

A NON-ASSOCIATIVE GENERALIZATION OF MV-ALGEBRAS

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ABSTRACT. We consider a non-associative generalization of MV-algebras. The underlying posets of our non-associative MV-algebras are not lattices, but they are related to so-called λ -lattices.

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1. Non-associative MV-algebras

As known, MV-algebras were introduced in the late-fifties by C. C. Chang as an algebraic semantics of the Łukasiewicz many-valued sentential logic (see [5], [6]). We recall the definition from [7] which is essentially due to P. Mangani [12]; Chang's original definition in [5] was a bit more complicated:

An *MV-algebra* is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities:

$$(MV1) \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z,$$

$$(MV2) \quad x \oplus y = y \oplus x,$$

$$(MV3) \quad x \oplus 0 = x,$$

$$(MV4) \quad \neg\neg x = x,$$

$$(MV5) \quad x \oplus \neg 0 = \neg 0 \text{ (the element } \neg 0 \text{ is denoted by } 1),$$

$$(MV6) \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

The prototypical example of an MV-algebra is the algebra $\Gamma(G, u) = ([0, u], \oplus, \neg, 0)$, where $(G, +, -, 0, \vee, \wedge)$ is an Abelian lattice-ordered group, $0 < u \in G$ and $[0, u] = \{x \in G : 0 \leq x \leq u\}$, and the operations \oplus and \neg are defined via $x \oplus y := (x + y) \wedge u$ and $\neg x := u - x$, respectively. D. Mundici

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proved in [13] (see also [7]) that every MV-algebra A is isomorphic to (up to isomorphism) unique MV-algebra $\Gamma(G, u)$.

Another well-known fact is that for any MV-algebra A , the relation \leq given by

$$x \leq y : \Longleftrightarrow \neg x \oplus y = 1 \quad (1)$$

is a lattice order on A with $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. Obviously, if $A = \Gamma(G, u)$, then \leq is the restriction of the group order to the interval $[0, u]$.

In the recent years, non-commutative generalizations of MV-algebras were considered by G. Georgescu and A. Iorgulescu [9] as *pseudo MV-algebras* and independently by J. Rachůnek [14] as *GMV-algebras*. Although the respective definitions are slightly different, the resultant non-commutative MV-algebras are equivalent; they are algebras with a binary operation \oplus and two unary operations \neg and \sim , which coincide whenever \oplus is commutative.

We have to remark that the name GMV-algebra appears e.g. in [2], [8] in a different sense. Here a *GMV-algebra* is a residuated lattice (in general non-commutative and unbounded) satisfying certain additional identities and bounded GMV-algebras correspond to pseudo MV-algebras.

In the paper we generalize MV-algebras omitting associativity of \oplus , but in such a way that the relation defined by (1) is still a partial order. However, without the identity (MV1) we would not be able to show that \leq is transitive. Therefore we replace (MV1) by another two axioms which hold in all MV-algebras and which force \leq to be transitive.

DEFINITION 1. An algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ is called a *non-associative MV-algebra* or an *NMV-algebra* for short if it satisfies the identities (MV2)–(MV6) and

$$\neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = 1, \quad (\text{WA})$$

$$\neg x \oplus (x \oplus y) = 1. \quad (\text{H})$$

If we put $y = 0$ in (H), we have $\neg x \oplus x = 1$, so \leq is reflexive. It follows easily from (MV6) that it is antisymmetric. Finally, if $\neg x \oplus y = 1$ and $\neg y \oplus z = 1$, then (WA) entails $\neg x \oplus z = 1$, thus \leq is also transitive. Altogether, \leq is a partial order as desired. In addition, using (MV6) and (WA) with $z = 0$ it can be seen that $\neg(\neg x \oplus y) \oplus y$ is a common upper bound of x, y , but in contrast to MV-algebras, it need not be their supremum.

* * *

As usual, given a partially ordered set (P, \leq) , we write $L(x, y) = \{a \in P : a \leq x \text{ and } a \leq y\}$ and $U(x, y) = \{a \in P : a \geq x \text{ and } a \geq y\}$ for any $x, y \in P$. If

$U(x, y) \neq \emptyset$ for all $x, y \in P$, then (P, \leq) is called an *upwards directed set*, and (P, \leq) is called a *directed set* provided both $L(x, y)$ and $U(x, y)$ are non-empty.

V. Snášel in his unpublished thesis [15] (see also [16]) introduced the concept of a λ -lattice as a generalization of lattices:

An algebra (L, \cup, \cap) of type $(2, 2)$ is called a λ -lattice if it satisfies the identities

$$(L1) \quad x \cap x = x, \quad x \cup x = x,$$

$$(L2) \quad x \cap y = y \cap x, \quad x \cup y = y \cup x,$$

$$(L3) \quad x \cap ((x \cap y) \cap z) = (x \cap y) \cap z, \quad x \cup ((x \cup y) \cup z) = (x \cup y) \cup z,$$

$$(L4) \quad x \cap (x \cup y) = x, \quad x \cup (x \cap y) = x.$$

If we put $x \leq y$ iff $x \cap y = x$, or equivalently, $x \leq y$ iff $x \cup y = y$, then (L, \leq) is a directed set and $x \cap y \in L(x, y)$ and $x \cup y \in U(x, y)$.

We can analogously introduce λ -semilattices (cf. [11]): An *upper λ -semilattice* is an algebra (S, \cup) of type (2) satisfying the identities

$$(S1) \quad x \cup x = x,$$

$$(S2) \quad x \cup y = y \cup x,$$

$$(S3) \quad x \cup ((x \cup y) \cup z) = (x \cup y) \cup z.$$

If we define $x \leq y$ iff $x \cup y = y$, then the relation \leq is a partial order on S such that $x \cup y \in U(x, y)$, so (S, \leq) is an upwards directed set.

The notion of a *lower λ -semilattice* can be defined dually, but we restrict ourselves to upper ones only, hence whenever we refer to a λ -semilattice we mean an upper λ -semilattice.

We notice that our λ -semilattices are equivalent to *commutative directoids* which were considered by J. Ježek and R. Quackenbush [10].

THEOREM 2. *Let $(A, \oplus, \neg, 0)$ be an NMV-algebra. Then upon defining $x \cup y := \neg(\neg x \oplus y) \oplus y$ and $x \cap y := \neg(\neg x \cup \neg y)$, (A, \cup, \cap) is a bounded λ -lattice with 0 at the bottom and 1 at the top.*

Proof. Putting $y = 0$ in (H) we obtain $\neg x \oplus x = 1$, so $x \cup x = \neg(\neg x \oplus x) \oplus x = \neg 1 \oplus x = x$. Clearly, $x \cup y = y \cup x$ by (MV6). Further, by (WA) we have $\neg x \oplus ((x \cup y) \cup z) = 1$ whence $x \cup ((x \cup y) \cup z) = \neg(\neg x \oplus ((x \cup y) \cup z)) \oplus ((x \cup y) \cup z) = (x \cup y) \cup z$. It is plain that $x \cup 0 = x$ and $x \cup 1 = 1$ for every $x \in A$. Thus (A, \cup) is a bounded λ -semilattice.

Further, observe that $x \oplus \neg(x \cap y) = x \oplus (\neg x \cup \neg y) = x \oplus (\neg(x \oplus \neg y) \oplus \neg y) = 1$ when we put $z = 0$ in (WA), whence it follows $x \cup (x \cap y) = \neg(\neg(x \cap y) \oplus x) \oplus x = x$. Using the definition of \cap and just proved properties of \cup it is straightforward to verify the remaining equations of (L1)–(L4). \square

2. λ -semilattices with involutions

A λ -semilattice with involutions is a λ -semilattice (S, \cup) with the greatest element 1, where every interval $[a, 1] \subseteq S$ (so-called *section*) has an involution f_a with $f_a(1) = a$. We write simply x^a for $f_a(x)$. Clearly, a λ -semilattice with involutions can be considered as a structure $(S, \cup, (^a)_{a \in S}, 1)$.

A λ -lattice with involutions is defined analogously as a system $(L, \cup, \cap, (^a)_{a \in L}, 1)$.

Let $(S, \cup, (^a)_{a \in L}, 1)$ be a λ -semilattice with involutions. In order to overcome the difficulties concerning the number of partial unary operations $^a: [a, 1] \rightarrow [a, 1]$, we define a new total binary operation \rightarrow on S via

$$x \rightarrow y := (x \cup y)^y. \quad (2)$$

LEMMA 3. *A λ -semilattice (S, \cup) with the top element 1 is a λ -semilattice with involutions if and only if there exists a binary operation \rightarrow on S that has the following properties, for all $x, y \in S$:*

- (a) $1 \rightarrow x = x$,
- (b) $x \cup y = (x \rightarrow y) \rightarrow y$,
- (c) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$.

In this case, $x^a = x \rightarrow a$ for $x \in [a, 1]$, $a \in S$.

Proof. Let S be a λ -semilattice with involutions and let \rightarrow be the operation given by (2). Then $1 \rightarrow x = (1 \cup x)^x = 1^x = x$, $(x \rightarrow y) \rightarrow y = ((x \cup y)^y \cup y)^y = (x \cup y)^{yy} = x \cup y$ and $((x \rightarrow y) \rightarrow y) \rightarrow y = (x \cup y) \rightarrow y = ((x \cup y) \cup y)^y = (x \cup y)^y = x \rightarrow y$. Obviously, $x^a = (x \cup a)^a = x \rightarrow a$ for every $x \in [a, 1]$.

Conversely, if \rightarrow satisfies (a), (b) and (c), then we define $f_a(x) = x^a := x \rightarrow a$ for $x \in [a, 1]$, $a \in S$. By (b) and (c), $(x \rightarrow a) \cup a = ((x \rightarrow a) \rightarrow a) \rightarrow a = x \rightarrow a$, i.e. $a \leq x \rightarrow a$ and $x^a \in [a, 1]$. Further, we have $x^{aa} = (x \rightarrow a) \rightarrow a = x \cup a = x$, so f_a is an involution on $[a, 1]$, and $1^a = 1 \rightarrow a = a$. Thus S is a λ -semilattice with involutions. Moreover, due to (c) and (b) we obtain $x \rightarrow y = ((x \rightarrow y) \rightarrow y) \rightarrow y = (x \cup y) \rightarrow y = (x \cup y)^y$. \square

Consequently, λ -(semi)lattices can be treated as algebras $(S, \cup, \rightarrow, 1)$ of type $(2, 2, 0)$ or $(L, \cup, \cap, \rightarrow, 1)$ of type $(2, 2, 2, 0)$, respectively.

Remark 4. Note that the partial order \leq can be retrieved via $x \leq y$ iff $x \rightarrow y = 1$, however, the operation \rightarrow does not determine \cup . To be more precise, if \rightarrow is a total binary operation satisfying all the equations in the language $\{\rightarrow, 1\}$ which are derivable in λ -semilattices with involutions, in particular, $1 \rightarrow x = x$ and $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, then $(x \rightarrow y) \rightarrow y$ need not be equal to $x \cup y$.

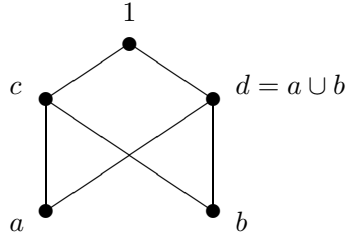


FIGURE 1

Example 5. Let (S, \cup) be a λ -semilattice as shown in Fig. 1. Let the involutions f_a and f_b in the non-trivial sections $[a, 1]$ and $[b, 1]$, respectively, be defined as follows: $f_a(c) = c$, $f_a(d) = d$ and $f_b(c) = d$, $f_b(d) = c$. The operation \rightarrow is then given by Table 1. However, the operation \rightsquigarrow given by Table 2 also fulfils the equations $1 \rightsquigarrow x = x$ and $(x \rightsquigarrow y) \rightsquigarrow y = (y \rightsquigarrow x) \rightsquigarrow x$, but $(a \rightsquigarrow b) \rightsquigarrow b = c \neq d = a \cup b$. Observe that \rightsquigarrow is obtained by (2) when $a \cup b$ is defined as c .

\rightarrow	a	b	c	d	1
a	1	c	1	1	1
b	d	1	1	1	1
c	c	d	1	d	1
d	d	c	c	1	1
1	a	b	c	d	1

TABLE 1

\rightsquigarrow	a	b	c	d	1
a	1	d	1	1	1
b	c	1	1	1	1
c	c	d	1	d	1
d	d	c	c	1	1
1	a	b	c	d	1

TABLE 2

LEMMA 6. *Let $(S, \cup, \rightarrow, 1)$ be a λ -semilattice with involutions. Then for all $x, y \in S$,*

- (i) $x \rightarrow 1 = 1$, $x \rightarrow x = 1$,
- (ii) $y \leq x \rightarrow y$.

Proof.

(i) We have $x \rightarrow 1 = (x \cup 1)^1 = 1^1 = 1$ and $x \rightarrow x = (x \cup x)^x = x^x = 1$.

(ii) This is obvious since $x \rightarrow y = (x \cup y)^y \geq y$. \square

THEOREM 7. *The variety of all λ -lattices with involutions is regular and arithmetical.*

Proof. Let \mathcal{V} be the variety of λ -lattices with involutions.

\mathcal{V} is regular: Let

$$t_1(x, y, z) = ((x \rightarrow y) \cap (y \rightarrow x)) \cap z,$$

$$t_2(x, y, z) = ((x \rightarrow y) \rightarrow z) \cup ((y \rightarrow x) \rightarrow z).$$

We show that $t_1(x, y, z) = t_2(x, y, z) = z$ iff $x = y$.

Obviously, $t_1(x, x, z) = z$ and $t_2(x, x, z) = z$. Conversely, let $t_1(x, y, z) = t_2(x, y, z) = z$. Then $z \leq x \rightarrow y, y \rightarrow x$ and $z \geq (x \rightarrow y) \rightarrow z, (y \rightarrow x) \rightarrow z$. But by Lemma 6(ii) we have $(x \rightarrow y) \rightarrow z, (y \rightarrow x) \rightarrow z \geq z$, so that $(x \rightarrow y) \rightarrow z = z = (y \rightarrow x) \rightarrow z$, whence $x \rightarrow y = (x \rightarrow y) \cup z = ((x \rightarrow y) \rightarrow z) \rightarrow z = z \rightarrow z = 1$, so $x \leq y$. Similarly $y \leq x$, and hence $x = y$.

\mathcal{V} is arithmetical: Let

$$m(x, y, z) = (((x \rightarrow y) \rightarrow z) \cap ((z \rightarrow y) \rightarrow x)) \cap (x \cup z).$$

We prove that $m(x, y, y) = m(x, y, x) = m(y, y, x) = x$.

We have $m(x, y, y) = (((x \rightarrow y) \rightarrow y) \cap ((y \rightarrow y) \rightarrow x)) \cap (x \cup y) = ((x \cup y) \cap x) \cap (x \cup y) = x$, $m(x, y, x) = (((x \rightarrow y) \rightarrow x) \cap ((x \rightarrow y) \rightarrow x)) \cap (x \cup x) = ((x \rightarrow y) \rightarrow x) \cap x = x$ since $(x \rightarrow y) \rightarrow x \geq x$ by Lemma 6, and $m(y, y, x) = (((y \rightarrow y) \rightarrow x) \cap ((x \rightarrow y) \rightarrow y)) \cap (y \cup x) = (x \cap (x \cup y)) \cap (y \cup x) = x$. \square

* * *

There is a one-to-one correspondence between NMV-algebras and bounded λ -(semi)lattices with involutions that satisfy a simple additional identity:

THEOREM 8.

(i) Let $(A, \oplus, \neg, 0)$ be an NMV-algebra. Define $x \cup y := \neg(\neg x \oplus y) \oplus y$ and $x \rightarrow y := \neg x \oplus y$. Then $\phi(A) = (A, \cup, \rightarrow, 0, 1)$ is a bounded λ -semilattice with involutions that satisfies the identity

$$x \rightarrow (y \rightarrow 0) = y \rightarrow (x \rightarrow 0). \quad (\text{WE})$$

(ii) Let $(S, \cup, \rightarrow, 0, 1)$ be a bounded λ -semilattice with involutions satisfying (WE). If we define $x \oplus y := (x \rightarrow 0) \rightarrow y$ and $\neg x := x \rightarrow 0$, then $\psi(S) = (S, \oplus, \neg, 0)$ is an NMV-algebra.

(iii) For any NMV-algebra A and any bounded λ -semilattice with involutions S satisfying (WE), $\psi(\phi(A)) = A$ and $\phi(\psi(S)) = S$.

Proof.

(i) We already know from Theorem 2 that (A, \cup) is a bounded λ -semilattice. We show that the conditions (a), (b) and (c) of Lemma 3 are satisfied. It is obvious that $1 \rightarrow x = \neg 1 \oplus x = x$ and $x \cup y = \neg(\neg x \oplus y) \oplus y = (x \rightarrow y) \rightarrow y$. Now, due to the axiom (H), we have $y \leq y \oplus \neg x = \neg x \oplus y$ whence

$$((x \rightarrow y) \rightarrow y) \rightarrow y = \neg(\neg(\neg x \oplus y) \oplus y) \oplus y = (\neg x \oplus y) \cup y = \neg x \oplus y = x \rightarrow y$$

verifying (c). So by Lemma 3, $\phi(A) = (A, \cup, \rightarrow, 0, 1)$ is a bounded λ -semilattice with involutions. Finally, $\phi(A)$ fulfils (WE) since

$$\begin{aligned} x \rightarrow (y \rightarrow 0) &= \neg x \oplus (\neg y \oplus 0) = \neg x \oplus \neg y = \neg y \oplus \neg x \\ &= \neg y \oplus (\neg x \oplus 0) = y \rightarrow (x \rightarrow 0). \end{aligned}$$

(ii) Let $(S, \cup, \rightarrow, 0, 1)$ be a bounded λ -semilattice with involutions that satisfies (WE). It is worth noticing that $\neg x \oplus y = ((x \rightarrow 0) \rightarrow 0) \rightarrow y = (x \cup 0) \rightarrow y = x \rightarrow y$.

$$\begin{aligned} \text{(MV2): } x \oplus y &= (x \rightarrow 0) \rightarrow y = (x \rightarrow 0) \rightarrow ((y \rightarrow 0) \rightarrow 0) = (y \rightarrow 0) \rightarrow \\ &((x \rightarrow 0) \rightarrow 0) = (y \rightarrow 0) \rightarrow x = y \oplus x \text{ by (WE).} \end{aligned}$$

$$\text{(MV3): } x \oplus 0 = (x \rightarrow 0) \rightarrow 0 = x.$$

$$\text{(MV4): } \neg \neg x = (x \rightarrow 0) \rightarrow 0 = x.$$

$$\text{(MV5): } x \oplus 1 = (x \rightarrow 0) \rightarrow 1 = 1.$$

$$\begin{aligned} \text{(MV6): } \neg(\neg x \oplus y) \oplus y &= (x \rightarrow y) \rightarrow y = x \cup y = (y \rightarrow x) \rightarrow x = \\ &\neg(\neg y \oplus x) \oplus x. \end{aligned}$$

$$\begin{aligned} \text{(WA): } \neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) &= \\ x \rightarrow (((x \rightarrow y) \rightarrow y) \rightarrow z) \rightarrow z &= x \rightarrow ((x \cup y) \cup z) = 1 \\ \text{since } x \leq (x \cup y) \cup z &\text{ by (S3).} \end{aligned}$$

$$\begin{aligned} \text{(H): } \neg x \oplus (x \oplus y) &= x \rightarrow ((x \rightarrow 0) \rightarrow y) = 1 \text{ since } x \leq (y \rightarrow 0) \rightarrow x = \\ (x \rightarrow 0) \rightarrow y &\text{ by Lemma 6 (ii).} \end{aligned}$$

(iii) Let $(A, \oplus, \neg, 0)$ be an NMV-algebra. Define $\phi(A) = (A, \cup, \rightarrow, 0, 1)$ and $\psi(\phi(A)) = (A, \oplus', \neg', 0)$. We have $x \oplus' y = (x \rightarrow 0) \rightarrow y = \neg(\neg x \oplus 0) \oplus y = x \oplus y$ and $\neg' x = x \rightarrow 0 = \neg x \oplus 0 = \neg x$. Thus $\psi(\phi(A)) = A$.

Conversely, let $(S, \cup, \rightarrow, 0, 1)$ be a bounded λ -semilattice with involutions that fulfils (WE). Define $\psi(S) = (S, \oplus, \neg, 0)$ and $\phi(\psi(S)) = (S, \cup', \rightarrow', 0, 1')$. We have $x \cup' y = \neg(\neg x \oplus y) \oplus y = (x \rightarrow y) \rightarrow y = x \cup y$, $x \rightarrow' y = \neg x \oplus y = x \rightarrow y$ and $1' = \neg 0 = 0 \rightarrow 0 = 1$, so that $\phi(\psi(S)) = S$. \square

COROLLARY 9. *Let $(S, \cup, \rightarrow, 0, 1)$ be a bounded λ -semilattice with involutions satisfying (WE). Then $(S, \cup, \cap, \rightarrow, 0, 1)$, where $x \cap y = ((x \rightarrow y) \rightarrow (x \rightarrow 0)) \rightarrow 0$, is a bounded λ -lattice with involutions.*

Proof. By Theorem 8(ii), $(S, \oplus, \neg, 0)$ is an NMV-algebra and by Theorem 2 we know that (S, \cup, \cap) is a bounded λ -lattice in which

$$\begin{aligned} x \cap y &= \neg(\neg x \cup \neg y) \\ &= (((y \rightarrow 0) \rightarrow (x \rightarrow 0)) \rightarrow (x \rightarrow 0)) \rightarrow 0 \\ &= ((x \rightarrow ((y \rightarrow 0) \rightarrow 0)) \rightarrow (x \rightarrow 0)) \rightarrow 0 \\ &= ((x \rightarrow y) \rightarrow (x \rightarrow 0)) \rightarrow 0. \end{aligned}$$

□

Remark 10. Though every NMV-algebra, as well as every bounded λ -semi-lattice with involutions satisfying (WE), is a λ -lattice, *Theorem 8 does not hold for λ -lattices*. The reason is that $x \cap y$ need not be the greatest lower bound of $\{x, y\}$, and consequently, the operation \cap defined in Corollary 9 is not the only possible one which makes $(S, \cup, \rightarrow, 0, 1)$ into a λ -lattice:

Example 11. Consider the λ -lattice (S, \cup, \cap_1) from Figure 2. Let the involutions f_0, f_a and f_b in the non-trivial sections be given as follows:

- $f_0(a) = d, f_0(b) = c, f_0(c) = b$ and $f_0(d) = a$,
- $f_a(c) = c$ and $f_a(d) = d$,
- $f_b(c) = d$ and $f_b(d) = c$.

The operation \rightarrow is given by Table 3, so that $(S, \cup, \cap_1, \rightarrow, 0, 1)$ is a bounded λ -lattice with involutions. A straightforward verification yields that \rightarrow obeys (WE), and hence $(S, \oplus, \neg, 0)$ is an NMV-algebra, where the operations \oplus and \neg are given by Table 4. Now, upon setting $x \cap y := \neg(\neg x \cup \neg y)$, (S, \cup, \cap) is a λ -lattice, but \cap does not agree with the initial \cap_1 . Indeed, we have $c \cap d = \neg(\neg c \cup \neg d) = \neg c = b \neq a = c \cap_1 d$. Therefore, the part (iii) of Theorem 8 does not work in the case of λ -lattices with involutions.

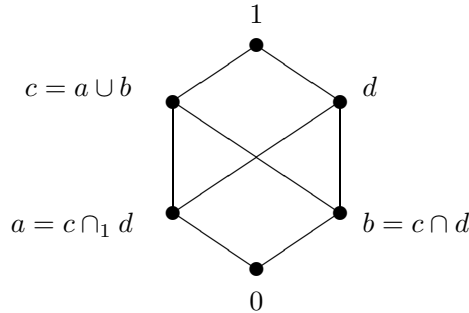


FIGURE 2

By Theorem 7 and Theorem 8 (i) we get

\rightarrow	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	d	c	c	1	1
1	0	a	b	c	d	1

TABLE 3

\oplus	0	a	b	c	d	1	\neg
0	0	a	b	c	d	1	1
a	a	d	c	c	1	1	d
b	b	c	d	1	d	1	c
c	c	c	1	1	1	1	b
d	d	1	d	1	1	1	a
1	1	1	1	1	1	1	0

TABLE 4

COROLLARY 12. *The variety of all NMV-algebras is regular and arithmetical.*

3. Implication reducts

There exist several equivalent counterparts of MV-algebras; for instance, MV-algebras are term equivalent to bounded weak implication algebras which were introduced in [4] as a generalization of J. C. Abbott's implication algebras (see [1]). We recall that an *implication algebra* is an algebra (A, \rightarrow) satisfying the equations

- (I1) $(x \rightarrow y) \rightarrow x = x$,
- (I2) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
- (I3) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

These axioms capture the basic properties of the implication in the classical propositional calculus. Starting from the implication in the Łukasiewicz logic, we obtain weak implication algebras: An algebra $(A, \rightarrow, 1)$ with a binary operation \rightarrow and a constant 1 is called a *weak implication algebra* if it fulfils (I2), (I3) and

- (I0) $x \rightarrow 1 = 1, 1 \rightarrow x = x$.

It is not hard to show that if $(A, \oplus, \neg, 0)$ is an MV-algebra then $(A, \rightarrow, 1)$ is a weak implication algebra, where $x \rightarrow y$ is defined as $\neg x \oplus y$.

Every weak implication algebra is a join-semilattice with 1 at the top with respect to the partial order given by $x \leq y$ iff $x \rightarrow y = 1$; $x \vee y = (x \rightarrow y) \rightarrow y$ is the supremum of any pair x, y .

A *bounded weak implication algebra* is a structure $(A, \rightarrow, 0, 1)$ such that $(A, \rightarrow, 1)$ is a weak implication algebra with the least element 0. Clearly, this is equivalent to the identity $0 \rightarrow x = 1$. Bounded weak implication algebras are known in the literature under the name *bounded commutative BCK-algebras* (see e.g. [7]).

This motivates us to describe the generalization of weak implication algebras which corresponds to our NMV-algebras.

DEFINITION 13. An *NMV-implication algebra* is an algebra $(A, \rightarrow, 0, 1)$ of type $(2, 0, 0)$ that satisfies the following identities:

- (NI1) $x \rightarrow 1 = 1, 1 \rightarrow x = x$ and $0 \rightarrow x = 1$,
- (NI2) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
- (NI3) $x \rightarrow (y \rightarrow 0) = y \rightarrow (x \rightarrow 0)$,
- (NI4) $x \rightarrow (((x \rightarrow y) \rightarrow y) \rightarrow z) \rightarrow z = 1$,
- (NI5) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$.

Comparing the above axioms with those of (weak) implication algebras, (NI1) includes (I0), (NI2) is precisely (I2) and (NI3) is another name for (WE) and rises as a weakening of (I3) by replacing z by 0. Furthermore, (NI4) captures (WA) and (NI5) is just (c) of Lemma 3.

Weak implication algebras are a particular case of NMV-implication ones. Indeed, any weak implication algebra fulfils (NI4) and (NI5) since in weak implication algebras we have $x \rightarrow (((x \rightarrow y) \rightarrow y) \rightarrow z) \rightarrow z = x \rightarrow (x \vee y \vee z) = 1$ and $((x \rightarrow y) \rightarrow y) \rightarrow y = (x \rightarrow y) \vee y = x \rightarrow y$.

Let us note that from (NI1) we can easily infer $x \rightarrow x = 1$.

THEOREM 14. Let $(A, \oplus, \neg, 0)$ be an NMV-algebra. If we define $x \rightarrow y := \neg x \oplus y$, then $(A, \rightarrow, 0, 1)$ is an NMV-implication algebra.

Conversely, if $(A, \rightarrow, 0, 1)$ is an NMV-implication algebra and if we put $x \oplus y := (x \rightarrow 0) \rightarrow y$ and $\neg x := x \rightarrow 0$, then $(A, \oplus, \neg, 0)$ is an NMV-algebra.

Proof. It is obvious at once that for each NMV-algebra $(A, \oplus, \neg, 0)$, the operation \rightarrow satisfies all the identities (NI1)–(NI5), so $(A, \rightarrow, 0, 1)$ is an NMV-implication algebra.

Conversely, assume that $(A, \rightarrow, 0, 1)$ is an NMV-implication algebra. First, we note that for any $x \in A$ we have $(x \rightarrow 0) \rightarrow 0 = (0 \rightarrow x) \rightarrow x = 1 \rightarrow x = x$ by (NI2) and (NI1), and hence $\neg x \oplus y = ((x \rightarrow 0) \rightarrow 0) \rightarrow y = x \rightarrow y$.

$$(MV2): x \oplus y = (x \rightarrow 0) \rightarrow y = (x \rightarrow 0) \rightarrow ((y \rightarrow 0) \rightarrow 0) = (y \rightarrow 0) \rightarrow ((x \rightarrow 0) \rightarrow 0) = (y \rightarrow 0) \rightarrow x = y \oplus x.$$

$$(MV3): x \oplus 0 = (x \rightarrow 0) \rightarrow 0 = x.$$

$$(MV4): \neg\neg x = (x \rightarrow 0) \rightarrow 0 = x.$$

$$(MV5): x \oplus 1 = (x \rightarrow 0) \rightarrow 1 = 1.$$

$$(MV6): \text{Using } \neg x \oplus y = x \rightarrow y \text{ we obtain } \neg(\neg x \oplus y) \oplus y = (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x = \neg(\neg y \oplus x) \oplus x \text{ by (NI2).}$$

□

$$(WA): \neg x \oplus (\neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus z) = x \rightarrow (((x \rightarrow y) \rightarrow y) \rightarrow z) \rightarrow z = 1 \text{ by (NI4).}$$

(H): We have $\neg x \oplus (x \oplus y) = x \rightarrow ((x \rightarrow 0) \rightarrow y) = x \rightarrow ((y \rightarrow 0) \rightarrow x)$, hence it is enough to show that $x \rightarrow (y \rightarrow x) = 1$ for all $x, y \in A$. This follows from (NI5), (NI2) and (NI1): $x \rightarrow (y \rightarrow x) = ((x \rightarrow (y \rightarrow x)) \rightarrow (y \rightarrow x)) \rightarrow (y \rightarrow x) = (((y \rightarrow x) \rightarrow x) \rightarrow x) \rightarrow (y \rightarrow x) = (y \rightarrow x) \rightarrow (y \rightarrow x) = 1$.

REFERENCES

- [1] ABBOTT, J. C.: *Semi-boolean algebra*, Mat. Vesnik **4** (1967), 177–198.
- [2] BAHLS, P.—COLE, J.—GALATOS, N.—JIPSEN, P.—TSINAKIS, C.: *Cancellative residuated lattices*, Algebra Universalis **50** (2003), 83–106.
- [3] CHAJDA, I.—HALAŠ, R.—KÜHR, J.: *Distributive lattices with sectionally antitone involutions*, Acta Sci. Math. (Szeged) **71** (2005), 19–33.
- [4] CHAJDA, I.—HALAŠ, R.—KÜHR, J.: *Implication in MV-algebras*, Algebra Universalis **52** (2004), 377–382.
- [5] CHANG, C. C.: *Algebraic analysis of many-valued logic*, Trans. Amer. Math. Soc. **88** (1958), 467–490.
- [6] CHANG, C. C.: *A new proof of the completeness of the Lukasiewicz axioms*, Trans. Amer. Math. Soc. **93** (1959), 74–80.
- [7] CIGNOLI, R. L. O.—D’OTTAVIANO, I. M. L.—MUNDICI, D.: *Algebraic Foundations of Many-valued Reasoning*, Kluwer Acad. Publ., Dordrecht-Boston-London, 2000.
- [8] GALATOS, N.—TSINAKIS, C.: *Generalized MV-algebras*, J. Algebra **283** (2005), 254–291.
- [9] GEORGESCU, G.—IORGULESCU, A.: *Pseudo-MV algebras*, Mult.-Valued Log. **6** (2001), 95–135.
- [10] JEŽEK, J.—QUACKENBUSH, R.: *Directoids: algebraic models of up-directed sets*, Algebra Universalis **27** (1990), 49–69.
- [11] KARÁSEK, J.: *Rotations of λ -lattices*, Math. Bohem. **121** (1996), 293–300.
- [12] MANGANI, P.: *Su certe algebre connesse con logiche a più valori*, Boll. Unione Mat. Ital. Ser. IV. **8** (1973), 68–78.
- [13] MUNDICI, D.: *Interpretation of AF C^* -algebras in Lukasiewicz sentential calculus*, J. Funct. Anal. **65** (1986), 15–63.
- [14] RACHŮNEK, J.: *A non-commutative generalization of MV-algebras*, Czechoslovak Math. J. **52** (2002), 255–273.
- [15] SNÁŠEL, V.: *λ -lattices*. Ph.D. Thesis, Masaryk Univ., Brno, 1991.

[16] SNÁŠEL, V.: λ -lattices, Math. Bohem. **122** (1997), 267–272.

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