gamma(\mathbf{1}))(j \neq k \implies B(j) \neq B(k))$.
2. $(\forall \gamma', \gamma'' < 2^{2^\kappa})(\gamma' \neq \gamma'' \implies (\forall j \in S_{\gamma'}(\mathbf{1}))(\forall k \in S_{\gamma''}(\mathbf{1}))(B(j) \neq B(k)))$.

We start with the selector S_0 . Recall that $S_0(x_0)$ is a member of \mathbb{F}_{x_0} and assume for induction that all values $S_0(x_\alpha) \in \mathbb{F}_{x_\alpha}$ have already been defined for $\alpha < \beta < \text{card}(\mathbf{1})$ in such a way that $B(S_0(x_{\alpha_1})) \neq B(S_0(x_{\alpha_2}))$ whenever

$$\alpha_1, \alpha_2 < \beta \text{ and } \alpha_1 \neq \alpha_2. \text{ By Lemma 8, } \text{card} \left(\bigcup_{\alpha < \beta} B(S_0(x_\alpha)) \right) \leq 2^\kappa \text{card}(\beta)$$

$< 2^{2^\kappa}$ while $\text{card}(\beta\kappa) = 2^{2^\kappa}$, by a theorem of Pospišil ([3, pp. 146]). Thus, there exists an ultrafilter $q \in \beta\kappa \setminus \bigcup_{\alpha < \beta} B(S_0(x_\alpha))$ and, by Lemma 12, there exists $j \in \mathbb{F}_{x_\beta}$ such that $q \in B(j)$. Putting $S_0(x_\beta) = j$ we complete our inductive definition of a selector S_0 satisfying the condition 1 for $\gamma = 0$.

Next, let us assume that $0 < \beta < 2^{2^\kappa}$ and that selectors S_α have already been defined for all $\alpha < \beta$ in such a way that the following conditions are satisfied:

- 1'. $(\forall \alpha < \beta) (\forall j, k \in S_\alpha(\mathbf{1})) (j \neq k \implies B(j) \neq B(k))$.
- 2'. $(\forall \alpha', \alpha'' < \beta) (\alpha' \neq \alpha'' \implies (\forall j \in S_{\alpha'}(\mathbf{1})) (\forall k \in S_{\alpha''}(\mathbf{1})) (B(j) \neq B(k)))$.

Note that by Lemma 8 and the assumptions of our Theorem it follows that $\text{card}\left(\bigcup_{\alpha < \beta} \bigcup_{x \in \mathbf{1}} B(S_\alpha(x))\right) \leq \text{card}(\beta) \text{card}(\mathbf{1}) 2^\kappa \leq \text{card}(\beta) 2^{2^\kappa} < 2^{2^{2^\kappa}}$. Thus, as before, there exists an ultrafilter $q \in \beta\kappa \setminus \bigcup_{\alpha < \beta} \bigcup_{x \in \mathbf{1}} B(S_\alpha(x))$ and again we can use

Lemma 12 to complete an inductive definition of S_β and thereby of the sequence of selectors $(S_\gamma)_{\gamma < 2^{2^\kappa}}$ which satisfy the conditions 1 and 2, by construction.

Note that topologies T_{S_γ} determined by selectors S_γ , $\gamma < 2^{2^\kappa}$ are density topologies, by virtue of Corollary 4. Moreover, by Corollary 5, all these topologies have only trivial autohomeomorphisms and no two of them are homeomorphic, if determined by distinct selectors.

Finally, it is worth to note that the assumptions of our theorem are satisfied by any standard density topology T of the space $\mathbf{1} = (0, 1)$ with an acceptable cardinal ω , no matter whether T is defined by means of measure or by means of category (see [4], [5]). The same is true if T is the Euclidean topology of this space.

Let us notice, that ω is not the only cardinal number for which the main theorem holds.

THEOREM 2. *Let κ denote any infinite cardinal. Then there exists regular topological space $\langle \mathbf{1}, T \rangle$ fulfilling the condition $\text{card}(\mathbf{1}) = \text{card}(L_T/\Delta_T) = 2^{2^\kappa}$ such that 2^κ is acceptable for this space.*

Proof. Let \mathcal{W} denote the set of all open intervals included in $(-1, 1)$. Let \mathcal{E} denote the family of all functions $e: 2^\kappa \rightarrow P((-1, 1))$, which fulfil the following conditions:

- $(\forall \gamma < 2^\kappa) (e(\gamma) \in \mathcal{W})$
- $\text{card}(\{\gamma < 2^\kappa : e(\gamma) \neq (-1, 1)\}) \leq \kappa$.

Let $\mathbf{1} = (-1, 1)^{2^\kappa}$. The topology determined on the set $\mathbf{1}$ by the basis $\mathcal{B} = \left\{ \prod_{\gamma < 2^\kappa} e(\gamma) : e \in \mathcal{E} \right\}$ will be further denoted by T . It is easy to check that the topological space $\langle \mathbf{1}, T \rangle$ is regular and that $\text{card}(\mathbf{1}) = \text{card}(L_T/\Delta_T) = 2^{2^\kappa}$. To

prove that 2^\varkappa is acceptable assume that $x = (x_\gamma)_{\gamma < 2^\varkappa} \in \mathbf{1}$, $\alpha \in \{-1, 1\}^\varkappa$. Let us suppose that for arbitrary $\gamma < 2^\varkappa$

$$W_\gamma^\alpha(x) = \begin{cases} (-1, 1) & \text{if } \gamma \geq \varkappa \\ (-1, x_\gamma) & \text{if } \gamma < \varkappa \text{ and } \alpha(\gamma) = -1 \\ (x_\gamma, 1) & \text{if } \gamma < \varkappa \text{ and } \alpha(\gamma) = 1. \end{cases}$$

Let $W^\alpha(x) = \prod_{\gamma < 2^\varkappa} W_\gamma^\alpha(x)$. From the definitions given above we can infer that all the sets $W^\alpha(x)$ belong to \mathcal{B} and that $W^\alpha(x) \cap W^\beta(x) = \emptyset$ while $\alpha, \beta \in \{-1, 1\}^\varkappa$ and $\alpha \neq \beta$. Moreover for arbitrary $\alpha \in \{-1, 1\}^\varkappa$, $U \in \mathcal{B}$, and $x \in \mathbf{1}$, such that $x \in U$, we have that $U \cap W^\alpha(x) \in \mathcal{B}$. Since $\text{card}(\{-1, 1\}^\varkappa) = 2^\varkappa$, we can construct a function f which fulfils the condition (A) of the cardinal 2^\varkappa , by the use of the family $\{[W^\alpha(x)]\}_{\alpha \in \{-1, 1\}^\varkappa}$. \square

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