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OSCILLATION AND STABILITY OF NONLINEAR DISCRETE MODELS EXHIBITING THE ALLEE EFFECT

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ABSTRACT. In this paper, we consider the discrete nonlinear delay population model exhibiting the Allee effect

$$x_{n+1} = x_n \exp\left(a + bx^p_{n-\tau} - cx^q_{n-\tau}\right), \tag{$*$}$$

where a, b and c are constants and p, q and τ are positive integers. First, we study the local stability of the equilibrium points. Next, we establish some oscillation results of nonlinear delay difference equations with positive and negative coefficients and apply them to investigate the oscillatory character of all positive solutions of equation $(*)$ about the positive steady state x^* and prove that every nonoscillatory solution tends to x^*.

1. Introduction

The so-called Allee effect refers to a population which has a maximal per capita growth rate at intermediate density. This occurs when the per capita growth rate increases as density increases and decreases after the density passes a certain value. This is certainly not the case in the delayed logistic equation,

$$N'(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right]$$

where the per capita growth rate is a decreasing function of the density. For an important case, aggregation and associated cooperative and social characteristics among members of species had been extensively studied in animal population by Allee [3], [4]. When the density of population becomes too large, the positive

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feedback effect of aggregation and cooperation may be dominated by density dependent stabilizing negative feedback effect due to interspecific competition due to excessive crowding and the ensuing shortage of resources. Gopalsamy and Ladas [6] studied the following delay Lotka-Volterra type single species population growth model,

$$x'(t) = x(t)[a + bx(t-\tau) - cx^2(t-\tau)] \tag{1}$$

where

$$a, b, c, \tau \in (0, \infty) \qquad \text{with} \quad c > b. \tag{2}$$

When $\tau = 0$, the per capita growth is $g(x) = a + bx - cx^2$. Then $\acute{g}(0) = b > 0$ and $g(x)$ achieves its maximum at $x = \frac{b}{2c}$, thus exhibiting the Allee effect. When $b < 0$, $g(x)$ is decreasing function and therefore there is no Allee effect. Elabbasy, Saker and Saif [9] proved that if (2) holds, and

$$(2ck^2 - bk)\tau > \frac{1}{e},$$

then every positive solution of equation (1) oscillates about the unique positive equilibrium point $k = \frac{1}{2c}[b + \sqrt{b^2 + 4ac}\,]$. They also proved that every nonoscillatory solution of equation (1) tends to k when t tends to infinity. They extended these results to the more general equation

$$x'(t) = x(t)[a + bx^p(t-\tau) - cx^q(t-\tau)], \tag{3}$$

where

$$a, b, c, \tau \in (0, \infty) \qquad \text{with} \quad c > b, \quad q > p$$

and some additional conditions on p and q. For a given differential equation, a difference equation approximation would be most acceptable if the solution of the difference equation is the same as the differential equation at the discrete points. But unless we can explicitly solve both equations, it is impossible to satisfy this requirement. Most of the time, it is desirable that a difference equation, when derived from a differential equation, preserves the dynamical features of the corresponding continuous time model such as equilibria, oscillation, their local and global stability characteristics and bifurcation behaviors. If such discrete models can be derived from continuous models, then the discrete time models can be used without loss of any functional similarity to the continuous-time models and it will preserve the considered realities; such discrete time models can be called "Dynamically consistent" with the continuous time models.

There is no unique way of deriving discrete time version of dynamical systems corresponding to continuous time formulations. One of the ways of deriving difference equations modeling the dynamic of populations with nonoverlapping generations is based on appropriate modifications of models with overlapping generations. In this approach, differential equations with piecewise constant arguments have been useful, see for example the paper by Liu and Gopalsamy [13]. Recently the method that has been established by Liu and Gopalsamy has been used by some authors to find the discrete analogy of some mathematical models. Using the technique that has been used in [13], we will derive the discrete analogy of equation (3). Thinking of differential equations with piecewise constant arguments, we can go on with the discrete analogy of equation (3). Let us assume that the average growth rate in (3) changes at regular intervals of time, then we can incorporate this aspect in (3) and obtain the following modified equation

$$\frac{1}{x(t)} x'(t) = a + bx^p[t-\tau] - cx^q[t-\tau],$$

where $[t]$ denotes the integer part of t, $t \in (0, \infty)$. Equation of this type is known as differential equation with piecewise with constant argument and this equation occupy a position midway between differential and difference equation. By a solution of this equation, we mean a function $x(t)$ which is defined for $t \in (0, \infty)$ and satisfy the properties:

(a) x is continuous on $[0, \infty)$.

(b) The derivative $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ exists at each point $t \in (0, \infty)$ with the possible exception of the points $t \in \{0, 1, 2, \dots\}$, where left side derivative exists.

(c) The equation (3) is satisfied on each interval $[n, n+1)$ with $n = 0, 1, 2, \dots$.

By integrating the last equation on any interval of the form $[n, n+1)$, $n = 0, 1, 2, \dots$ we obtain

$$x(t) = x(n) \exp\big([a + bx^p(n-\tau) - cx^q(n-\tau)]\,(t-n)\big).$$

Letting $t \mapsto n+1$, we obtain that

$$x(n+1) = x(n) \exp\left[a + bx^p(n-\tau) - cx^q(n-\tau)\right] \tag{4}$$

where

$$\begin{aligned} &a, b, c \in (0, \infty) \ \text{ with } \ c > b, \qquad\qquad \text{and} \\ &p,\ q,\ \tau \ \text{ are positive integers with } \ q > p, \end{aligned} \tag{5}$$

which is a discrete time analogy of (3). We note that the equilibrium points of (4) are the same as of system (3). So the derived discrete analogy preserves on the equilibria.

In recent years, the investigation of the theory of difference equations has assumed a greater importance as well deserved discipline. Many results in the theory of difference equations have been obtained as more or less natural discrete analogous of corresponding results of differential equations ([1], [2]). Nevertheless, the theory of difference equations is richer than the corresponding theory of differential equations. For example, a simple difference equation resulting from the first order differential equation exhibits the chaotic behavior which can only happen in higher order differential equations. We remark that in recent years oscillation and global attractivity of nonlinear delay discrete models have become a very popular subject. In fact, different models have been studied in [7], [11] and the references cited therein.

By a solution of equation (4), we mean a sequence $\{x_n\}$ which is defined for $n \geq -\tau$ and satisfies equation (4) for $n \geq 0$. Association with equation (4), we consider the initial condition

$$x(i) = a_i > 0 \qquad \text{for} \quad i = -\tau, \dots, 0. \tag{6}$$

The exponential form of equation (4) assures that the solution $\{x_n\}$ with respect to any initial condition (6) remains positive.

Now we mention some definitions that will be useful in our investigation of equation (4).

Definition 1.1. A solution $\{x_n\}$ of equation (4) is said to be *oscillatory about* x^* if the terms $x_n - x^*$ of the sequence $\{x_n - x^*\}$ are neither all eventually positive nor all eventually negative.

Consider the more general difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), \qquad n = 0, 1, \dots. \tag{7}$$

Definition 1.2. Let I be an interval of real positive numbers.

(i) The equilibrium point $\overline{x}$ of equation (7) is *locally stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all

$$x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I,$$

with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \dots + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \varepsilon \qquad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of equation (7) is *locally asymptotically stable* if $\overline{x}$ is locally stable solution of equation (7) and there exists $\gamma > 0$, such that for all

$$x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I,$$

with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of equation (7) is *global attractor* if for all

$$x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of equation (7) is *globally asymptotically stable* if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of equation (7).

(v) The equilibrium point $\overline{x}$ of equation (7) is *unstable* if $\overline{x}$ is not locally stable.

The linearized equation of equation (7) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, \dots, \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{8}$$

The following well known theorem, called the *Linearized Stability Theorem*, is very useful in determining the local stability character of the equilibrium solution $\overline{x}$, of equation (7).

Theorem A. *Assume that $p_i \in \mathbb{R}$ and $k \in \{1, 2, \dots\}$. Then*

$$\sum_{i=1}^{k} |p_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \qquad n = 0, 1, \dots .$$

Theorem B. *Assume $p \in \mathbb{R}$ and $k \in \mathbb{N}$. Then all roots of the equation $m^{k+1} - m^k + p = 0$ lie inside the unit ball, $|m| < 1$, if and only if*

$$0 < p < 2\cos\left(\frac{k\pi}{2k+1}\right).$$

The paper is organized as follows:
In Section 2, we study the local stability of the equilibrium points of equation (4). In Section 3, we study the oscillation of nonlinear delay difference equations with positive and negative coefficients, and apply our results to the equation (4) to give a sufficient condition for oscillation of all positive solutions about the positive steady state x^*. Also we prove that every nonoscillatory solution of equation (4) tends to x^*.

2. Local stability of equation (4)

In this section we study the local asymptotic stability of the equilibrium points of equation (4).

First we show that equation (4) has a unique positive equilibrium point. Observe that the equilibrium points of equation (4) are the solutions of the equation

$$x^* = x^* \exp\left(a + bx^{*p} - cx^{*q}\right).$$

So

$$x^* = 0, \qquad \text{or} \qquad a + bx^{*p} - cx^{*q} = 0. \tag{I}$$

Set

$$f(x) = a + bx^p - cx^q \qquad \text{for} \quad x \neq 0.$$

Now,

$$f(0) = a, \qquad \lim_{x\to\infty} f(x) = -\infty$$

and

$$f'(x) = bpx^{p-1} - qcx^{q-1}.$$

It follows from (I) that

$$f'(x^*) = \frac{1}{x^*}\left[-qa - bx^{*p}(q-p)\right] < 0.$$

This means that for every x^* satisfying the relation (I), we have $f'(x^*)$ is always negative, therefore equation (4) has exactly one positive solution x^*.

Theorem 2.1. *The following statements are true:*

(i) *The zero equilibrium of equation* (4) *is unstable.*

(ii) *The positive equilibrium point of equation* (4) *is locally stable if*

$$ap + (q-p)cx^{*q} < 2\cos\left(\frac{k\pi}{2k+1}\right).$$

Proof.

(i) The linearized equation of equation (4) about the zero equilibrium point is

$$y_{n+1} - \mathrm{e}^a y_n = 0.$$

Then its characteristic root is $\mathrm{e}^a > 1$. Hence by Theorem A the zero equilibrium point of equation (4) is unstable.

(ii) The linearized equation of equation (4) about the positive equilibrium point x^* is

$$z_{n+1} - z_n + \left[ap + (q-p)cx^{*q}\right] z_{n-\tau} = 0.$$

The result follows by Theorem B. □

3. Oscillation of nonlinear difference equations with positive and negative coefficients and application to equation (4)

In this section, we establish some new sufficient conditions for oscillation of nonlinear delay difference equations with positive and negative coefficients.

Consider the nonlinear delay difference equation,

$$\Delta x(n) + P(n)H_1(x(n-\sigma)) - Q(n)H_2(x(n-\tau)) = 0, \qquad n \geq n_0, \tag{9}$$

where the following hypotheses are satisfied:

(h_1) $\{P(n)\}$, $\{Q(n)\}$ are real positive sequences, $H_1, H_2 \in C[\mathbb{R}, \mathbb{R}]$, τ and σ, are positive integers with $\sigma \geq \tau$.

(h_2) $uH_i(u) > 0$ for $u \neq 0$, $i = 1, 2$, $\lim_{u\to 0} \frac{H_1(u)}{u} = 1$, $H_1 \geq H_2$ and there exists a positive constant δ such that, either

$$H_1(u) \leq u \qquad \text{for} \quad u \in [0, \delta],$$

or

$$H_1(u) \geq u \qquad \text{for} \quad u \in [-\delta, 0].$$

(h$_3$) $\lim_{n\to\infty} P(n) = \overline{p}$, $Q(n) \le \overline{q}$, $\overline{p} > \overline{q}$,
$P(n) - Q(n+\tau-\sigma) \ge \varepsilon > 0$ for $n \ge n_0 - \tau + \sigma$, and
$\sum_{n-\sigma}^{n-\tau-1} Q(n+\tau) \le 1$ for $n \ge n_0 + \sigma$.

(h$_4$) There exists a positive constant M such that
$\frac{H_2(u)}{u} \le M$ and $1 - M\overline{q}(\sigma-\tau) > 0$.

Theorem 3.1. *Assume that* (h$_1$)–(h$_4$) *are satisfied. Then every nonoscillatory solution of equation* (9) *tends to zero as* $n \to \infty$.

P r o o f. Let $x(n)$ be a nonoscillatory solution of equation (9). We will assume that $x(n)$ is eventually positive (the case where $x(n)$ is eventually negative is similar and will be omitted). Assume that $n_1 \ge n_0 + \sigma$ is such that $x(n) > 0$ for $n \ge n_1 - \sigma$. Set

$$z(n) = x(n) - \sum_{s=n-\sigma}^{n-\tau-1} Q(s+\tau)H_2((x(s)), \qquad n \ge n_0 + \sigma - \tau. \tag{10}$$

First, we show that $z(n)$ is nonincreasing. We see from equation (10) that

$$\Delta z(n) = \Delta x(n) - Q(n)H_2((x(n-\tau)) + Q(n+\tau-\sigma)H_2(x(n-\sigma)). \tag{11}$$

From equations (9) and (11) we get

$$\Delta z(n) = -P(n)H_1((x(n-\sigma)) + Q(n+\tau-\sigma)H_2(x(n-\sigma)).$$

Hence from (h$_2$) and (h$_3$) we find

$$\Delta z(n) \le -(P(n) - Q(n+\tau-\sigma))H_1(x(n-\sigma)) \le 0. \tag{12}$$

This yields that $z(n)$ is nonincreasing. Next, we prove that $z(n)$ is positive. If it is not the case, then eventually $z(n) < 0$, and so there exist $n_2 > n_1$ and $\alpha > 0$ such that $z(n) \le -\alpha < 0$ for $n \ge n_2$, that is

$$x(n) \le -\alpha + \sum_{i=n-\sigma}^{n-1-\tau} Q(i+\tau)H_2(x(i)) \le 0, \qquad n \ge n_2. \tag{13}$$

We consider the following two possible cases.

(i) If $\{x_n\}$ is unbounded, that is $\limsup_{n\to\infty} x(n) = \infty$, then there exists a sequence of points $\{s_i\}_{i=1}^{\infty}$ such that $s_i \geq n_3 + \sigma$, $i = 1, 2, 3, \ldots$, $s_i \to \infty$, $x(s_i) \to \infty$ as $i \to \infty$, and $x(s_i) = \max\{x_n : \ n_3 \leq n \leq s_i\}$. From ($h_2$), ($h_3$) and (13), we find that

$$x(s_i) \leq -\alpha + \sum_{i=n-\sigma}^{n-1-\tau} Q(i+\tau)H_2(x(i)) \leq -\alpha + x(s_i)$$

which is a contradiction.

(ii) if $\{x(n)\}$ is bounded that is $\limsup_{n\to\infty} x(n) = a < \infty$. Let $\{s_i\}_{i=1}^{\infty}$ be a sequence of points such that $s_i \to \infty$, as $i \to \infty$ and $x(s_i) \to a$ as $i \to \infty$. Let ξ_i be such that $x(\xi_i) = \max\big\{x(s) : \ s_i - \sigma \leq s \leq s_i - \tau\big\}$, $s_i - \sigma \leq \xi_i \leq s_i - \tau$, $i = 1, 2, \ldots$. Then $\xi_i \to \infty$, as $i \to \infty$. From (h_2), (h_3) and (13), we get

$$x(s_i) \leq -\alpha + \sum_{s=s_i-\sigma}^{s_i-1-\tau} Q(s+\tau)H_2(x(s)) \leq -\alpha + x(\xi_i).$$

Taking the superior limit as $i \to \infty$, we obtain

$$a \leq -\alpha + a,$$

which is also a contradiction. Combining (i) and (ii) we have $z(n)$ is positive and $x(n) \geq z(n)$ for $n \geq n_2$.

Now we show that $x(n)$ is bounded. Otherwise there exists a sequence of points $\{n_l\}$ such that, $\lim_{l\to\infty} n_l = \infty$, $\lim_{l\to\infty} x(n_l) = \infty$ and $x(n_l) = \max_{s\leq n_l} x(s)$. From equation (10) we have

$$z(n_l) = x(n_l) - \sum_{s=n_l-\sigma}^{n_l-\tau-1} Q(s+\tau)H_2(x(s)) = x(n_l) - \sum_{s=n_l-\sigma}^{n_l-\tau-1} Q(s+\tau)\frac{H_2(x(s))}{x(s)}x(s).$$

Then, from (h_3) and (h_4) we find that

$$z(n_l) \geq x(n_l)[1 - \overline{q}M(\sigma-\tau)] \to \infty \qquad \text{as} \quad l \to \infty,$$

which contradicts (12). Then from (10) and (12) we see that $z(n)$ is also bounded, and $\lim_{n\to\infty} z(n) = k \in \mathbb{R}$. By summing both sides of (12) from n_2 to ∞ we obtain

$$k - z(n_2) \leq -\sum_{i=n_2}^{\infty} \{(P(i) - Q(i+\tau-\sigma))\}H_1(x(i-\sigma)). \tag{14}$$

We claim that $\liminf_{n\to\infty} x(n) = 0$. Otherwise there exists a positive constant β and $n_3 \geq n_2$ such that $x(n) \geq \beta$ for $n \geq n_3$. Since $x(n) > 0$ and $x(n)$ is bounded from above, (h_3) implies that $H_1(x(n-\sigma)) \geq \beta'$ for some constant β'. Also, (h_3) implies that

$$(P(n) - Q(n+\tau-\sigma)) \geq (P(n) - \overline{q}) \to \overline{p} - \overline{q} \qquad \text{as} \quad n \to \infty.$$

Then for n sufficiently large, $(P(n) - Q(n+\tau-\sigma))H_1(x(n-\sigma))$ is bounded below by a positive constant. This contradicts (14). Hence $\liminf_{n\to\infty} x(n) = 0$. We prove that $\lim_{n\to\infty} x(n) = 0$. Otherwise, let $\limsup_{n\to\infty} x(n) = \mu$. From equation (10), since $z(n_l) \leq x(n_l)$, we get

$$k \leq 0.$$

Now from equation (10) and by using (h_3) and (h_4) we find that

$$z(n_l) \geq x(n_l) - \overline{q}M \sum_{s=n_l-\sigma}^{n_l-\tau-1} x(s).$$

Choose $\varepsilon_0 > 0$ and sufficiently small, we get from the last inequality that

$$z(n_l) \geq x(n_l) - \overline{q}M(\sigma-\tau)(\mu+\varepsilon_0).$$

By taking the limit as $n \to \infty$ we obtain

$$k \geq \mu - \overline{q}M(\sigma-\tau)(\mu+\varepsilon_0).$$

As ε_0 is arbitrary, we conclude

$$0 \geq k \geq \mu[1 - \overline{q}M(\sigma-\tau)] \geq \mu.$$

This implies that $k = \mu = 0$. Hence $\lim_{n\to\infty} x(n) = 0$ and then $\lim_{n\to\infty} z(n) = 0$. The proof is complete. □

Theorem 3.2. *Assume that* (h_1)–(h_4) *hold. If every solution of the delay difference equation*

$$\Delta z(n) + (P(n) - Q(n+\tau-\sigma))(1-\varepsilon)z(n-\sigma) = 0 \tag{15}$$

oscillates, where $\varepsilon > 0$ is arbitrarily small, then every solution of equation (9) *oscillates.*

P r o o f. Assume that (h_2) holds, with

$$H_1(u) \leq u \quad for \quad 0 \leq u \leq \delta.$$

The case where

$$H_1(u) \geq u \quad for \quad -\delta \leq u \leq 0,$$

is similar and will be omitted. Now assume, for the sake of contradiction that equation (9) has a nonoscillatory solution. We will assume that $x(n)$ is eventually positive solution of equation (9) (the case where $x(n)$ is eventually negative is similar and will be omitted), i.e., there exists n_1 sufficiently large such that $x(n) > 0$, $x(n-\tau) > 0$, and $x(n-\sigma) > 0$ for $n \geq n_1$. Set

$$z(n) = x(n) - \sum_{s=n-\sigma}^{n-\tau-1} Q(s+\tau)H_2((x(s)), \qquad n \geq n_1.$$

Then as in the proof of Theorem 3.1 we have

$$\Delta z(n) + (P(n) - Q(n+\tau-\sigma))H_1(x(n-\sigma)) \leq 0. \tag{16}$$

Since $\lim_{n\to\infty} x(n) = 0$, it follows by Theorem 3.1 and (h_2) that

$$\lim_{n\to\infty} \frac{H_1(x(n-\sigma))}{x(n-\sigma)} = 1.$$

Then there exist $\varepsilon \in (0,1)$ and n_ε such that for $n \geq n_\varepsilon$, $x(n-\sigma) > 0$ and

$$H_1(x(n-\sigma)) \geq (1-\varepsilon)x(n-\sigma).$$

We obtain from (16) that

$$\Delta z(n) + (P(n) - Q(n+\tau-\sigma))(1-\varepsilon)x(n-\sigma) \leq 0.$$

It follows by Theorem 3.1, since $x(n) \geq z(n)$ for $n \geq n_0 + \sigma - \tau$, that $z(n)$ is positive and satisfies

$$\Delta z(n) + (P(n) - Q(n+\tau-\sigma))(1-\varepsilon)z(n-\sigma) \leq 0. \tag{17}$$

Then by [16, Lemma 1], the delay difference equation (15) has an eventually positive solution also, which contradicts the assumption that every solution of equation (15) oscillates. Then every solution of equation (9) oscillates. The proof is complete. □

The oscillation of the linear delay difference equation (15) has been studied by many authors. By using the oscillation results in [5], [8], [10], [15], we get the following results.

Theorem 3.3. *Assume that one of the following statements is true:*

(i)

$$\limsup_{n\to\infty} \sum_{i=0}^{\sigma} \Theta(n-i) > 1, \tag{18}$$

(ii)

$$\liminf_{n\to\infty} \Theta(n) > \frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}}, \tag{19}$$

(iii)

$$\liminf_{n\to\infty} \frac{1}{\sigma} \sum_{i=1}^{\sigma} \Theta(n-i) > \frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}}, \tag{20}$$

where

$$\Theta(n) = (P(n) - Q(n+\tau-\sigma))(1-\varepsilon).$$

Then every solution of equation (15) *is oscillatory.*

Remark 3.1. Clearly, if the strict inequalities hold in (18), (19) and (20) for $\varepsilon = 0$, then the same result is also true for all sufficiently small $\varepsilon > 0$. Thus, we can restate Theorem 3.3 as follows:

Corollary 1. *Assume that one of the following statements is true*

(i)

$$\limsup_{n\to\infty} \sum_{i=0}^{\sigma} \Lambda(n-i) > 1, \tag{21}$$

(ii)

$$\liminf_{n\to\infty} \Lambda(n) > \frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}}, \tag{22}$$

(iii)

$$\liminf_{n\to\infty} \frac{1}{\sigma} \sum_{i=1}^{\sigma} \Lambda(n-i) > \frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}}, \tag{23}$$

where

$$\Lambda(n) = P(n) - Q(n+\tau-\sigma),$$

then every solution of equation (15) *is oscillatory.*

Theorem 3.4. *Assume that* $(\mathrm{h}_1) - (\mathrm{h}_4)$ *hold. If*

$$\liminf_{n\to\infty} \sum_{i=1}^{\sigma} \Theta(n-i) > L > 0,$$

and

$$\limsup_{n\to\infty} \Theta(n) > 1 - \frac{L^2}{4}, \tag{24}$$

then every solution of equation (15) *is oscillatory.*

Theorem 3.5. *Assume that one of the following is true*

(i)

$$0 \le \alpha = \liminf_{n\to\infty} \sum_{i=1}^{\sigma} \Theta(n-i) \le \frac{\sigma^{\sigma+1}}{(\sigma+1)^{\sigma+1}},$$

and

$$\limsup_{n\to\infty} \sum_{i=0}^{\sigma} \Theta(n-i) > 1 - \frac{\alpha^2}{4}. \tag{25}$$

Then every solution of equation (15) *is oscillatory.*

Theorem 3.6. *Assume that*

(i)

$$0 \le \alpha = \liminf_{n\to\infty} \sum_{i=1}^{\sigma} \Theta(n-i) \le \frac{\sigma^{\sigma+1}}{(\sigma+1)^{\sigma+1}},$$

(ii)

$$\limsup_{n\to\infty} \sum_{i=0}^{\sigma} \Theta(n-i) > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{26}$$

Then every solution of equation (15) *is oscillatory.*

Remark 3.2. Theorem 3.2 shows that the oscillation of equation (9) is equivalent to the oscillation of the delay difference equation (15). Now, by applying the above results we have the following oscillation criteria for oscillation of the nonlinear delay difference equation (9).

Theorem 3.7. *Assume that* $(\mathrm{h}_1) - (\mathrm{h}_4)$ *hold. Furthermore, assume that one of the conditions* (21), (22) *or* (23) *holds. Then every solution of equation* (9) *oscillates.*

Theorem 3.8. *Assume that* (h_1)–(h_4) *hold. Furthermore, assume that the assumptions of* Theorem 3.2 *hold. Then every solution of equation* (9) *oscillates.*

Theorem 3.9. *Assume that* (h_1)–(h_4) *hold. Furthermore, assume that the assumptions of* Theorem 3.5 *hold. Then every solution of equation* (9) *oscillates.*

Theorem 3.10. *Assume that* $(\mathrm{h}_1) - (\mathrm{h}_4)$ *hold. Furthermore, assume that the assumptions of* Theorem 3.6 *hold. Then every solution of equation* (9) *oscillates.*

Remark 3.3. In the above results some of the conditions (h_1)–(h_4) may be weakened. In particular, from the proofs of Theorems 3.1 and 3.2 it is clear that if $\sigma = \tau$, then condition (h_4) in all the above results may be dropped.

In the following theorem, we apply our oscillation results to establish the sufficient condition for oscillation of all positive solutions of equation (4) about x^*.

Theorem 3.11. *Assume that* (5) *holds and*

$$qc(x^*)^q - bp(x^*)^p > \frac{\tau^\tau}{(\tau+1)^{\tau+1}}. \tag{27}$$

Then every positive solution of equation (4) *oscillates about* x^*.

Proof. Let $x(n)$ be an arbitrary positive solution of equation (4) (we consider the case where $x_n > x^*$ since the case where $x_n < x^*$ is similar and will be omitted). Set

$$x_n = x^* \exp\{z_n\}. \tag{28}$$

Clearly, $z(n)$ is positive, and satisfies the nonlinear difference equation

$$\Delta z_n + qc(x^*)^q H_1(z(n-\tau)) - bp(x^*)^p H_2(z(n-\tau)) = 0, \tag{29}$$

with

$$H_1(u) = \frac{\mathrm{e}^{qu}-1}{q}, \qquad \text{and} \qquad H_2(u) = \frac{\mathrm{e}^{pu}-1}{p}.$$

Observe that

$$qc(x^*)^q > bp(x^*)^p.$$

Since $\sigma = \tau$, in view of Remark 3.2 it is easy to see that all the hypotheses (h_1)–(h_3) are satisfied for equation (29). Then by the condition (27) every solution of equation (29) oscillates about zero, and this implies that every solution of equation (4) oscillates about x^*. The proof is complete. □

We note that

$$(qc(x^*)^q - bp(x^*)^p)(\tau+1) > 1/\big(1+\tfrac{1}{\tau}\big)^\tau \to 1/\mathrm{e} \qquad \text{as} \quad \tau \to \infty.$$

Therefore one may think of the condition (27) of Theorem 3.11 as being the discrete analogy of equation (4) with the delay $\tau + 1$. So the derived discrete analogy preserves the oscillation condition.

In the following we give an attractivity result of all nonoscillatory solutions of equation (4).

Theorem 3.12. *Assume that* (5) *holds. Then every positive nonoscillatory solution of equation* (4) *converges to the positive equilibrium point* x^*.

Proof. Let $x(n)$ be a positive solution of equation (4) which does not oscillate about x^*. Without loss of generality we might assume that $x(n)$ is eventually grater than or equal to x^* (the case where the solution is less than x^* is similar and is left to the reader). It is clear that

$$(x - x^*)f(x) < 0 \qquad \text{for } x \text{ from some neighborhood of } x^*, \tag{30}$$

where $f(x)$ is defined as before by

$$f(x) = a + bx^p - cx^q.$$

Now let N be a nonnegative integer number and let

$$x_{n-\tau} \geq x^* \qquad \text{for all} \quad n \geq N > \tau.$$

It follows from equation (4) and (30) that

$$x_{n+1} = x_n \exp(a + bx^p_{n-\tau} - cx^q_{n-\tau}) \leq x_n \qquad \text{for all} \quad n \geq N.$$

Thus the sequence $x(n)$ is non-increasing and bounded from below by x^* and since x^* is the only equilibrium point of equation (4),

$$\lim_{n\to\infty} x(n) = x^*.$$

This completes the proof of the theorem. □

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