VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-007-0029-0
Math. Slovaca **57** (2007), No. 4, 349–358

RATE OF APPROXIMATION FOR THE BÉZIER VARIANT OF BALAZS KANTOROVICH OPERATORS

Vijay Gupta* — X. M. Zeng**

(*Communicated by Michal Zajac*)

ABSTRACT. In the present paper we study the Bézier variant of the well known Balazs-Kantorovich operators $L_{n,\alpha}(f,x)$, $\alpha \geq 1$. We establish the rate of convergence for functions of bounded variation. For particular value $\alpha = 1$, our main theorem completes a result due to Agratini [Math. Notes (Miskolc) 2 (2001), 3–10].

1. Introduction

For a real valued function f defined on the interval $[0,\infty)$, Balazs [2] introduced the Bernstein type rational functions, which are defined by

$$R_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{b_n}\right), \qquad \text{where} \quad p_{n,k}(x) = \binom{n}{k} \frac{(a_n x)^k}{(1+a_n x)^n} \tag{1}$$

and a_n, b_n are suitably chosen positive numbers independent of x. The weighted estimates and uniform convergence for the special case $a_n = n^{\beta-1}$, $b_n = n^{\beta}$, $0 < \beta \leq 2/3$ were investigated in [3]. Actually the operators defined by (1) are just Bernstein type rational functions, but the approximation properties of these operators are different from the usual Bernstein polynomials.

Zeng and Piriou [12] were the first who introduce and study the Bézier variant of the Bernstein operators. After this the rates of convergence for the several integral type operators were obtained by Gupta and collaborators (see e.g. [4]–[10] etc). Actually Bézier curves play an important role in Computer

2000 Mathematics Subject Classification: Primary 41A30, 41A36.
Keywords: rate of convergence, Bézier variant, bounded variation, total variation, Balazs-Kantorovich operator.

Aided Geometric Design, this along with the different approximation properties of Balazs operators from usual Bernstein polynomials motivated us to study further in this direction.

Recently A g r a t i n i [1] defined the Kantorovich variant of the Balazs operators as

$$L_n(f,x) = na_n \sum_{k=0}^{n} p_{n,k}(x) \int_{I_{n,k}} f(t)\,\mathrm{d}t, \qquad n \in \mathbb{N}, \ \ x \geq 0, \tag{2}$$

where $I_{n,k} = [k/na_n, (k+1)/na_n]$, and $x \geq 0$, $0 \leq k \leq n$.

The Bézier variant of these Balazs-Kantorovich operators can be defined as:

$$L_{n,\alpha}(f,x) = na_n \sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) \int_{I_{n,k}} f(t)\,\mathrm{d}t, \qquad n \in \mathbb{N}, \ \ x \geq 0, \tag{3}$$

where $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $\alpha \geq 1$, and $J_{n,k}(x) = \sum\limits_{j=k}^{n} p_{n,j}(x)$ is the basis function.

Throughout the paper let

$$W_{n,\alpha}(x,t) = na_n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \chi_{n,k}(t),$$

where $\chi_{n,k}$ is the characteristic function of the interval $[k/na_n, (k+1)/na_n]$ with respect to $I \equiv [0,\infty)$. Thus with this definition it is obvious that

$$L_{n,\alpha}(f,x) = \int_0^{\infty} f(t) W_{n,\alpha}(x,t)\,\mathrm{d}t.$$

A g r a t i n i [1] obtained some approximation properties and the rate of convergence for the operators L_n but [1] does not explicitly contains the sign term which is important part in the proof of the rate of convergence. In the present paper we estimate the rate of convergence for the Bézier variant of Balazs-Kantorovich operators. For special value our main theorem provides the complete estimate on the rate of convergence for the ordinary Balazs-Kantorovich operators.

2. Auxiliary results

Lemma 1. *For $x \in (0,\infty)$, we have*

$$p_{n,k}(x) \leq \frac{1+a_nx}{\sqrt{2ena_nx}}.$$

Proof. Following [11], the optimum bound for Bernstein basis function is given by

$$\binom{n}{k} t^k(1-t)^{n-k} \leq \frac{1}{\sqrt{2ent(1-t)}}.$$

Substituting $t = \frac{a_nx}{1+a_nx}$ in the above inequality, we get

$$\binom{n}{k} \frac{(a_nx)^k}{(1+a_nx)^n} \leq \frac{1+a_nx}{\sqrt{2ena_nx}}.$$

□

Lemma 2. *For $x \in (0,\infty)$, we have*

$$\left| \sum_{k>na_nx/(a+a_nx)} p_{n,k}(x) - \frac{1}{2}\right| \leq \frac{[1+(a_nx)^2+0.5(1+a_nx)^2]}{(1+a_nx)\left[1+\sqrt{na_nx}\,\right]}.$$

Proof. Following [12, Lemma 2], for $t \in [0,1]$ we have

$$\left| \sum_{k>nt} \binom{n}{k} t^k(1-t)^{n-k} - \frac{1}{2}\right| < \frac{0.8(2t^2-2t+1)+1/2}{1+\sqrt{nt(1-t)}}.$$

Substituting $t = \frac{a_nx}{1+a_nx}$ in the above inequality, we get the required result. □

Lemma 3. ([1]) *For $e_i(t) = t^i$, $i = 0,1,2,\ldots,$ and for all $x \geq 0$, we have*

$$L_n(e_0,x) = 1, \qquad L_n(e_1,x) = \frac{x}{1+a_nx} + \frac{1}{2na_n} \tag{4}$$

and

$$L_n(e_2,x) = \frac{n-1}{n}\frac{x^2}{(1+a_nx)^2} + \frac{2x}{na_n(1+a_nx)} + \frac{1}{3n^2a_n^2}. \tag{5}$$

$$\begin{aligned} L_n((e_1-xe_0)^2,x) &= \frac{1}{3n^2a_n^2} + \frac{na_n^3x^4 - a_n^2x^3 - a_nx^2 + x}{na_n(1+a_nx)^2} \\ &\leq \frac{1+na_nx+n^2a_n^4x^4}{n^2a_n^2} = O((na_n)^{-1}) \end{aligned} \tag{6}$$

Proof. Equations (4) and (5) were obtained in [1]. By a direct computation from (4) and (5), we obtain

$$\begin{aligned} L_n((e_1 - xe_0)^2, x) &= \frac{1}{3n^2a_n^2} + \frac{na_n^3x^4 - a_n^2x^3 - a_nx^2 + x}{na_n(1+a_nx)^2} \\ &\le \frac{1}{n^2a_n^2} + \frac{na_n^3x^4 + x}{na_n} = \frac{1 + na_nx + n^2a_n^4x^4}{n^2a_n^2}. \end{aligned}$$

From the page 349 of the paper or the reference [3], we learn that

$$a_n = n^{\beta-1}, \qquad -1 < \beta - 1 \le -1/3.$$

Thus

$$\frac{1 + na_nx + n^2a_n^4x^4}{n^2a_n^2} = O((na_n)^{-1}).$$

Lemma 3 is proved. □

Lemma 4. *Let $x \in (0,\infty)$, then for sufficiently large n, we have*

(i) $\beta_{n,\alpha}(x,y) = \int_0^y W_{n,\alpha}(x,t)\,\mathrm{d}t \le \frac{\alpha}{(x-y)^2}\left(\frac{1+na_nx+n^2a_n^4x^4}{n^2a_n^2}\right)$, $0 \le y < x$,

and

(ii) $1 - \beta_{n,\alpha}(x,z) = \int_z^\infty W_{n,\alpha}(x,t)\,\mathrm{d}t \le \frac{\alpha}{(z-x)^2}\left(\frac{1+na_nx+n^2a_n^4x^4}{n^2a_n^2}\right)$, $x < z < \infty$.

Proof. We first prove (i). By (6), there holds

$$\begin{aligned} \int_0^y W_{n,\alpha}(x,t)\,\mathrm{d}t &\le \int_0^y W_{n,\alpha}(x,t)\frac{(x-t)^2}{(x-y)^2}\,\mathrm{d}t \le \alpha(x-y)^{-2}L_n((t-x)^2,x) \\ &\le \frac{\alpha}{(x-y)^2}\left(\frac{1+na_nx+n^2a_n^4x^4}{n^2a_n^2}\right), \qquad 0 \le y < x, \end{aligned}$$

where we have applied Lemma 3. The proof of (ii) is similar. □

3. Rate of convergence

Theorem. *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$ and let $V_a^b(g_x)$ be the total variation of g_x on $[a,b]$. If $\alpha \geq 1$, $x \in (0,\infty)$, $r > 1$ be given and $f(t) = O(t^r)$, $t \to \infty$, then for n sufficiently large*

$$\begin{aligned}
&\left| L_{n,\alpha}(f,x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right| \\
&\leq \alpha 2^\alpha \left[\frac{1 + (a_n x)^2 + 0.5(a_n x + 1)^2}{(1 + a_n x)\big[1 + \sqrt{n, a_n x}\,\big]} \right] |f(x+) - f(x-)| \\
&\quad + 2\alpha \left(\frac{1 + n a_n x + n a_n^2 x^2 + n^2 a_n^4 x^4}{n^2 a_n^2 x^2} \right) \sum_{k=1}^{n} V_{x - x/\sqrt{k}}^{x + x/\sqrt{k}}(g_x) + O((n a_n)^{-r}),
\end{aligned} \tag{7}$$

where

$$g_x(t) = \begin{cases} f(t) - f(x-), & 0 \leq t < x, \\ 0, & t = x, \\ f(t) - f(x+), & x < t < \infty. \end{cases}$$

P r o o f. Making use of identity for all n, we have

$$\begin{aligned}
f(t) = {} & \frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) + g_x(t) + \frac{f(x+) - f(x-)}{2^\alpha} \operatorname{sign}_x(t) \\
& + \delta_x(t) \left[f(x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right]
\end{aligned}$$

where

$$\operatorname{sign}_x(t) = \begin{cases} 2^\alpha - 1, & t > x, \\ 0, & t = x, \\ -1, & t < x, \end{cases} \qquad \text{and} \qquad \delta_x(t) = \begin{cases} l, & x = t, \\ 0, & x \neq t. \end{cases}$$

It follows that

$$\begin{aligned}
&\left| L_{n,\alpha}(f,x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right| \\
&\leq |L_{n,\alpha}(g_x, x)| + \left| \frac{f(x+) - f(x-)}{2^\alpha} L_{n,\alpha}(\operatorname{sign}_x(t), x) \right. \\
&\quad \left. + \left[f(x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] L_{n,\alpha}(\delta_x, x) \right|.
\end{aligned} \tag{8}$$

For these operators it is obvious that $L_{n,\alpha}(\delta_x, x) = 0$. We first estimate $L_{n,\alpha}(\mathrm{sign}_x(t), x)$. Choosing k' such that $x \in [k'/na_n, (k'+1)/na_n]$, then

$$\begin{aligned} L_{n,\alpha}(\mathrm{sign}_x(t), x) &= \sum_{k=0}^{k'-1}(-1)Q_{n,k}^{(\alpha)}(x) + \left(\frac{Q_{n,k'}^{(\alpha)}(x)}{\int\limits_{I_{n,k'}} \mathrm{d}t}\right) \int\limits_{k'/na_n}^{x} (-1)\,\mathrm{d}t \\ &\quad + \left(\frac{Q_{n,k'}^{(\alpha)}(x)}{\int\limits_{I_{n,k'}} \mathrm{d}t}\right) \int\limits_{x}^{(k'+1)/na_n} (2^\alpha - 1)\,\mathrm{d}t + \sum_{k=k'+1}^{n}(2^\alpha - 1)Q_{n,k}^{(\alpha)}(x) \\ &= \sum_{k=k'+1}^{n} 2^\alpha Q_{n,k}^{(\alpha)}(x) + \left(\frac{Q_{n,k'}^{(\alpha)}(x)}{\int\limits_{I_{n,k'}} \mathrm{d}t}\right) \int\limits_{x}^{(k'+1)/na_n} 2^\alpha\,\mathrm{d}t - 1. \end{aligned}$$

Note that $0 \le \left(\frac{Q_{n,k'}^{(\alpha)}(x)}{\int\limits_{I_{n,k'}} \mathrm{d}t}\right) \int\limits_{x}^{(k'+1)/na_n} 2^\alpha\,\mathrm{d}t \le 2^\alpha Q_{n,k'}^{(\alpha)}(x)$, we conclude

$$\begin{aligned} |L_{n,\alpha}(\mathrm{sign}_x(t), x)| &\le \left| \sum_{k=k'+1}^{n} 2^\alpha Q_{n,k}^{(\alpha}(x) - 1 \right| + 2^\alpha Q_{n,k'}^{(\alpha)}(x) \\ &= |2^\alpha J_{n,k'+1}^\alpha(x) - 1| + 2^\alpha Q_{n,k'}^{(\alpha)}(x). \end{aligned}$$

Applying the inequality $|a^\alpha - b^\alpha| \le \alpha|a-b|$ for $0 \le a$, $b \le 1$ and $\alpha \ge 1$ yields

$$|2^\alpha J_{n,k'+1}^\alpha(x) - 1| \le \alpha 2^\alpha \left| J_{n,k'+1}(x) - \frac{1}{2} \right| \le \alpha 2^\alpha \left| \sum_{k > na_n x/(1+a_n x)} p_{n,k}(x) - \frac{1}{2} \right|.$$

Therefore by Lemma 1 and Lemma 2, we get

$$|L_{n,\alpha}(\mathrm{sign}_x(t), x)| \le \alpha 2^\alpha \left[\frac{[1 + (a_n x)^2 + 0.5(1 + a_n x)^2]}{(1 + a_n x)[1 + \sqrt{na_n x}]} + \frac{1 + a_n x}{\sqrt{2ena_n x}} \right], \tag{9}$$

Next we estimate $L_{n,\alpha}(g_x, x)$ as follows:

$$\begin{aligned} &L_{n,\alpha}(g_x, x) \\ &= \int\limits_0^\infty g_x(t) W_{n,\alpha}(x,t)\,\mathrm{d}t \\ &= \left(\int\limits_0^{x - x/\sqrt{n}} + \int\limits_{x - x/\sqrt{n}}^{x + x/\sqrt{n}} + \int\limits_{x + x/\sqrt{n}}^{\infty} \right) W_{n,\alpha}(x,t) g_x(t)\,\mathrm{d}t = E_1 + E_2 + E_3, \quad \text{say.} \end{aligned} \tag{10}$$

We start with E_2. For $t \in \left[x - x/\sqrt{n}, x + x/\sqrt{n}\,\right]$, we have

$$|g_x(t)| \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \leq \frac{1}{n}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)$$

and thus

$$|E_2| \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \leq \frac{1}{n}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x). \tag{11}$$

Next we estimate E_1. Setting $y = x - x/\sqrt{n}$ and integrating by parts, we have

$$E_1 = \int_0^y g_x(t)\,\mathrm{d}_t(\beta_{n,\alpha}(x,t)) = g_x(y)\beta_{n,\alpha}(x,y) - \int_0^y \beta_{n,\alpha}(x,t)\,\mathrm{d}_t(g_x(t)).$$

Since $|g_x(y)| \leq V_y^t(g_x)$, we conclude

$$|E_1| \leq V_y^x(g_x)\beta_{n,\alpha}(x,y) + \int_0^y \beta_{n,\alpha}(x,t)\,\mathrm{d}_t(-V_t^x(g_x)).$$

Also $y = x - x/\sqrt{n} \leq x$, therefore by Lemma 4, we have for n sufficiently large

$$\begin{aligned}|E_1| &\leq \frac{\alpha}{(x-y)^2}\left(\frac{1 + na_nx + n^2a_n^4x^4}{n^2a_n^2}\right)V_y^x(g_x)\\ &\quad + \alpha\left(\frac{1 + na_nx + n^2a_n^4x^4}{n^2a_n^2}\right)\int_0^y \frac{1}{(x-t)^2}\,\mathrm{d}_t(-V_t^x(g_x)).\end{aligned}$$

Integrating by parts the last integral, we obtain

$$|E_1| \leq \alpha\left(\frac{1 + na_nx + n^2a_n^4x^4}{n^2a_n^2}\right)\left(x^{-2}V_0^x(g_x) + 2\int_0^y \frac{V_t^x(g_x)\,\mathrm{d}t}{(x-t)^3}\right).$$

Replacing the variable y in the last integral by $x - x/\sqrt{n}$, we get

$$\begin{aligned}\int_0^{x-x/\sqrt{n}} V_t^x(g_x)(x-t)^{-3}\,\mathrm{d}t &= \frac{1}{2x^2}\int_1^n V_{x-x/\sqrt{t}}^x(g_x)\,\mathrm{d}t\\ &= \frac{1}{2x^2}\sum_{k=1}^{n-1}\int_k^{k+1} V_{x-x/\sqrt{t}}^x(g_x)\,\mathrm{d}t\\ &\leq \frac{1}{2x^2}\sum_{k=1}^{n-1} V_{x-x/\sqrt{k}}^x(g_x)\\ &\leq \frac{1}{2x^2}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^x(g_x).\end{aligned}$$

Hence

$$|E_1| \le \frac{2\alpha}{x^2}\left(\frac{1+na_nx+n^2a_n^4x^4}{n^2a_n^2}\right)\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x). \tag{12}$$

Finally we estimate E_3, defining

$$\hat{g}_x(t)=\begin{cases} g_x(t) & \text{if } 0\le t\le 2x,\\ g_x(2x) & \text{if } 2x<t<\infty.\end{cases}$$

We split E_3 as follows:

$$E_3=E_{31}+E_{32},$$

where

$$E_{31}=\int_{x+x/\sqrt{n}}^{\infty} W_{n,\alpha}(x,t)\hat{g}_x(t)\,\mathrm{d}t \quad \text{and} \quad E_{32}=\int_{2x}^{\infty} W_{n,\alpha}(x,t)[g_x(t)-g_x(2x)]\,\mathrm{d}t$$

with $y=x+x/\sqrt{n}$ the first integral can be written in the form

$$E_{31}=\lim_{R\to+\infty}\left\{g_x(y)[1-\beta_{n,\alpha}(x,y)]+\hat{g}_x(R)[\beta_{n,\alpha}(x,R)-1]+\int_y^R[1-\beta_{n,\alpha}(x,t)]\,\mathrm{d}_t\hat{g}_x(t)\right\}$$

By Lemma 4, we conclude for n sufficiently large

$$\begin{aligned}|E_{31}| &\le \alpha\left(\frac{1+na_nx+n^2a_n^4x^4}{n^2a_n^2}\right)\cdot\\ &\qquad\cdot\lim_{R\to+\infty}\left\{\frac{V_x^y(g_x)}{(y-x)^2}+\frac{\hat{g}_x(R)}{(R-x)^2}+\int_y^R\frac{1}{(t-x)^2}\,\mathrm{d}_t(V_x^t(\hat{g}_x))\right\}\\ &=\alpha\left(\frac{1+na_nx+n^2a_n^4x^4}{n^2a_n^2}\right)\left\{\frac{V_x^y(g_x)}{(y-x)^2}+\int_y^{2x}\frac{1}{(t-x)^2}\,\mathrm{d}_t(V_x^t(g_x))\right\}.\end{aligned}$$

Using a similar method as above, we get

$$\int_y^{2x}\frac{1}{(t-x)^2}\,\mathrm{d}_t(V_x^t(g_x))\le x^{-2}V_x^{2x}(g_x)-\frac{V_x^y(g_x)}{(y-x)^2}+x^{-2}\sum_{k=1}^{n-1}V_x^{x+x/\sqrt{k}}(g_x),$$

which implies the estimate

$$|E_{31}|\le\frac{2\alpha}{x^2}\left(\frac{1+na_nx+n^2a_n^4x^4}{n^2a_n^2}\right)\sum_{k=1}^{n}V_x^{x+x/\sqrt{k}}(g_x). \tag{13}$$

Lastly we estimate E_{32}. By assumption there exists an integer $r > 1$ such that $f(t) = O(t^{2r})$, $t \to \infty$. Thus for certain constant $M > 0$ depending only on f, x, r, we have

$$\begin{aligned}|E_{32}| &\le M_1 n a_n \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{2x}^{\infty} \chi_{n,k}(t) t^{2r}\,\mathrm{d}t \\ &\le M_1 n a_n \alpha \sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} \chi_{n,k}(t) t^{2r}\,\mathrm{d}t.\end{aligned}$$

By Lemma 3, we have

$$|E_{32}| \le \alpha 2^r M \cdot L_n((t-x)^{2r}, x) = O((n a_n)^{-r}), \qquad n \to \infty. \tag{14}$$

Finally collecting the estimates of (8)–(14), we get (7). This completes the proof of the theorem. □

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Received 24. 6. 2005
Revised 9. 8. 2005

* *School of Applied Sciences*
Netaji Subhas Institute of Technology
Sector 3 Dwarka
New Delhi 110075
INDIA
E-mail: vijaygupta2001@hotmail.com

** *Department of Mathematics*
Xiamen University
Xiamen, 361005
P. R. CHINA
E-mail: xmzeng@jingxian.xmu.edu.cn