VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-007-0007-6
Math. Slovaca **57** (2007), No. 2, 171–178

ON A CLASS OF DIFFERENCE SEQUENCES RELATED TO THE ℓ^p SPACE DEFINED BY ORLICZ FUNCTIONS

Binod Chandra Tripathy — Sabita Mahanta

(*Communicated by Pavel Kostyrko*)

ABSTRACT. In this article we introduce the difference sequence space $m(M, \Delta, \phi)$ using the Orlicz function. We study its different properties like solidity, completeness etc. Also we obtain some inclusion relations involving the space $m(M, \Delta, \phi)$.

1. Introduction

Throughout the article w, ℓ_∞, ℓ^p denote the spaces of *all*, *bounded* and *p-absolutely summable* sequences respectively. The zero sequence is denoted by θ. The sequence space $m(\phi)$ was introduced by Sargent [12], who studied some of its properties and obtained its relationship with the space ℓ^p. Later on it was investigated from sequence space point of view by Rath and Tripathy [10], Tripathy [13], Tripathy and Sen [14], Tripathy and Mahanta [15] and others.

The notion of difference sequence space was introduced by Kizmaz [4]. He studied the properties of the difference sequence spaces

$$X(\Delta) = \left\{x = (x_k) \in w : \ (\Delta x_k) \in X\right\},$$

where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$ and for $X = \ell_\infty$, c and c_0.

2000 Mathematics Subject Classification: Primary 40A05, 46A45, 46E30.
Keywords: completeness, Orlicz function, difference sequence space, solid space, symmetric space.

An *Orlicz function* is a function $M\colon [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex, with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M is said to satisfy Δ_2-*condition* for all values of x, if there exists a constant $K > 0$, such that $M(2x) \le KM(x)$, for all values of $x \ge 0$.

If the convexity of Orlicz function is replaced by $M(x+y) \le M(x) + M(y)$, then this function is called the *modulus function*, introduced by Nakano [7]. It was further investigated from sequence space point of view by Ruckle [11] and many others.

Remark. An Orlicz function satisfies the inequality $M(\lambda x) \le \lambda M(x)$ for all λ with $0 < \lambda < 1$.

2. Definition and background

Let $\wp_s$ denotes the class of all subsets of $\mathbb{N}$, that do not contain more than s elements. Throughout (ϕ_n) represents a non-decreasing sequence of strictly positive real numbers such that $n\phi_{n+1} \le (n+1)\phi_n$ for all $n \in \mathbb{N}$. By Φ we denote the class of all these sequences (ϕ_n).

The sequence space $m(\phi)$ introduced by Sargent [12] is defined as follows:

$$m(\phi) = \Big\{(x_k) \in w : \ \|(x_k)\|_{m(\phi)} = \sup_{s\ge 1,\, \sigma\in\wp_s} \tfrac{1}{\phi_s} \sum_{k\in\sigma} |x_k| < \infty\Big\}.$$

Lindenstrauss and Tzafriri [5] used the notion of Orlicz function and introduced the sequence space

$$\ell_M = \Big\{(x_k) \in w : \ \sum_{k=1}^{\infty} M\Big(\tfrac{|x_k|}{\rho}\Big) < \infty \ \text{ for some } \ \rho > 0\Big\}.$$

The space ℓ_M with the norm

$$\|(x_k)\| = \inf\Big\{\rho > 0 : \ \sum_{k=1}^{\infty} M\Big(\tfrac{|x_k|}{\rho}\Big) \le 1\Big\}$$

becomes a Banach space, which is called an *Orlicz sequence space*. The space ℓ_M is closely related to the space ℓ^p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \le p < \infty$.

In the later stage different Orlicz sequence spaces were introduced and studied by Et [2], Esi and Et [1], Parashar and Choudhary [9], Nuray and Gülcü [8] and many others.

In this article we shall use the following known sequence spaces defined by Orlicz functions:

$$\ell_1(M,\Delta) = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|\Delta x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\},$$

$$\ell_\infty(M,\Delta) = \left\{ x = (x_k) \in w : \sup_{k\geq 1} M\left(\frac{|\Delta x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}$$

(see for instance Mursaleen et.al [6]).

In this article we introduce the following sequence spaces:

$$m(M,\Delta,\phi) = \left\{ (x_k) \in w : \sup_{\substack{s\geq 1,\\ \sigma\in\wp_s}} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\Delta x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

Taking $X = \mathbb{C}$, the set of complex numbers, from the section Particular Cases of Tripathy and Mahanta [15], we have that the space $m(M,\phi)$ is a Banach space under the norm

$$\|(x_k)\| = \inf\left\{ \rho > 0 : \sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

A sequence space E is said to be *solid* (or *normal*) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequences (α_k) of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

A sequence space E is a said to be *symmetric* if $(x_n) \in E$ implies $\left(x_{\pi(n)}\right) \in E$, where $\pi(n)$ is a permutation of the elements of $\mathbb{N}$.

The following results will be used for establishing some results of this article.

Lemma 1. (Sargent [12, Lemma 10]) *In order that $m(\phi) \subseteq m(\psi)$, it is necessary and sufficient that $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} < \infty$.*

Lemma 2. (Sargent [12, Lemma 11])

(a) $\ell_1 \subseteq m(\phi) \subseteq \ell_\infty$ *for all* $\phi \in \Phi$.

(b) $m(\phi) = \ell_1$ *if and only if* $\lim_{s\to\infty} \phi_s < \infty$.

(c) $m(\phi) = \ell_\infty$ *if and only if* $\lim_{s\to\infty} \frac{\phi_s}{s} > 0$.

Taking $m = 1$, i.e., considering only the first difference, we have the following results from Theorem 2.2 of Et and Nuray [3].

Lemma 3. *If X is a Banach space normed by $\|\cdot\|$, then $X(\Delta)$ is also a Banach space normed by $\|x\|_\Delta = |x_1| + \|\Delta x\|$.*

3. Main results

In this section we prove some results involving the sequence space $m(M, \Delta, \phi)$.

Theorem 1. *The space $m(M, \Delta, \phi)$ is a linear space.*

Proof. Let $(x_k), (y_k) \in m(M, \Delta, \phi)$ and $\alpha, \beta \in \mathbb{C}$. Then there exists positive numbers ρ_1, ρ_2 such that

$$\sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\Delta x_k|}{\rho_1}\right) < \infty$$

and

$$\sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\Delta y_k|}{\rho_2}\right) < \infty\,.$$

Let $\rho_3 = \max\bigl(2|\alpha|\rho_1, 2|\beta|\rho_2\bigr)$. Since M is non-decreasing convex function,

$$\begin{aligned}
\sum_{k\in\sigma} M\left(\frac{|\Delta(\alpha x_k + \beta y_k)|}{\rho_3}\right) &\leq \sum_{k\in\sigma} M\left(\frac{|\alpha\Delta x_k|}{\rho_3} + \frac{|\beta\Delta y_k|}{\rho_3}\right)\\
&< \sum_{k\in\sigma} M\left(\frac{|\Delta x_k|}{\rho_1}\right) + \sum_{k\in\sigma} M\left(\frac{|\Delta y_k|}{\rho_2}\right)
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad & \sup_{\substack{s\geq 1,\\ \sigma\in\wp_s}} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\Delta(\alpha x_k + \beta y_k)|}{\rho_3}\right)\\
& \leq \sup_{\substack{s\geq 1,\\ \sigma\in\wp_s}} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\Delta x_k|}{\rho_1}\right) + \sup_{\substack{s\geq 1,\\ \sigma\in\wp_s}} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\Delta y_k|}{\rho_2}\right) < \infty\,.\\
\Longrightarrow\ & (\alpha x_k + \beta y_k) \in m(M, \Delta, \phi).
\end{aligned}$$

Hence $m(M, \Delta, \phi)$ is a linear space. □

Lemma 4. *The space $m(M, \Delta, \phi)$ is a normed linear space, normed by* $h_\Delta\bigl((x_k)\bigr) = |x_1| + \inf\Bigl\{\rho > 0 : \sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\Bigl(\frac{|\Delta x_k|}{\rho}\Bigr) \leq 1\Bigr\}$.

Proof. Clearly $h_\Delta\bigl((-x_k)\bigr) = h_\Delta\bigl((x_k)\bigr)$. Next $x = \theta$ implies $\Delta x_k = 0$ and as such $M(0) = 0$, therefore $h_\Delta(\theta) = 0$. It can be easily shown that $h_\Delta\bigl((x_k)\bigr) = 0 \Longrightarrow x = \theta$.

Next let $\rho_1 > 0$, $\rho_2 > 0$ be such that

$$\sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\Delta x_k|}{\rho_1}\right) \leq 1$$

and

$$\sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\Delta y_k|}{\rho_2}\right) \leq 1\,.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\Delta(x_k + y_k)|}{\rho}\right)$$

$$\leq \frac{\rho_1}{\rho_1+\rho_2} \cdot \sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\Delta x_k|}{\rho_1}\right) + \frac{\rho_2}{\rho_1+\rho_2} \cdot \sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\Delta y_k|}{\rho_2}\right).$$

Since $\rho_1 > 0$, $\rho_2 > 0$, so by the definition of h_Δ, we have

$$h_\Delta\big((x_k + y_k)\big) \leq h_\Delta\big((x_k)\big) + h_\Delta\big((y_k)\big)\,.$$

Finally the continuity of the scalar multiplication follows from the following equality,

$$\begin{aligned} h_\Delta\big(\lambda(x_k)\big) &= |\lambda x_1| + \inf\Big\{\rho > 0 : \sup_{s\geq 1,\, \sigma\in\wp_s} \tfrac{1}{\phi_s} \sum_{k\in\sigma} M\Big(\tfrac{|\Delta\lambda x_k|}{\rho}\Big) \leq 1\Big\} \\ &= |\lambda| h_\Delta\big((x_k)\big)\,. \end{aligned}$$

This completes the proof of the theorem. □

Theorem 2. *$m(M,\Delta,\phi) \subseteq m(M,\Delta,\psi)$ if and only if $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} < \infty$.*

P r o o f. By taking $y_k = M\Big(\frac{|\Delta x_k|}{\rho}\Big)$ in Lemma 1, it can be proved that $m(M,\Delta,\phi) \subseteq m(M,\Delta,\psi)$ if and only if $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} < \infty$. □

Corollary 3. *$m(M,\Delta,\phi) = m(M,\Delta,\psi)$ if and only if $\sup_{s\geq 1} \frac{\phi_s}{\psi_s} < \infty$ and*

$$\sup_{s\geq 1} \frac{\psi_s}{\phi_s} < \infty\,.$$

Theorem 4. *Let M, M_1, M_2 be Orlicz functions satisfying Δ_2-condition. Then*

(i) $m(M_1,\Delta,\phi) \subseteq m(M\circ M_1,\Delta,\phi)$.

(ii) $m(M_1,\Delta,\phi) \cap m(M_2,\Delta,\phi) \subseteq m(M_1{+}M_2,\Delta,\phi)$.

P r o o f.

(i) Let $(x_k) \in m(M_1, \Delta, \phi)$. Then there exists $\rho > 0$ such that

$$\sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M_1\left(\frac{|\Delta x_k|}{\rho}\right) < \infty\,.$$

Let $0 < \varepsilon < 1$ and δ, with $0 < \delta < 1$, such that $M(t) < \delta$ for $0 \leq t < \delta$.

Let $y_k = M_1\Big(\frac{|\Delta x_k|}{\rho}\Big)$ and for any $\sigma \in \wp_s$ let

$$\sum_{k\in\sigma} M(y_k) = \sum\nolimits_1 M(y_k) + \sum\nolimits_2 M(y_k)\,,$$

where the first summation is over $y_k \leq \delta$ and the second is over $y_k > \delta$.

By the Remark

$$\sum\nolimits_1 M(y_k) \leq M(1) \sum\nolimits_1 y_k \leq M(2) \sum\nolimits_1 y_k. \tag{1}$$

For $y_k > \delta$ we have,

$$y_k < \frac{y_k}{\delta} \leq 1 + \frac{y_k}{\delta}\,.$$

Since M is non-decreasing and convex, so

$$M(y_k) < M\left(1 + \frac{y_k}{\delta}\right) < \frac{1}{2}M(2) + \frac{1}{2}M\left(\frac{2y_k}{\delta}\right).$$

Since M satisfies Δ_2-condition, so

$$M(y_k) < \frac{1}{2}K\frac{y_k}{\delta}M(2) + \frac{1}{2}K\frac{y_k}{\delta}M(2) < K\frac{y_k}{\delta}M(2).$$

Hence

$$\sum\nolimits_2 M(y_k) \leq \max\Big(1, K\delta^{-1}M(2)\Big) \sum\nolimits_2 y_k. \tag{2}$$

From (1) and (2), it follows that $(x_k) \in m(M \circ M_1, \Delta, \phi)$.

(ii) Let $(x_k) \in m(M_1, \Delta, \phi) \cap m(M_2, \Delta, \phi)$. Then there exists $\rho_1 > 0$, $\rho_2 > 0$ such that

$$\sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M_1\left(\frac{|\Delta x_k|}{\rho_1}\right) < \infty$$

and

$$\sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M_2\left(\frac{|\Delta x_k|}{\rho_2}\right) < \infty\,.$$

Let $\rho_3 = \max(\rho_1, \rho_2)$.

The remaining part of the proof follows from the inequality

$$\sum_{k\in\sigma}(M_1 + M_2)\left(\frac{|\Delta x_k|}{\rho_3}\right) \leq \sum_{k\in\sigma} M_1\left(\frac{|\Delta x_k|}{\rho_1}\right) + \sum_{k\in\sigma} M_2\left(\frac{|\Delta x_k|}{\rho_2}\right) \qquad \square$$

Taking $M_1(x) = x$ in Theorem 5(i), we have the following result.

Corollary 5. *Let M be an Orlicz function satisfying Δ_2-condition. Then $m(\Delta, \phi) \subseteq m(M, \Delta, \phi)$.*

Theorem 6.

(a) $\ell_1(M, \Delta) \subseteq m(M, \Delta, \phi) \subseteq \ell_\infty(M, \Delta)$.

(b) $m(M, \Delta, \phi) = \ell_1(M, \Delta)$ *if and only if* $\sup\limits_{s\geq 1} \phi_s < \infty$.

(c) $m(M, \Delta, \phi) = \ell_\infty(M, \Delta)$ *if and only if* $\sup\limits_{s\geq 1} \frac{s}{\phi_s} < \infty$.

Proof.

(a) The result follows from Lemma 2, by taking $y_k = M\left(\frac{|\Delta x_k|}{\rho}\right)$.

(b) The result follows from the point of view of Lemma 2.

(c) The result follows from the point of view of Lemma 2. □

From Lemma 3 and the fact that $m(M, \phi)$ is a Banach space, the following result follows.

Theorem 7. *The space $m(M, \Delta, \phi)$ is complete.*

The following result is a routine work.

Proposition 8. *The space $m(M, \Delta, \phi)$ is a BK-space.*

Proposition 9. *The space $m(M, \Delta, \phi)$ is not solid in general.*

Proof. To show that the space is not solid in general, consider the following example. □

Example 1. Let $\phi_k = 1$ and $x_k = 1$ for all $k \in \mathbb{N}$. Consider $\lambda_k = (-1)^k$ for all $k \in \mathbb{N}$ and $M(x) = x$. Then $(x_k) \in m(M, \Delta, \phi)$ but $(\lambda_k x_k) \notin m(M, \Delta, \phi)$.

Proposition 10. *The space $m(M, \Delta, \phi)$ is not symmetric in general.*

Proof. To show that the space is not symmetric in general, consider the following example. □

Example 2. Let $M(x) = x$ and $\phi_k = k$ for all $k \in \mathbb{N}$. Then the sequence (x_k) define by $x_k = k$ for all $k \in \mathbb{N}$ is in $m(M, \Delta, \phi)$. Consider the sequence (y_k), the rearrangement of (x_k) defined as follows

$$(y_k) = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, x_{64}, x_{11}, -, -, -, -).$$

Then $(y_k) \notin m(M, \Delta, \phi)$. Hence the space $m(M, \Delta, \phi)$ is not symmetric in general.

Acknowledgement. The authors thank the referees for their comments.

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Received 13.9.2004
Revised 29.3.2005

Mathematical Sciences Division
Institute of Advanced Study
in Science and Technology
Paschim Boragaon
Garchuk
Guwahati–781 035
INDIA
E-mail: tripathybc@yahoo.com
tripathybc@rediffmail.com
sabitamahanta@yahoo.co.in