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THE LOWER BOUND OF NUMBER OF SOLUTIONS FOR THE SECOND ORDER NONLINEAR BOUNDARY VALUE PROBLEM VIA THE ROOT FUNCTIONS METHOD

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ABSTRACT. We consider a second order nonlinear differential equation with homogeneous Dirichlet boundary conditions. Using the root functions method we prove a relation between the number of zeros of some variational solutions and the number of solutions of our boundary value problem which follows into a lower bound of the number of its solutions.

0. Introduction

The problem of multiplicity of the nth order boundary value problem (BVP) has been investigated in many papers. There are many ways to handle this problem. One of them is the well-known Shooting Method. In this paper we will try (at least partially) to solve a problem of a lower bound of the number of solutions of second order BVP. Papers using the Shooting Method to bound the number of solutions of BVP are usually based on the same principle, which we call the *root functions method*. Roughly speaking we will try to show that the number of zeros of some variational problem has a connection with the number of solutions of BVP and this connection is made by root functions.

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We will consider the following 2nd order BVP with Dirichlet boundary conditions

$$x'' = f(t, x, x')\,, \tag{1}$$

$$x(0) = 0\,, \quad x(\pi) = 0\,, \tag{2}$$

where $f\colon\ [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ and $T \in (\pi, \infty]$.[1]

In the first paragraph we will set up the definition of a shooting function as a solution of some parameterized initial value problem (IVP). Then we define the root functions, which have been already mentioned in [DS]. We actually generalize this notion of root functions for our BVP with minimal requirements on the function f. Further we show properties of a shooting function and root functions, which will be useful in the next paragraphs.

In Paragraph 2 we will present a theorem, which under another assumptions on f gives a lower bound of the number of solutions. Then we prove a corollary of this theorem, which refines the lower bound of the number of solutions under an additional assumption on f. To achieve our purpose we are using technique similar to [GS]. We also show a non-trivial example where we use this theorem.

In the conclusion we will emphasize the importance of root functions behaviour analysis. A connection between the behaviour of $\frac{\partial f}{\partial x}$ (for $f = f(x)$) and the behaviour of root functions will be outlined.

In this article we will use the following notations:

$\|\cdot\|$ — norm in $\mathbb{R}^2$;

$\|x\|_1 = \sup\limits_{t\in[0,T]} \big\|\big(x(t), x'(t)\big)\big\|$, where $x \in \mathcal{C}^1\big([0,T]\big)$ ($\|\cdot\|_1$ is norm in $\mathcal{C}^1$);

$(a,b)^0 = (a,b)\setminus\{0\}$.

1. Definition of root functions

For the definition of root function we will consider IVP (1) with the initial conditions:

$$x(0) = 0, \quad x'(0) = \lambda, \ \ \lambda \in (\Lambda_1, \Lambda_2)\,, \tag{3}$$
$$-\infty \leqq \Lambda_1 < 0 < \Lambda_2 \leqq \infty\,.$$

We will suppose the following assumptions on f which will be called the *standard assumptions*:

(H1) f is continuous on its domain and the function $x(t) \equiv 0$ for $t \in [0,T]$ is the unique solution of the initial value problem (1) with the conditions: $x(\tilde t\,) = 0$, $x'(\tilde t\,) = 0$, for each $\tilde t \in [0,T]$ (it implies that $f(t,0,0) \equiv 0$).

[1] If $T = \infty$, then all intervals $[\,\cdot\,, T]$ have to be replaced by $[\,\cdot\,, \infty)$.

(H2) $\exists\,\Lambda_1 \in \mathbb{R}^- \cup \{-\infty\}\ \exists\,\Lambda_2 \in \mathbb{R}^+ \cup \{\infty\}\ \ \forall \lambda \in (\Lambda_1, \Lambda_2)$:
IVP (1), (3) has the unique solution defined on $[0,T]$.

Definition 1.1. The initial value problem (1), (3) is called the *shooting problem* (SP) associated with (1) (or SP (1), (3)) when it fulfils the following assumptions:

(a) for all $\lambda \in (\Lambda_1, \Lambda_2)$ there exists the unique classical solution of IVP (1), (3) and it can be extended on the whole interval $[0,T]$;
(b) IVP (1), (3) has the property of continuous dependence on parameter λ — it means: $x \in \mathcal{C}\big([0,T] \times (\Lambda_1, \Lambda_2)\big)$ and $x' \in \mathcal{C}\big([0,T] \times (\Lambda_1, \Lambda_2)\big)$.

Next we will define *shooting function* as a solution of SP (1), (3) with parameter λ and show its properties under *standard assumptions*.

Definition 1.2. The *shooting function* of SP (1), (3) is a function $S\colon\ [0,T] \times (\Lambda_1, \Lambda_2) \to \mathbb{R}$ such that:

for each $\tilde\lambda \in (\Lambda_1, \Lambda_2)$ function $S_{\tilde\lambda}(\cdot) := S(\cdot, \tilde\lambda)\colon\ [0,T] \to \mathbb{R}$ is a solution of SP (1), (3) with parameter $\lambda = \tilde\lambda$ — this function will be called the *shot with slope* $\tilde\lambda$ ($S'_{\tilde\lambda}(0) = \tilde\lambda$).

Lemma 1.1. *Let f fulfil standard assumptions. Then there exists a shooting function S of SP* (1), (3) *with the following properties:*

(i) $S \in \mathcal{C}\big([0,T] \times (\Lambda_1, \Lambda_2)\big)$, $S' \in \mathcal{C}\big([0,T] \times (\Lambda_1, \Lambda_2)\big)$, *where* $S' = \frac{\partial S}{\partial t}$;
(ii) $\forall\,\lambda \in (\Lambda_1, \Lambda_2)$*:* $S(\cdot, \lambda) \in \mathcal{C}^1\big([0,T]\big)$;
(iii) $\forall\,\lambda \in (\Lambda_1, \Lambda_2)^0$*:* $\|S(\cdot, \lambda)\|_1 > 0$.

P r o o f . The existence of a function S follows from the assumptions (H1) and (H2), which imply (a), (b) of Definition 1.1, see [Ka, p. 59]. The property (i) holds, since it is the same as (b) of Definition 1.1. The property (ii) is also fulfilled since the function $S_\lambda(\cdot) = S(\cdot, \lambda)$ ($\lambda \in (\Lambda_1, \Lambda_2)$) is a classical solution of SP (1), (3).

Let the property (iii) do not hold, then there exists $\tilde t \in [0,T]$ and $\tilde\lambda \in (\Lambda_1, \Lambda_2)^0$ such that $\|S_{\tilde\lambda}(\tilde t\,)\|_1 = 0$. From (H1) it follows that the only solution of (1), (3) with the property $x(\tilde t\,) = 0$, $x'(\tilde t\,) = 0$ is zero solution $x(t) \equiv 0$ for $t \in [0,T]$. This is a contradiction since $S_{\tilde\lambda}(t)$ is also a solution of (1), (3) and $\tilde\lambda \neq 0$. □

Let (t_1, λ_1) be the inner point of the set $(0,T) \times (\Lambda_1, \Lambda_2)^0$ such that $S(t_1, \lambda_1) = 0$. By Lemma 1.1 we have $S'(t_1, \lambda_1) \neq 0$, then from the Implicit Function Theorem we get a continuous function $t_k\colon \mathcal{O}_\delta(\lambda_1) \to \mathcal{O}_\varepsilon(t_1)$ (where $\mathcal{O}_\delta(\lambda_1) = (\lambda_1 - \delta, \lambda_1 + \delta) \subset (\Lambda_1, \Lambda_2)^0$, $\mathcal{O}_\varepsilon(t_1) = (t_1 - \varepsilon, t_1 + \varepsilon) \subset (0,T)$ and $\varepsilon, \delta > 0$) which fulfils: $S\big(t_k(\lambda), \lambda\big) = 0$ for $\lambda \in \mathcal{O}_\delta(\lambda_1)$.

The following definition of *root functions* is crucial in estimation of number of solutions.

Definition 1.3. Let (t,λ) be an inner point of the set $(0,T)\times(\Lambda_1,\Lambda_2)^0$ such that $S(t,\lambda)=0$, where S is the shooting function of SP (1), (3).

The *root function* of SP (1), (3) is a continuous function $t_R\colon \mathcal{O}(\lambda)\to(0,T)$, where $\mathcal{O}(\lambda)\subset(\Lambda_1,\Lambda_2)^0$ is a maximal open interval such that $S(t_R(\lambda),\lambda)=0$ for $\lambda\in\mathcal{O}(\lambda)$.

In the case that $\mathcal{D}(t_R)\subset(0,\Lambda_2)$ (resp. $\mathcal{D}(t_R)\subset(\Lambda_1,0)$) the root function t_R will be called the *right* (*resp. left*) *root function* and denoted as t_r (resp. t_l).

The root function $t_R(\lambda)\equiv 0$ for $\lambda\in(\Lambda_1,\Lambda_2)$ will be called the *trivial root function*.

Theorem 1.1. *Let function f fulfil standard assumptions, then every root function of SP* (1), (3) *has the following properties:*

(P1) *Through every point $(t,\lambda)\in(0,T)\times(\Lambda_1,\Lambda_2)^0$ there goes at most one root function* (*i.e., root functions cannot intersect among themselves*).

(P2) *For every compact $K\subset(\Lambda_1,\Lambda_2)^0$ there holds:*

$$\exists\,\delta>0\ \forall\,t_R\not\equiv 0\ \forall\,\lambda\in K\cap\mathcal{D}(t_R):\ t_R(\lambda)>\delta.$$

(P3) *Let $t_R\not\equiv 0$ and $\lambda_1\in\partial\mathcal{D}(t_R)$, then only one of the following possibilities can arise:*

(a) $\lambda_1=0$*;*

(b) $\lambda_1=\Lambda_1$ *or* $\lambda_1=\Lambda_2$*;*

(c) $\lambda_1\in(\Lambda_1,\Lambda_2)^0$ *and* $\lim\limits_{\lambda\to\lambda_1} t_R(\lambda)=T$.

P r o o f . By Lemma 1.1 we have the shooting function S of SP (1), (3) defined on $[0,T]\times(\Lambda_1,\Lambda_2)$ with properties (i)–(iii) of Lemma 1.1 and we have well-defined root functions (if there exists any).

Property (P1) follows directly from the Implicit Function Theorem.

(P2): Let the opposite hold, i.e., there exists a compact $K\subset(0,\Lambda_2)$ and a sequence of $t_n:=t_{r_n}(\lambda_n)$ such that $\lim\limits_{n\to\infty} t_n=0$. Since $S_{\lambda_n}(t_n)=0=S_{\lambda_n}(0)$, the Mean Value Theorem for every n gives $\tilde t_n\in(0,t_n)$ such that $S'_{\lambda_n}(\tilde t_n)=0$. It is easy to see that $\lim\limits_{n\to\infty}\tilde t_n=0$. Since K is a compact, there must be a point of accumulation of λ_n denoted as $\tilde\lambda$. It is obvious that $\tilde\lambda\in K\subset(0,\Lambda_2)$ and from the property (i) of the shooting function S and previous statements we finally have: $S_{\tilde\lambda}(0)=0=S'_{\tilde\lambda}(0)$. It is a contradiction since $S'_{\tilde\lambda}=\tilde\lambda>0$. (The statement for t_l can be proved analogously.)

(P3): Let $\lambda_1\in\partial\mathcal{D}(t_R)$ fulfil neither (a) nor (b). Hence $\lambda_1\in(\Lambda_1,\Lambda_2)^0$. If $\lim\limits_{\lambda\to\lambda_1} t_R(\lambda)$ does not exists, then from the continuity of t_R we get the discontinuity of shooting function S in points (t,λ_1) where $t \in \Big(\varliminf\limits_{\lambda\to\lambda_1} t_R(\lambda), \varlimsup\limits_{\lambda\to\lambda_1} t_R(\lambda)\Big)$ which is a contradiction. Let $\lim\limits_{\lambda\to\lambda_1} t_R(\lambda) = t_1$ and $t_1 < T$. From (P2) we can see that $t_1 > 0$. Hence (t_1,λ_1) is an inner point of $(0,T)\times(\Lambda_1,\Lambda_2)^0$. Using the Implicit Function Theorem we can extend the root function t_R on a greater connected set, which is a contradiction with maximality of $\mathcal{D}(t_R)$ (see Definition 1.3). Hence $\lim\limits_{\lambda\to\lambda_1} t_R(\lambda) = T$ and the theorem is proved. $\square$

Let function f of BVP (1), (2) fulfil the following assumption:

(H3) $\exists\, K > 0\ \forall\, t \in [0,T]\ \ \forall \vec{x} \in \mathbb{R}^2\colon\ \big|f(t,\vec{x})\big| \leqq K\|\vec{x}\|$.

Remark 1.1. Assumption (H3) together with assumption (H1) imply the extensibility of every shot S_λ ($\lambda \in (\Lambda_1,\Lambda_2)^0$) on the whole interval $[0,T]$.

Lemma 1.2. *Let standard assumptions and* (H3) *on the function f hold. Then the shooting function of SP* (1), (3) *fulfils:*

$$\exists\, M > 1\ \ \forall \lambda \in (\Lambda_1,\Lambda_2)^0:\ \ \frac{\|S_\lambda\|_1}{|\lambda|} < M\,.$$

P r o o f . Let us denote $u(t,\lambda) = S(t,\lambda)$ and $v(t,\lambda) = S'(t,\lambda)$ for $t \in [0,T]$, $\lambda \in (\Lambda_1,\Lambda_2)^0$. It is easy to see that functions (u,v) fulfil the following system of differential equations on interval $[0,T]$ for arbitrary $\lambda \in (\Lambda_1,\Lambda_2)^0$:

$$\begin{pmatrix} u'(t,\lambda) \\ v'(t,\lambda) \end{pmatrix} = \begin{pmatrix} v(t,\lambda) \\ f\big(t,u(t,\lambda),v(t,\lambda)\big) \end{pmatrix}; \qquad \left[u' = \frac{\partial u}{\partial t},\ v' = \frac{\partial v}{\partial t}\right],$$

and they also fulfil initial conditions: $\big(u(0,\lambda), v(0,\lambda)\big) = (0,\lambda)$. Integrating last equation and using standard norm in $\mathbb{R}^2$ we have:

$$\big\|U(t,\lambda)\big\| \leqq |\lambda| + \int_0^t \big\|F(\tau,u(\tau,\lambda),v(\tau,\lambda))\big\|\,\mathrm{d}\tau\,,$$

where

$$U(t,\lambda) = \begin{pmatrix} u(t,\lambda) \\ v(t,\lambda) \end{pmatrix}, \qquad F(t,u,v) = \begin{pmatrix} v \\ f(t,u,v) \end{pmatrix}.$$

Using assumption (H3) it gives:

$$\begin{aligned} \big\|U(t,\lambda)\big\| &\leqq |\lambda| + \int_0^t \sqrt{\big|v(\tau,\lambda)\big|^2 + \big|f(\tau,u(\tau,\lambda),v(\tau,\lambda))\big|^2}\,\mathrm{d}\tau \\ &\leqq |\lambda| + \int_0^t \sqrt{\big|v(\tau,\lambda)\big|^2 + K^2\Big(\big|u(\tau,\lambda)\big|^2 + \big|v(\tau,\lambda)\big|^2\Big)}\,\mathrm{d}\tau \end{aligned}$$

$$\leqq |\lambda| + \int\limits_0^t \sqrt{(1+K^2)\left(\left|u(\tau,\lambda)\right|^2 + \left|v(\tau,\lambda)\right|^2\right)}\,\mathrm{d}\tau$$
$$\leqq |\lambda| + \sqrt{(1+K^2)}\int\limits_0^t \left\|U(\tau,\lambda)\right\|\,\mathrm{d}\tau\,.$$

By the well-known Gronwall's Theorem we finally get:

$$\frac{\left\|U(t,\lambda)\right\|}{|\lambda|} \leqq \mathrm{e}^{\int_0^t \sqrt{(1+K^2)}d\tau} \leqq \mathrm{e}^{T\sqrt{(1+K^2)}} =: M\,,$$

which implies

$$\forall\,\lambda \in (\Lambda_1,\Lambda_2)^0:\ \frac{\|S_\lambda\|_1}{|\lambda|} \leqq M\,.$$

This concludes the proof of the lemma. □

2. Lower bound of the number of solutions

In this paragraph we will present additional conditions on function f in BVP (1), (2), which together with assumptions (H1)–(H3) give a lower bound of the number of its solutions. We will use the technique of variational solutions (see Definitions 2.1, 2.2 later). To put it simply, we will show connection between the number of zeros of variational solutions and the number of solutions of (1), (2).

In this section we take $\Lambda_1 = -\infty$ and $\Lambda_2 = \infty$. We will also suppose standard assumptions ((H1), (H2)) on f and in addition:

(H4) There is $g\colon\ [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ continuous on its domain locally Lipschitz and positively homogeneous[2] in $\vec{x} := (x,y)$ which fulfils the following property:

$$\lim_{\|\vec{x}\|\to 0} \frac{\left|f(t,\vec{x}) - g(t,\vec{x})\right|}{\|\,\vec{x}\,\|} = 0 \qquad \text{uniformly in} \quad t\in[0,T]\ .$$

Definition 2.1. A solution $h_r\colon\ [0,T]\to\mathbb{R}$ ($h_l\colon\ [0,T]\to\mathbb{R}$) of variational problem:

$$x'' = g(t,x,x')\,, \tag{4}$$
$$x(0) = 0\,,\qquad x'(0) = 1\,, \tag{5}$$
$$(x(0) = 0\,,\qquad x'(0) = -1)\,, \tag{5'}$$

where function g fulfils conditions of (H4), will be denoted as the *right* (*left*) 0-*variational solution* of SP (1), (3).

[2]Positive homogeneity of g means: $\forall\,\lambda>0$: $g(t,\lambda\vec{x}) = \lambda g(t,\vec{x})$ for all $(t,\vec{x})\in[0,T]\times\mathbb{R}^2$.

Remark 2.1. If the function $f(t,\vec{x}) = f(t,x,y)$ has continuous partial derivatives by x and y in a neighborhood of point $(t,0,0)$ for all $t \in [0,T]$, then we can use $g(t,x,y) = \frac{\partial f}{\partial x}(t,0,0)x + \frac{\partial f}{\partial y}(t,0,0)y$.

We recall that positive homogeneity and continuity of g from (H4) imply:

$$\exists\, A > 0 \;\; \forall\, t \in [0,T] \;\; \forall\, \vec{x} \in \mathbb{R}^2 : \; \bigl|g(t,\vec{x})\bigr| \leqq A\|\,\vec{x}\,\| \,.$$

Let a solution h_r of problem (4), (5) be defined on $[0,T_1)$ for $T_1 < T$. Then for arbitrary $t_1, t_2 \in (0,T_1)$ we have:

$$\bigl|h_r'(t_1) - h_r'(t_2)\bigr| \leqq \int_{t_2}^{t_1} \bigl|g\bigl(t, h_r(t), h_r'(t)\bigr)\bigr| \,\mathrm{d}t \leqq A\|h_r\|_1|t_1 - t_2|.$$

Using the same technique as in Lemma 1.2 we get:

$$\exists\, N > 1 \colon \; \|h_r\|_1 \leqq N \;\; \wedge \;\; \bigl|h_r'(t_1) - h_r'(t_2)\bigr| \leqq AN|t_1 - t_2| \,.$$

Hence there exists $\lim\limits_{t\to T_1} h_r'(t) < \infty$ and also $\lim\limits_{t\to T_1} h_r(t) < \infty$. This implies the extensibility of h_r to point T_1. Since g is continuous and locally Lipschitz it can be extended on the whole interval $[0,T]$ in a unique way. A similar conclusion holds for h_l. Hence functions h_r and h_l (from Definition 2.1) are well-defined.

Lemma 2.1. *Under* (H3), (H4) *and standard assumptions on the function f the shooting function S fulfils the following properties:*

(i) $\lim\limits_{\lambda\to 0+} \bigl\|\frac{S_\lambda}{|\lambda|} - h_r\bigr\|_1 = 0$;

(ii) $\lim\limits_{\lambda\to 0-} \bigl\|\frac{S_\lambda}{|\lambda|} - h_l\bigr\|_1 = 0$.

P r o o f . We will prove only the case (i) where $\lambda > 0$. The case (ii) can be proved analogously.

Let us denote $v_\lambda(t) = \frac{S(t,\lambda)}{|\lambda|} - h_r(t)$ for $t \in [0,T]$ and $\lambda \in (0,\infty)$. We can see that the function $v_\lambda(t) = \frac{S_\lambda(t)}{|\lambda|} - h_r(t)$ fulfils the following equation for all $\lambda \in (0,\infty)$ and $t \in [0,T]$:

$$v_\lambda'' = \frac{f\,(t, S_\lambda, S_\lambda')}{|\lambda|} - g\,(t, h_r, h_r') \,.$$

Due to property (iii) of the shooting function S we can rewrite the previous equation in a form:

$$v_\lambda'' = \frac{g(t,S_\lambda,S_\lambda')}{|\lambda|} - g(t,h_r,h_r') + \frac{\|S_\lambda\|_1}{|\lambda|}\,\frac{f(t,S_\lambda,S_\lambda') - g(t,S_\lambda,S_\lambda')}{\|S_\lambda\|_1}.$$

Using positive homogeneity of function g we have:

$$v_\lambda'' = \tilde{g}(t, v_\lambda, v_\lambda') + H_\lambda(t)\,, \tag{6}$$

where
$$\tilde{g}(t, v_\lambda, v'_\lambda) = g(t, v_\lambda + h_r, v'_\lambda + h'_r) - g(t, h_r, h'_r)$$
and
$$H_\lambda(t) = \frac{\|S_\lambda\|_1}{|\lambda|} \frac{f(t, S_\lambda, S'_\lambda) - g(t, S_\lambda, S'_\lambda)}{\|S_\lambda\|_1}.$$
It is easy to see that function v_λ fulfils the following initial properties for $\lambda > 0$:
$$v_\lambda(0) = 0\,, \qquad v'_\lambda(0) = \frac{\lambda}{|\lambda|} - h'_r(0) = 0\,. \tag{7}$$
One can see that (by (H4)) function $\tilde{g}$ is continuous on $[0,T] \times \mathbb{R}^2$ and locally Lipschitz in 2nd and 3rd variable. Hence $v^0_\lambda \equiv 0$ is the unique solution of IVP (6), (7) with perturbation $H_\lambda(t) \equiv 0$. If we show that $\lim_{\lambda\to 0+} \big|H_\lambda(t)\big| = 0$ uniformly in $t \in [0,T]$, then by [BL, p. 119, Lemma 2.6.4] we get:
$$\lim_{\lambda\to 0+} \|v_\lambda\|_1 = 0.$$
By Lemma 1.2 we have such $M \in (1,\infty)$ that $\frac{\|S_\lambda\|_1}{|\lambda|} \leqq M$ for all $\lambda \in (0,\infty)$ and from property (i) of the shooting function we also know that:
$$\lim_{\lambda\to 0+} \|S_\lambda\|_1 = 0,$$
which together with a limit property of function g (in (H4)) finally gives:
$$\lim_{\lambda\to 0+} \big|H_\lambda(t)\big| = 0 \qquad \text{uniformly in} \quad t \in [0,T]\,.$$
Hence Lemma 2.1 is proved. □

Let the function f fulfil the following assumption:

(H5) There is $G\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ continuous on $[0,T] \times \mathbb{R}^2$ locally Lipschitz and positively homogeneous in $\vec{x} := (x,y) \in \mathbb{R}^2$ which fulfils the following property:
$$\lim_{\|\vec{x}\|\to\infty} \frac{\big|f(t,\vec{x}) - G(t,\vec{x})\big|}{\|\,\vec{x}\,\|} = 0 \qquad \text{uniformly in} \quad t \in [0,T]\,.$$

Definition 2.2. A solution $z_r\colon [0,T] \to \mathbb{R}$ ($z_l\colon [0,T] \to \mathbb{R}$) of the problem:
$$z'' = G(t,\, z, z')\,, \tag{8}$$
$$z(0) = 0\,, \qquad z'(0) = 1\,, \tag{9}$$
$$(z(0) = 0\,, \qquad z'(0) = -1)\,, \tag{9'}$$
where the function G fulfils conditions of (H5), will be denoted as the *right* (*left*) ∞*-variational solution* of SP (1), (3).

Remark 2.2. Let us suppose $f = f(t,x)$ and let there exist numbers $f_\infty, f_{-\infty} \in \mathbb{R}$ such that:

$$\lim_{x\to\infty} \frac{f(t,x)}{x} = f_\infty \quad \text{and} \quad \lim_{x\to-\infty} \frac{f(t,x)}{x} = f_{-\infty} \qquad \text{uniformly in} \quad t \in [0,T]\,.$$

Then we can take $G(x) := f_\infty x^+ - f_{-\infty} x^-$ which fulfils (H5).[3]

Positive homogeneity and continuity of G from (H5) implies:

$$\exists\, B > 0 \ \ \forall\, t \in [0,T] \ \ \forall\, \vec{x} \in \mathbb{R}^2 : \ |G(t,\vec{x})| \leqq B \|\, \vec{x}\, \|$$

which together with locally Lipschitz condition of G gives the extensibility of the solution of IVP (8), (9) and (8), (9') on the whole interval $[0,T]$ in a unique way (see Remark 2.1). Hence functions z_r and z_l are well-defined.

Lemma 2.2. *Under* (H3), (H5) *and standard assumptions on the function f the shooting function S fulfils the following properties:*

(i) $\lim\limits_{\lambda\to\infty} \left\| \frac{S_\lambda}{|\lambda|} - z_r(t) \right\|_1 = 0$;

(ii) $\lim\limits_{\lambda\to-\infty} \left\| \frac{S_\lambda}{|\lambda|} - z_l(t) \right\|_1 = 0$.

P r o o f . The proof is the same as that of Lemma 2.1, but instead of (H4) we use (H5) and instead of the fact that $\lim\limits_{\lambda\to 0} \|S_\lambda\|_1 = 0$ we use:

$$\lim_{|\lambda|\to\infty} \|S_\lambda\|_1 = \infty,$$

which follows from [BL, p. 118, Lemma 2.6.3] or [Kr, p. 179, Lemma 15.1]. $\square$

Further we define variational indices for estimation of number of solutions for BVP (1), (2).

Definition 2.3. Let h_r (h_l) be the right (left) 0-variational solution of SP (1), (3). The index i_r^0 (i_l^0) of problem (1), (2), (3) will denote the number of zeros of h_r (h_l) in interval $(0,\pi)$.

Let z_r (z_l) be the right (left) ∞-variational solution of SP (1), (3). The index i_r^∞ (i_l^∞) of problem (1), (2), (3) will denote the number of zeros of z_r (z_l) in interval $(0,\pi)$.

Next we define additional adjusting indices of problem (1), (2), (3):

$$\delta_l^0 = \begin{cases} 1, & \text{if } h_l(\pi) = 0\,, \\ 0, & \text{if } h_l(\pi) \neq 0\,, \end{cases} \qquad \text{and} \qquad \delta_r^0 = \begin{cases} 1, & \text{if } h_r(\pi) = 0\,, \\ 0, & \text{if } h_r(\pi) \neq 0\,, \end{cases}$$

$$\delta_l^\infty = \begin{cases} 1, & \text{if } z_l(\pi) = 0\,, \\ 0, & \text{if } z_l(\pi) \neq 0\,, \end{cases} \qquad \text{and} \qquad \delta_r^\infty = \begin{cases} 1, & \text{if } z_r(\pi) = 0\,, \\ 0, & \text{if } z_r(\pi) \neq 0\,. \end{cases}$$

[3] Where $x^+ := \max\{x,0\}$ and $x^- := \max\{-x,0\}$.

Finally the *left variational index* I_l and the *right variational index* I_r of problem (1), (2), (3) are defined as follows:

$$I_l = \begin{cases} i_l^0 - i_l^\infty - \delta_l^\infty, & \text{if } i_l^0 > i_l^\infty\,, \\ i_l^\infty - i_l^0 - \delta_l^0, & \text{if } i_l^\infty > i_l^0\,, \\ 0\,, & \text{if } i_l^\infty = i_l^0\,, \end{cases} \quad \text{and} \quad I_r = \begin{cases} i_r^0 - i_r^\infty - \delta_r^\infty\,, & \text{if } i_r^0 > i_r^\infty\,, \\ i_r^\infty - i_r^0 - \delta_r^0, & \text{if } i_r^\infty > i_r^0\,, \\ 0\,, & \text{if } i_r^\infty = i_r^0\,. \end{cases}$$

Note. On an example we will try to explain the meaning of variational indices and its connection to root functions. Let the right 0-variational solution h_r have $k \in \mathbb{N}$ zeros in $(0,\pi)$ and let $h_r(\pi) \neq 0$. Further let the right ∞-variational solution z_r have $k-1$ zeros in $(0,\pi)$ and let $z_r(\pi) \neq 0$. Then we have defined (see Lemma 2.1) k right root functions $\{t_r^n\}_1^k$ smaller than π near $\lambda = 0$. If one of them passes through line $t = \pi$ in point $\lambda_1 > 0$, then we have $S_{\lambda_1}\big(t_r(\lambda_1)\big) = 0$ which means that shot S_{λ_1} is a non-trivial solution of BVP (1), (2). Let all these right root functions stay below the line $t = \pi$. Then by Theorem 1.1 and Lemma 2.2 there should exist:

$$\lim_{\lambda\to\infty} t_r^n(\lambda) = T_n \in [0,\pi]\,, \qquad n = 1,\dots,k,$$

where $\{T_n\}_1^k$ and $T_0 = 0$ are indeed zeros of z_r in $[0,\pi]$ (T_0 is a limit of the trivial root function). Since z_r has only single zeros in $[0,\pi]$ we know that all $\{T_n\}_1^k$ are different from each other and greater than $T_0 = 0$ which is in contradiction with our assumption on number of zeros of z_r. Hence there is (at least) one right root function (the greatest from t_r^n) which must pass through line $t = \pi$. Therefore BVP (1), (2) has at least $I_r = k-(k-1)-0 = 1$ non-trivial solutions with $x'(0) > 0$.

In the case $z_r(\pi) = 0$ there need not be any solution, because there could be a root function denoted as $\tilde{t}_r$ which fulfil: $\lim\limits_{\lambda\to\infty} \tilde{t}_r(\lambda) = \pi$ and $\tilde{t}_r(\lambda) < \pi$ for $\lambda \in (0,\infty)$. Hence we may say that there are at least $I_r = k - (k-1) - 1 = 0$ non-trivial solutions.

In the case when $z_r(\pi) \neq 0$ and $h_r(\pi) = 0$, we do not know whether there are k right root functions below π in the close right neighbourhood of $\lambda = 0$ or there are $k+1$ of them. Indeed, there could be $\tilde{t}_r$ which fulfil: $\lim\limits_{\lambda\to 0} \tilde{t}_r(\lambda) = \pi$ and $\tilde{t}_r(\lambda) > \pi$ for $\lambda \in (0,\infty)$. Hence this zero of h_r need not give a next solution and we may just say that there is at least $I_r = k - (k-1) - 0 = 1$ solution.

In other words, zero of variational solution in a right boundary π can steal one solution but need not give another one. Therefore the right (left) variational index, which estimates a lower bound of the number of solutions of BVP (1), (2) with $x'(0) > 0$ ($x'(0) < 0$), must be defined in such a way.

Remark 2.3. If we can choose G as in Remark 2.2, then indices i_r^∞, i_l^∞ of problem (1), (2), (3) can be computed via the position of $(f_\infty, f_{-\infty})$ with respect to the Fučík spectrum of equation:

$$x'' = f_\infty x^+ - f_{-\infty} x^-, \qquad x(0) = 0.$$

Hence if $f_\infty, f_{-\infty} \leqq -1$, we can express adjusted right and left ∞-variational indices in the following form:

$$i_r^\infty + \delta_r^\infty = \max\left(2\left[\frac{\sqrt{|f_\infty||f_{-\infty}|}}{\sqrt{|f_\infty|}+\sqrt{|f_{-\infty}|}}\right],\ 2\left[\frac{(\sqrt{|f_\infty|}-1)\sqrt{|f_{-\infty}|}}{\sqrt{|f_\infty|}+\sqrt{|f_{-\infty}|}}\right]+1\right),$$
$$i_l^\infty + \delta_l^\infty = \max\left(2\left[\frac{\sqrt{|f_\infty||f_{-\infty}|}}{\sqrt{|f_\infty|}+\sqrt{|f_{-\infty}|}}\right],\ 2\left[\frac{(\sqrt{|f_{-\infty}|}-1)\sqrt{|f_\infty|}}{\sqrt{|f_\infty|}+\sqrt{|f_{-\infty}|}}\right]+1\right).$$

If $f_\infty, f_{-\infty} > -1$ we have indices:

$$i_r^\infty + \delta_r^\infty = 0\,,$$
$$i_l^\infty + \delta_l^\infty = 0\,.$$

Here $[\cdot]$ means the integer part of a number. (For more details see the proof of [FK, p. 278, Lemma 35.4].)

Lemma 2.3. *Indices I_r and I_l are finite for every function g (resp. G) satisfying conditions of* (H4) (*resp.* (H5)).

Proof. Let there exist $t_1 \in (0, T]$ — a point of accumulation of zeros of function h_r. It is easy to see that t_1 is also a point of accumulation of zeros of function h_r'. Then by Remark 2.1 this solution of (4), (5) can be smoothly extended through t_1 up to T. It means:

$$\lim_{t\to t_1} h_r(t) = 0 \qquad \text{and} \qquad \lim_{t\to t_1} h_r'(t) = 0,$$

further from positive homogeneity and continuity of g it follows that $\lim_{t\to t_1} h_r''(t) = 0$. Hence by uniqueness of the solution of (4), (5) we know that function $h_r(t) \equiv 0$ for $t \in [t_1, T]$ and h_r is different from the solution $x(t) \equiv 0$ of (4) with the same initial conditions $x(t_1) = 0$, $x'(t_1) = 0$. This is a contradiction, since (H4) (continuity and the locally Lipschitz condition of g) implies the global uniqueness of such IVP on $[0, T]$. Since $t = 0$ is an isolated zero of h_r, there can be only finite number of zeros of h_r on compact $[0, T]$. $\square$

Now we are able to present and prove the main theorem of this paragraph.

Theorem 2.1. *Let assumptions* (H1)–(H5) *on the function* f *hold. Then BVP* (1), (2) *has at least* I_r *non-trivial solutions with* $x'(0) > 0$ *and at least* I_l *non-trivial solutions with* $x'(0) < 0$.

P r o o f . We prove only the first part of this statement. The second part can be proved analogously.

By Lemma 2.1 there exists such $\varepsilon > 0$ that shot S_ε has at most $i_r^0 + \delta_r^0$ and at least i_r^0 zeros in $(0, \pi]$. It means there are exactly $i_0 \in \{i_r^0,\, i_r^0 + \delta_r^0\}$ non-trivial right root functions defined in the right neighbourhood of $\lambda = \varepsilon$ and not greater than $t = \pi$. By Lemma 2.2 there exists such $\Lambda \gg 1$ $(\Lambda < \infty)$ that shot $S_\Lambda(\cdot)$ has at most $i_r^\infty + \delta_r^\infty$ and at least i_r^∞ zeros in $(0, \pi]$ and there are exactly $i_\infty \in \{i_r^\infty,\, i_r^\infty + \delta_r^\infty\}$ non-trivial right root functions defined in the left neighbourhood of $\lambda = \Lambda$ and not greater than $t = \pi$. Property (P3) (see Theorem 1.1) implies that every right root function t_r, taken in rectangle $[\varepsilon, \Lambda] \times [0, \pi]$, is either defined on the whole interval $[\varepsilon, \Lambda]$ or there exists $\lambda_1 \in (\varepsilon, \Lambda)$ such that $\lim\limits_{\lambda \to \lambda_1} t_r(\lambda) = \pi$. It means that shot S_{λ_1} is a solution of BVP (1), (2).

Root functions cannot intersect among themselves (see (P1)) and they are greater than $t_R \equiv 0$ on compact $[\varepsilon, \Lambda]$ (see (P2)). Hence there are at most $\min\{i_0, i_\infty\}$ non-trivial right root functions smaller than π on interval $[\varepsilon, \Lambda]$. Anyway, there are at least I_r $(\leqq |i_0 - i_\infty|)$ non-trivial right root functions intersecting the line $t = \pi$ (each one for different $\lambda > 0$) and giving new non-trivial solutions of BVP (1), (2) with $x'(0) > 0$. It concludes the first part of the proof of the theorem.

Note. We do not know if there is a shot S_ε $(\varepsilon \in (0,1))$ with the same number of zeros as h_r in the interval $(0, \pi]$ unless the strict equal monotonicity of right root functions is fulfilled on the interval $(0,1)$ (similarly in case with S_Λ, $\Lambda \gg 1$). Therefore generally we can say there are only at least I_r solutions (with $x'(0) > 0$) and not $|\, i_0 - i_\infty|$, which can be greater by 1 than I_r.

Actually in the case $f(t, x, x') = -x$ we have one $t_r \equiv \pi$ on $\mathbb{R}^+$ and the claim of Theorem 2.1 gives nothing, even though there are infinitely many solutions of BVP (1), (2) in the form: $x(t) = \lambda \sin t$ for $\lambda \in \mathbb{R}$. But when, e.g., $f(t, x, x') = -\frac{3}{2}x$, the claim of Theorem 2.1 gives the "existence" of 0 non-trivial solutions, which is true because there is no non-trivial solution of such BVP. □

Remark 2.4. One can verify that continuity of f and assumptions (H4), (H5) imply assumption (H3).

Corollary 2.1. *Let assumptions of* Theorem 2.1 *hold and let there be* $\lambda_0 > 0$ *such that shot* S_{λ_0} *of SP* (1), (3) *fulfils:*

$$\forall\, t \in (0, \pi] : \ S_{\lambda_0}(t) > 0.$$

Then BVP (1), (2) *has at least* $i_r^0 + i_r^\infty$ *non-trivial solutions with* $x'(0) > 0$.

P r o o f . From Lemma 2.1 and Lemma 2.2 we have at least i_r^0 (i_r^∞) non-trivial right root functions smaller than π sufficiently close to point $\lambda = 0$ ($\lambda = \infty$). Due to the existence of solution (1), (3) — S_{λ_0}, which has no zero in $(0,\pi]$, we can see that there is no non-trivial right root function defined on the whole interval $(0,\infty)$ and smaller than π. Hence all i_r^0 (i_r^∞) non-trivial right root functions, which start (end) below line $t = \pi$, intersect it (each one for different $\lambda > 0$) and give at least $i_r^0 + i_r^\infty$ non-trivial solutions of BVP (1), (2) with $x'(0) > 0$ (see proof of Theorem 1.1). So this corollary is proved. □

Remark 2.5. A similar corollary can be formulated for $\lambda_0 < 0$.

Corollary 2.2. *Let assumptions of* Theorem 2.1 *hold and function* f *in BVP* (1), (2) *fulfil the following assumption:*

$$\exists\, x_0 > 0 \;\; \forall\, t \in [0,\pi]:\;\; f(t, x_0, 0) = 0.$$

Then BVP (1), (2) *has at least* $i_r^0 + i_r^\infty$ *non-trivial solutions with* $x'(0) > 0$.

P r o o f . It is obvious that we use Corollary 2.1 to prove this one. Hence we have to show the existence of $\lambda_0 > 0$ which allows us to use it.

Let the opposite hold. It means:

$$\forall\, \lambda > 0 \;\; \exists\, t_\lambda \in (0,\pi] \;\; \forall\, t \in (0, t_\lambda):\;\; S_\lambda(t) > 0 \;\;\wedge\;\; S_\lambda(t_\lambda) = 0. \tag{10}$$

Then for all $\lambda > 0$ there is $\tilde{t}_\lambda \in (0, t_\lambda)$ — maximum of S_λ on the interval $(0, t_\lambda)$ where $S'_\lambda(\tilde{t}_\lambda) = 0$. From property (i) in Lemma 1.1 and the fact that $S_0(t) = S(t,0) \equiv 0$ we know:

$$\lim_{\lambda\to 0+} \left|S_\lambda(\tilde{t}_\lambda)\right| = 0.$$

By [BL, p. 118, Lemma 2.6.3] we have:

$$\lim_{\lambda\to\infty} \sqrt{S_\lambda(\tilde{t}_\lambda)^2 + S'_\lambda(\tilde{t}_\lambda)^2} = \infty.$$

Hence by property (i) in Lemma 1.1 of shooting function S the mapping $M(\lambda) := S_\lambda(\tilde{t}_\lambda)$ is a continuous surjection from $(0,\infty)$ to $(0,\infty)$ and therefore we have $\lambda_0 > 0$ which fulfils:

$$S_{\lambda_0}(\tilde{t}_{\lambda_0}) = x_0 \qquad \text{and} \qquad S'_{\lambda_0}(\tilde{t}_{\lambda_0}) = 0.$$

It implies $f\left(\tilde{t}_{\lambda_0}, S_{\lambda_0}(\tilde{t}_{\lambda_0}), S'_{\lambda_0}(\tilde{t}_{\lambda_0})\right) = 0$. One can see that function $\tilde{x}(t) \equiv x_0$ for $t \in \left(\tilde{t}_{\lambda_0}, \pi\right]$ is a solution of (1). If we also define $\tilde{x} = S_{\lambda_0}$ on interval $\left[0, \tilde{t}_{\lambda_0}\right]$ we have a solution of (1), (3) on the interval $[0,\pi]$. From (H2) (uniqueness of solutions of SP (1), (3)) it follows that $S_{\lambda_0}(t) = \tilde{x}(t)$ for $t \in [0,\pi]$ and so S_{λ_0} fulfils:

$$\forall\, t \in (0,\pi]:\;\; S_{\lambda_0}(t) > 0,$$

which is a contradiction with (10). Hence by Corollary 2.1, this one is also proved. □

Now we present a simple example of nonlinear BVP which fulfils assumptions of Theorem 2.1.

Example 2.1. Let us have the following BVP:

$$x'' = -m^2 \sin x - n^2 \frac{2}{\pi} |x| \arctan x \,, \qquad m, n \in \mathbb{N},$$
$$x(0) = 0 \,, \quad x(\pi) = 0.$$

Before we use Theorem 2.1, we have to verify assumptions (H1)–(H5) for $\lambda \in \mathbb{R}$ and $T > \pi$.

It is easy to see that function $f(x) = -m^2 \sin x - n^2 \frac{2}{\pi} |x| \arctan x$ is continuous and locally Lipschitz on $\mathbb{R}$, fulfils assumption $f(0) = 0$ and (H3) (for $K = m^2 + n^2$), which imply (H1) and (H2) for $\lambda \in \mathbb{R}$ an $T = \infty$. Because $f \in \mathcal{C}^1(\mathbb{R})$, we can take $g(x) := f'(0)x = -m^2 x$ in (H4) (see Remark 2.1). If we denote $G(x) := -n^2 x^+ - (-n^2) x^- = -n^2 x$ (see Remark 2.2), we have:

$$\lim_{\sqrt{x^2+y^2} \to \infty} \frac{\left| n^2 x - m^2 \sin x - n^2 \frac{2}{\pi} |x| \arctan x \right|}{\sqrt{x^2 + y^2}} = 0,$$

hence assumption (H5) holds, too.

Solving left and right (linear) variational problems from Definitions 2.1 and 2.2 (for our functions g, G) we get the following solutions:

(i) $h_r(t) = \frac{1}{m} \sin mt$ — solution of (4), (5),
(ii) $z_r(t) = \frac{1}{n} \sin nt$ — solution of (8), (9),
(iii) $h_l(t) = -\frac{1}{m} \sin mt$ — solution of (4), (5'),
(iv) $z_l(t) = -\frac{1}{n} \sin nt$ — solution of (8), (9').

Hence indices of our problem are: $i_r^0 = m-1$, $i_l^0 = m-1$, $i_r^\infty = n-1$, $i_l^\infty = n-1$ and $\delta_l^0 = 1$, $\delta_r^0 = 1$, $\delta_l^\infty = 1$ $\delta_r^\infty = 1$ (see Definition 2.3). Using Theorem 2.1 we can say that our BVP has at least $I_r = \max\{|m-n|-1, 0\}$ non-trivial solutions with $x'(0) > 0$ and at least $I_l = \max\{|m-n|-1, 0\}$ non-trivial solutions with $x'(0) < 0$.

It is easy to verify that $f(\pi) < 0$ and $f(4.5) > 0$ when $m = 2n$. It implies that there exists such $x_0 \in (\pi, 4.5)$ that $f(x_0) = 0$. Hence by Corollary 2.2 there exist at least $3n - 2$ non-trivial solutions of our BVP with $x'(0) > 0$ in case $m = 2n$. Function f is odd so that $f(-x_0) = -f(x_0) = 0$ and there exist also at least $3n - 2$ non-trivial solutions of our BVP with $x'(0) < 0$ in case $m = 2n$.

3. Conclusion

We showed a connection between the number of zeros of variational problems and the number of solutions (Theorem 2.1) (i.e., the lower bound of the number of solutions of BVP (1), (2) depends on the number of root functions taking values under right boundary π near points $\lambda \in \{-\infty, 0, \infty\}$). To get an upper bound of number of solutions we have to put another assumptions on the function f. Actually for arbitrary finite $k > 0$ we are able to construct a function $f(x)\colon \mathbb{R} \to \mathbb{R}$ fulfilling assumptions of Theorem 2.1 such that BVP (1), (2) has k non-trivial solutions and its variational indices are $I_r = 0 = I_l$. We outline a procedure of such construction, because it shows us how the behaviour of root functions depends on the derivative of function f.

It is easy to see that for $f(x) := f_0(x) = -\frac{1}{2}x$ there is no root function of SP (1), (3) which takes some values under π. Now we break f_0 in $x = 1$ such a way that:

$$f_1(x) := \begin{cases} f_0(x)\,, & \text{for } |x| \leqq 1\,, \\ -\frac{3}{2}x + \operatorname{sgn}(x)\,, & \text{for } |x| > 1\,, \end{cases}$$

and indices of problem (1), (2), (3) with $f = f_1$ are:

$$i_r^\infty = i_l^\infty = 1\,; \quad i_r^0 = i_l^0 = 0\,; \quad \delta_r^\infty = \delta_l^\infty = \delta_r^0 = \delta_l^0 = 0.$$

One can see (Lemma 2.1 and 2.2) that there is one root function which takes values under π, close to $\lambda \in \{-\infty, \infty\}$ for SP (1), (3) with $f = f_1$ and there is not a root function with such property near $\lambda = 0$. Hence by Theorem 2.1 there exists shot S_{λ_1} ($\lambda_1 > 0$) which is a non-trivial solution of BVP (1), (2) where $f = f_1$. (Note: $S_{-\lambda_1} = -S_{\lambda_1}$ is also a solution.)

Let us denote $M := \max\Big\{ \sup\limits_{t \in [0,\pi]} S_{\lambda_1}(t), 1 \Big\}$. Now we break f_1 and get:

$$f_2(x) = \begin{cases} f_1(x)\,, & \text{for } |x| \leqq M\,, \\ -\frac{1}{2}x - \operatorname{sgn}(x)(M-1)\,, & \text{for } |x| > M\,. \end{cases}$$

It is obvious that shots S_{λ_1} and $S_{-\lambda_1}$ stay solutions of BVP (1), (2) where $f = f_2$, which variational indices are $I_r = 0 = I_l$. Using this procedure arbitrary many times we can construct the function f with required number of solutions but with indexes $I_r = 0 = I_l$.

From this procedure of construction of appropriate f we see that the behaviour of root functions depends on the behaviour of $f' = \frac{\partial f}{\partial x}$. Simply said if $f'(x) < -n^2$, $n \in \mathbb{N}$, for $|x| \in I \subset (0, \infty)$ where I is a sufficiently large interval, then there exists shot S_{λ_0} with n zeros in $(0, \pi)$ (i.e., there are at least n root functions smaller than π in $\lambda = \lambda_0$). This consideration implies a question about the number of solutions when $\lim\limits_{|x| \to \infty} f'(x) = -\infty$, which is answered by [FK, p. 293, Theorem 37.2] giving infinitely many solutions of such BVP (1), (2).

At the end we would like to remind the possibility to formulate theorems similar to Theorem 2.1 for Neumann's condition. Another interesting problem is a generalization of the *root functions method* for nth order BVP. Unfortunately, there is one big barrier we have to deal with — multiple zeros of shooting function.

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