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PROBLEMS OF AN ADDITIONAL EXPERIMENT

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Dedicated to Professor B. Riečan on the occasion of his 70th birthday

ABSTRACT. In some situations estimates of unknown parameters must be corrected by additional measurements. It is in principle no problem to calculate the corrected estimates, however, it is of more interest to find formulae for correction itself. The formulae enable us to design an additional experiment and to judge its usefulness.

The aim of the paper is to find such formulae for several situations.

1. Introduction

An influence of additional experiment on estimators is interesting not only from pure mathematical point of view but from practical requirement of many professions, e.g. geodesy, physics, chemistry, technical science, etc.

The following example can serve as a motivation. Coordinates of several points of the Earth surface had been determined by a measurement for a mapping purpose. After some time either the value of the distance between two chosen points, or the azimuth between them must be known more precisely than the original measurement offers (e.g. for a construction of a bridge, a tunnel, etc.). Therefore an additional measurement must be realized. This new measurement together with the original one produce new, more precise, coordinates of the points. In practice, it is suitable to calculate directly differences among the original and new coordinates instead the new coordinates themselves.

Besides such kind of problems also pure mathematical interest leads to problem of an additional experiment, cf. the third fundamental theorem of the least square theory (for more detail see Lemma 3.6).

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The problem of additional experiments is closely related to problems often referred to as "updating in regression estimation" or "influential observations in regression". To design properly an additional experiment it must be taken into account a knowledge on the last mentioned influential observations. However this class of problems is not investigated in the paper.

Our aim is to find explicit corrections of model parameters estimators and to study a problem of unknown variance components.

2. Notation and auxiliary statements

Let an n-dimensional random vector $\boldsymbol{Y}$ with an affiliated class of probability measures $\mathcal{F} = \{P_\Theta : \ \boldsymbol{\Theta} \in \underline{\Theta}\} \ll \mu$ be under consideration. Here P_Θ is a probability measure parametrized by the vector parameter $\boldsymbol{\Theta}$, $\underline{\Theta}$ is a set of admissible values of the vector $\boldsymbol{\Theta}$, μ is a dominating σ-finite measure, $\mathrm{d}P_\Theta/\mathrm{d}\mu = f(\cdot, \boldsymbol{\Theta})$ (the Radon-Nikodym derivative). The vector $\boldsymbol{\Theta}$ is decomposed into two vectors $\boldsymbol{\beta}$ and $\boldsymbol{\vartheta}$, i.e. $\boldsymbol{\Theta}' = (\boldsymbol{\beta}', \boldsymbol{\vartheta}')$.

The class $\mathcal{F}$ is assumed to have two properties:

(i) $(\forall \boldsymbol{\Theta} \in \underline{\Theta}\,)\Big(E_\Theta(\boldsymbol{Y}) = \int_{\mathbb{R}^n} \boldsymbol{u} f(\boldsymbol{u}, \boldsymbol{\Theta})\,\mathrm{d}\mu(\boldsymbol{u}) = \mathbf{X}\boldsymbol{\beta}\Big)$, i.e. the mean value of the vector $\boldsymbol{Y}$ does not depend on the parameter $\boldsymbol{\vartheta}$ and

(ii) $(\forall \boldsymbol{\Theta} \in \underline{\Theta})\Big(\mathrm{Var}(\boldsymbol{Y}) = \int_{\mathbb{R}^n} (\boldsymbol{u} - \mathbf{X}\boldsymbol{\beta})(\boldsymbol{u} - \mathbf{X}\boldsymbol{\beta})' f(\boldsymbol{u}, \boldsymbol{\Theta})\,\mathrm{d}\mu(\boldsymbol{u}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i = \boldsymbol{\Sigma}(\boldsymbol{\vartheta})\Big)$, i.e. the covariance matrix of the vector $\boldsymbol{Y}$ does not depend on the parameter $\boldsymbol{\beta}$.

Here $\mathbb{R}^n$ is n-dimensional Euclidean space, $\mathbf{X}$ is an $n \times k$ known matrix, $\boldsymbol{\beta}$ is an unknown k-dimensional parameter, $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_p)'$ is an unknown parameter and $\mathbf{V}_1, \dots, \mathbf{V}_p$ are given symmetric $n \times n$ matrices. In the following text it is assumed $\underline{\Theta} = \underline{\beta} \times \underline{\vartheta}$, where $\underline{\beta}$ is a linear manifold in $\mathbb{R}^k$ and $\underline{\vartheta}$ is an open set in $\mathbb{R}^p$.

Such situation will be denoted as $\boldsymbol{Y} \sim_n \big[\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}(\boldsymbol{\vartheta})\big]$, $\boldsymbol{\beta} \in \underline{\beta}$, $\boldsymbol{\vartheta} \in \underline{\vartheta}$. The notation $\boldsymbol{Y} \sim N_n\big[\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}(\boldsymbol{\vartheta})\big]$ means that $\boldsymbol{Y}$ is normally distributed.

The following two lemmas are well known and therefore they are given without proofs (in more detail cf. [13] and [15]).

Lemma 2.1. *Let $\boldsymbol{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, where the rank of the known matrix $\mathbf{X}$ is $r(\mathbf{X}) = k < n$ and the known matrix $\boldsymbol{\Sigma}$ is positive definite (p.d.). Then the best* *linear unbiased estimator (BLUE) is*

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y} \sim_k \left[\boldsymbol{\beta}, (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\right].$$

If $\boldsymbol{\Sigma}$ is of the form $\boldsymbol{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$, where $\vartheta_1, \dots, \vartheta_p$, are unknown parameters the one possibility how to estimate the vector $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_p)'$, is to use MINQUE ([15]). There are also other possibilities, e.g. REML (restricted maximum likelihood) estimator (for more detail cf. [1]). For the sake of simplicity in the following text the MINQUE is chosen for a demonstration how to proceed with results of an additional experiment.

Lemma 2.2. *Let $\boldsymbol{Y}$ be the random vector from* Lemma 2.1 *however $\boldsymbol{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$. Let $g(\boldsymbol{\vartheta}) = \boldsymbol{g}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\vartheta}$, be such function of $\boldsymbol{\vartheta}$ that $\boldsymbol{g} \in \mathcal{M}\Big(\mathbf{S}_{(M_{\mathbf{X}}\Sigma_0 M_{\mathbf{X}})^+}\Big)$. Then MINQUE (minimum norm quadratic unbiased estimator) of $g(\cdot)$ is*

$$\widehat{\boldsymbol{g}'\boldsymbol{\vartheta}} = \sum_{i=1}^{p} \lambda_i \boldsymbol{Y}'(\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_{\mathbf{X}})^+ \mathbf{V}_i (\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_{\mathbf{X}})^+ \boldsymbol{Y},$$

$$\mathbf{S}_{(M_{\mathbf{X}}\Sigma_0 M_{\mathbf{X}})^+}\boldsymbol{\lambda} = \boldsymbol{g},$$

$$\left\{\mathbf{S}_{(M_{\mathbf{X}}\Sigma_0 M_{\mathbf{X}})^+}\right\}_{i,j} = \mathrm{Tr}\left[(\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_{\mathbf{X}})^+ \mathbf{V}_i (\mathbf{M}_X \boldsymbol{\Sigma}_0 \mathbf{M}_{\mathbf{X}})^+ \mathbf{V}_j\right], \qquad i, j = 1, \dots, p.$$

Here $\mathcal{M}\Big(\mathbf{S}_{(M_{\mathbf{X}}\Sigma_0 M_{\mathbf{X}})^+}\Big) = \Big\{\mathbf{S}_{(M_{\mathbf{X}}\Sigma_0 M_{\mathbf{X}})^+}\boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^p\Big\}$, $^+$ denotes the Moore-Penrose generalized inverse ([14]) *of the matrix, $\mathbf{M}_{\mathbf{X}} = \mathbf{I} - \mathbf{P}_{\mathbf{X}}$, $\mathbf{P}_{\mathbf{X}} = \mathbf{X}\mathbf{X}^+$, $\boldsymbol{\Sigma}_0 = \sum_{i=1}^{p} \vartheta_i^{(0)} \mathbf{V}_i$, $\boldsymbol{\vartheta}^{(0)} = \big(\vartheta_1^{(0)}, \dots, \vartheta_p^{(0)}\big)'$ is an approximation of the actual value of the vector $\boldsymbol{\vartheta}$.*

If $p = 1$, then the estimator $\hat{\vartheta}$ is $\hat{\vartheta} = \boldsymbol{Y}'(\mathbf{M}_{\mathbf{X}}\mathbf{V}\mathbf{M}_{\mathbf{X}})^+\boldsymbol{Y}/(n-k) = \big(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}\big)'\mathbf{V}^{-1}\big(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}\big)/(n-k)$, where $\hat{\boldsymbol{\beta}} = \big(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}\big)^{-1}\mathbf{X}'\mathbf{V}^{-1}\boldsymbol{Y}$.

Lemma 2.3. *The estimator $\widehat{\mathbf{g}'\boldsymbol{\vartheta}}$ from* Lemma 2.2 *can be expressed as*

$$\widehat{\boldsymbol{g}'\boldsymbol{\vartheta}} = \sum_{i=1}^{p} \lambda_i (\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}})'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}), \qquad \mathbf{S}_{(M_{\mathbf{X}}\Sigma_0 M_{\mathbf{X}})^+}\boldsymbol{\lambda} = \boldsymbol{g}.$$

If $\mathbf{S}_{(M_{\mathbf{X}}\Sigma_0 M_{\mathbf{X}})^+}$ is regular, then

$$\hat{\boldsymbol{\vartheta}} = \mathbf{S}^{-1}_{(M_{\mathbf{X}}\Sigma_0 M_{\mathbf{X}})^+} \begin{pmatrix} (\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}})'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_1\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}) \\ \vdots \\ (\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}})'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_p\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}) \end{pmatrix}.$$

P r o o f. It is implied by the relationship

$$(\mathbf{M_X}\boldsymbol{\Sigma}_0\mathbf{M_X})^+\boldsymbol{Y} = \boldsymbol{\Sigma}_0^{-1}\big[\mathbf{I} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\big]\boldsymbol{Y} = \boldsymbol{\Sigma}_0^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}})$$

and by Lemma 2.2. □

In practice an iteration procedure is used for the estimation of $\boldsymbol{\vartheta}$; in the first step some value of $\boldsymbol{\vartheta}$ is chosen arbitrarily, in the second step the $\boldsymbol{\vartheta}$-MINQUE $\hat{\boldsymbol{\vartheta}}$ is chosen instead of $\boldsymbol{\vartheta}$, etc.. In the following text the notation $\boldsymbol{\vartheta}_0$-MINQUE is used, where $\boldsymbol{\vartheta}_0$ is a chosen vector in the small neighbourhood of the last iterated value of the estimator.

3. Additional experiment in nonsingular model without constraints

Let the original experiment be characterized by the model

$$\boldsymbol{Y}_1 \sim_{n_1} (\mathbf{X}_1\boldsymbol{\beta}, \boldsymbol{\Sigma}_1)\,, \qquad \boldsymbol{\beta} \in \mathbb{R}^k\,, \;\; r(\mathbf{X}_1) = k < n_1\,, \;\; \boldsymbol{\Sigma}_1 \text{ p.d.}$$

and the additional one by the model

$$\boldsymbol{Y}_2 \sim_{n_2} (\mathbf{X}_2\boldsymbol{\beta}, \boldsymbol{\Sigma}_2)\,, \qquad \boldsymbol{\beta} \in \mathbb{R}^k\,, \;\; \boldsymbol{\Sigma}_2 \text{ p.d.}$$

The vectors $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ are uncorrelated.

Let $\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)$ be the estimator based on the observation vector $\boldsymbol{Y}_1$, i.e. $\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1) = (\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\boldsymbol{Y}_1$, and $\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ be the estimator based on both vectors, i.e. it is the corrected estimator.

Lemma 3.1. *Then the BLUE $\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ based on the results of both experiments can be expressed as*

$$\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1) + \boldsymbol{k}\,,$$

where the correction $\boldsymbol{k}$ is

$$\begin{aligned} \boldsymbol{k} &= (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\big[\boldsymbol{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)\big] \\ &= (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\big[\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\widehat{\mathbf{X}_2\boldsymbol{\beta}(\boldsymbol{Y}_2)} - \mathbf{C}_2\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)\big]\,. \end{aligned}$$

Here

$$\mathbf{C}_i = \mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i\,, \quad i = 1, 2\,, \qquad \widehat{\mathbf{X}_2\boldsymbol{\beta}(\boldsymbol{Y}_2)} = \mathbf{P}_{\mathbf{X}_2}^{\Sigma_2^{-1}}\boldsymbol{Y}_2\,,$$

$$\mathbf{P}_{\mathbf{X}_2}^{\Sigma_2^{-1}} = \mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2)^{-}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}$$

and $^-$ denotes generalized inverse ([14]) *of the matrix (i.e. $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$).*

P r o o f. Since

$$\hat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) = (\mathbf{C}_1 + \mathbf{C}_2)^{-1}(\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2)$$

(with respect to Lemma 2.1), it is sufficient to use the relationships

$$\begin{aligned}(\mathbf{C}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2)^{-1} &= \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{C}_1^{-1}\,,\\ \mathbf{C}_1^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_2')^{-1} &= (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\end{aligned} \tag{1}$$

and

$$\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2 = \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2)^{-}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2\,.$$

(It is to be remarked that $\hat{\boldsymbol{\beta}}(\mathbf{Y}_2)$ need not exist, since $\mathbf{X}_2$ is a matrix which need not have full rank in columns, however the estimator $\widehat{\mathbf{X}_2\boldsymbol{\beta}(\mathbf{Y}_2)} = \mathbf{P}_{\mathbf{X}_2}^{\Sigma_2^{-1}}\mathbf{Y}_2$ exists. Here $\mathbf{P}_{\mathbf{X}_2}^{\Sigma_2^{-1}} = \mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2)^{-}\mathbf{X}'\boldsymbol{\Sigma}_2^{-1}$.) □

Remark 3.2. A measure of concordance between original and an additional experiment can be characterized either by the vector

$$\mathbf{w}_2 = \mathbf{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}(\mathbf{Y}_1)\,, \tag{2}$$

or by the vector $\widehat{\mathbf{X}_2\boldsymbol{\beta}(\mathbf{Y}_2)} - \mathbf{X}_2\hat{\boldsymbol{\beta}}(\mathbf{Y}_1)$. If the original and the additional experiment are in concordance, i.e. $E(\mathbf{w}_2) = \mathbf{0}$, then in the case of the normality of the vector $\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix}$, it must hold

$$P\Big\{\mathbf{w}_2'\big[\mathrm{Var}(\mathbf{w}_2)\big]^{-1}\mathbf{w}_2 \le \chi^2_{n_2}(0; 1-\alpha)\Big\} = 1 - \alpha\,,$$

where $\chi^2_{n_2}(0; 1-\alpha)$ is the $(1-\alpha)$-quantile of the central chi-square distribution with n_2 degrees of freedom and $\mathrm{Var}(\mathbf{w}_2) = \boldsymbol{\Sigma}_2 + \mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_2'$.

Lemma 3.3. *Since $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are uncorrelated, then*

$$\mathrm{Var}\big[\hat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)\big] = (\mathbf{C}_1 + \mathbf{C}_2)^{-1} = \mathrm{Var}\big[\hat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big] - \mathbf{K}\,,$$

where the correction matrix $\mathbf{K}$ is

$$\mathbf{K} = \mathrm{Var}\big[\hat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big]\mathbf{X}_2'\big[\mathrm{Var}(\mathbf{w}_2)\big]^{-1}\mathbf{X}_2\,\mathrm{Var}\big[\hat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big] \tag{3}$$

P r o o f. It is implied by the relationships (1)

$$\mathrm{Var}\big[\hat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big] = \mathbf{C}_1^{-1} \qquad \text{and} \qquad \mathrm{Var}(\mathbf{w}_2) = \boldsymbol{\Sigma}_2 + \mathbf{X}_2\mathbf{C}_1^{-1}\mathbf{X}_2'\,.$$

□

Thus it can be judged the influence of the additional experiment on the accuracy of the estimator $\widehat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)$ which is characterized by $\operatorname{Var}\big[\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)\big] = \mathbf{C}_1^{-1}$. Since the matrix $\mathbf{K}$ from (3) can be calculated in advance, the additional experiment can be designed in such a way that $\operatorname{Var}\big[\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\big]$ attains sufficiently small (prescribed) values.

If a single additional measurement is done, i.e. $Y_2 \sim_1 (\boldsymbol{f}'\boldsymbol{\beta}, \vartheta_2)$, then

$$\operatorname{Var}\big[\hat{\beta}(\boldsymbol{Y}_1, Y_2)\big] = \operatorname{Var}\big[\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)\big] - \mathbf{C}_1^{-1}\boldsymbol{f}\boldsymbol{f}'\mathbf{C}_1^{-1}/(\vartheta_2 + \boldsymbol{f}'\mathbf{C}_1^{-1}\boldsymbol{f}).$$

Lemma 3.4. *Let the model*

$$\begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{pmatrix} \sim N_{n_1+n_2}\left[\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}\boldsymbol{\beta}, \vartheta \begin{pmatrix} \mathbf{V}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{V}_2 \end{pmatrix}\right], \tag{4}$$

where the matrix $\mathbf{X}_1$ is of the full rank in columns and the matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ are p.d., be under consideration. Then the best estimator (i.e. unbiased and with the minimum variance) of ϑ is

$$\begin{aligned}\hat{\vartheta}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) &= (\boldsymbol{Y}_1', \boldsymbol{Y}_2')\begin{pmatrix} \boxed{\alpha\alpha}, & \boxed{\alpha\beta} \\ \boxed{\beta\alpha}, & \boxed{\beta\beta} \end{pmatrix}\begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{pmatrix}/(n_1 + n_2 - k) \\ &\sim \vartheta\chi^2_{n_1+n_2-k}(0)/(n_1 + n_2 - k) \sim_1 \big[\vartheta, 2\vartheta^2/(n_1 + n_2 - k)\big],\end{aligned}$$

where

$$\begin{aligned}\boxed{\alpha\alpha} &= \mathbf{V}_1^{-1} - \mathbf{V}_1^{-1}\mathbf{X}_1(\mathbf{H}_1 + \mathbf{H}_2)^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}, \\ \boxed{\alpha\beta} &= -\mathbf{V}_1^{-1}\mathbf{X}_1(\mathbf{H}_1 + \mathbf{H}_2)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1} = \boxed{\beta\alpha}', \\ \boxed{\beta\beta} &= \mathbf{V}_2^{-1} - \mathbf{V}_2^{-1}\mathbf{X}_2(\mathbf{H}_1 + \mathbf{H}_2)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1}, \\ \mathbf{H}_i &= \mathbf{X}_i'\mathbf{V}_i^{-1}\mathbf{X}_i, \qquad i = 1, 2.\end{aligned}$$

P r o o f. The result is an obvious transcription of Lemma 2.2 (the case $p = 1$). In [6, Theorem IV.1] it is proved that this estimator is the best one. □

Theorem 3.5. *The best estimator $\hat{\vartheta}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ from* Lemma 3.4 *can be expressed as*

$$\hat{\vartheta}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = \frac{Q_2(\boldsymbol{Y}_1, \boldsymbol{Y}_2)}{n_1 + n_2 - k} = \frac{Q_1(\boldsymbol{Y}_1)}{n_1 - k} + \kappa,$$

where

$$Q_1(\boldsymbol{Y}_1) = \boldsymbol{Y}_1' \left(\mathbf{M}_{\mathbf{X}_1}\mathbf{V}_1\mathbf{M}_{\mathbf{X}_1}\right)^+ \boldsymbol{Y}_1\,,$$

$$Q_2(\boldsymbol{Y}_1,\boldsymbol{Y}_2) = \begin{pmatrix}\boldsymbol{Y}_1\\ \boldsymbol{Y}_2\end{pmatrix}' \left[\mathbf{M}_{\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}} \begin{pmatrix}\mathbf{V}_1, & \mathbf{0}\\ \mathbf{0}, & \mathbf{V}_2\end{pmatrix} \mathbf{M}_{\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}}\right]^+ \begin{pmatrix}\boldsymbol{Y}_1\\ \boldsymbol{Y}_2\end{pmatrix}$$

$$= \begin{pmatrix}\mathbf{K}_1\boldsymbol{Y}_1\\ \mathbf{K}_2\boldsymbol{Y}_2\end{pmatrix}' \mathbf{M}_{\begin{pmatrix}\mathbf{K}_1\mathbf{X}_1\\ \mathbf{K}_2\mathbf{X}_2\end{pmatrix}} \begin{pmatrix}\mathbf{K}_1\boldsymbol{Y}_1\\ \mathbf{K}_2\boldsymbol{Y}_2\end{pmatrix},$$

$$\kappa = \frac{1}{n_1+n_2-k}\Big\{-n_2\hat{\vartheta}(\boldsymbol{Y}_1) + \boldsymbol{w}_2'\big[\mathbf{V}_2^{-1} - \mathbf{V}_2^{-1}\mathbf{X}_2(\mathbf{H}_1+\mathbf{H}_2)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1}\big]\boldsymbol{w}_2\Big\},$$

$$\mathbf{V}_1^{-1} = \mathbf{K}_1\mathbf{K}_1'\,, \qquad \mathbf{V}_2^{-1} = \mathbf{K}_2\mathbf{K}_2'$$

and $\boldsymbol{w}_2$ *is defined in* (2).

P r o o f. Let

$$\begin{aligned} \boldsymbol{v}_1(\boldsymbol{Y}_1) &= \boldsymbol{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)\,,\\ \boldsymbol{v}_1(\boldsymbol{Y}_1,\boldsymbol{Y}_2) &= \boldsymbol{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\,,\\ \boldsymbol{v}_2(\boldsymbol{Y}_1,\boldsymbol{Y}_2) &= \boldsymbol{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\,. \end{aligned}$$

Then expression for $\hat{\vartheta}(\boldsymbol{Y}_1,\boldsymbol{Y}_2)$ can be written as

$$\hat{\vartheta}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) = \frac{\big[\boldsymbol{v}_1'(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\mathbf{V}_1^{-1}\boldsymbol{v}_1(\boldsymbol{Y}_1,\boldsymbol{Y}_2) + \boldsymbol{v}_2'(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\mathbf{V}_2^{-1}\boldsymbol{v}_2(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\big]}{n_1+n_2-k}\,.$$

Since (Lemma 3.1) $\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) = \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1) + (\mathbf{H}_1+\mathbf{H}_2)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1}\boldsymbol{w}_2$,

$$\begin{aligned} \boldsymbol{v}_1'(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\mathbf{V}_1^{-1}\boldsymbol{v}_1(\boldsymbol{Y}_1,\boldsymbol{Y}_2) &= \boldsymbol{v}_1'(\boldsymbol{Y}_1)\mathbf{V}_1^{-1}\boldsymbol{v}_1(\boldsymbol{Y}_1) + \boldsymbol{w}_2'\mathbf{V}_2^{-1}\mathbf{X}_2(\mathbf{H}_1+\mathbf{H}_2)^{-1}\times\\ &\quad \times\mathbf{H}_1(\mathbf{H}_1+\mathbf{H}_2)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1}\boldsymbol{w}_2\,,\\ \boldsymbol{v}_2'(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\mathbf{V}_2^{-1}\boldsymbol{v}_2(\boldsymbol{Y}_1,\boldsymbol{Y}_2) &= \boldsymbol{w}_2'\big[\mathbf{I} - \mathbf{X}_2(\mathbf{H}_1+\mathbf{H}_2)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1}\big]'\mathbf{V}_2^{-1}\times\\ &\quad \times\big[\mathbf{I} - \mathbf{X}_2(\mathbf{H}_1+\mathbf{H}_2)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1}\big]\boldsymbol{w}_2 \end{aligned}$$

is valid. Here the equality $\boldsymbol{v}_1'(\boldsymbol{Y}_1)\mathbf{V}_1^{-1}\mathbf{X}_1 = \boldsymbol{0}$ was utilized. Since

$$\mathbf{V}_2^{-1}\mathbf{X}_2(\mathbf{H}_1+\mathbf{H}_2)^{-1}\mathbf{H}_1(\mathbf{H}_1+\mathbf{H}_2)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1} + [\mathbf{I} - \mathbf{V}_2^{-1}\mathbf{X}_2(\mathbf{H}_1+\mathbf{H}_2)^{-1}\mathbf{X}_2']\times$$
$$\times\mathbf{V}_2^{-1}[\mathbf{I} - \mathbf{X}_2(\mathbf{H}_1+\mathbf{H}_2)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1}] = \mathbf{V}_2^{-1} - \mathbf{V}_2^{-1}\mathbf{X}_2(\mathbf{H}_1+\mathbf{H}_2)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1},$$

the proof is finished. □

The influence of the additional experiment on the residual quadratic form, i.e. $\big[\boldsymbol{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)\big]'\mathbf{V}_1^{-1}\big[\boldsymbol{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1)\big]$, is characterized by the following lemma (cf. also [13, p. 157]).

Lemma 3.6 (Third fundamental theorem of the least square theory)**.** *In the model* (4)*, where* $r(\mathbf{X}_{1,(n_1,k)}) = k < n_1$*,* $\mathbf{V}_1$*,* $\mathbf{V}_2$ *are p.d., it holds that*

$$Q_1(\mathbf{Y}_1)/Q_2(\mathbf{Y}_1, \mathbf{Y}_2) \sim B\left(\tfrac{n_1-k}{2}, \tfrac{n_2}{2}\right).$$

Here $B\left(\frac{n_1-k}{2}, \frac{n_2}{2}\right)$ *means the beta distribution with parameters equal to* $\frac{n_1-k}{2}$ *and* $\frac{n_2}{2}$ *and* $Q_1(\mathbf{Y}_1)$ *and* $Q_2(\mathbf{Y}_1, \mathbf{Y}_2)$ *are defined in* Theorem 3.5.

Proof. Let $\mathbf{Z}_1$ and $\mathbf{Z}_2$ be nonsingular matrices such that $\mathbf{Z}_1\mathbf{V}_1\mathbf{Z}_1' = \mathbf{I}_{n_1,n_1}$ and $\mathbf{Z}_2\mathbf{V}_2\mathbf{Z}_2' = \mathbf{I}_{n_2,n_2}$, respectively. Then

$$\begin{aligned} Q_2(\mathbf{Y}_1, \mathbf{Y}_2) &= \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}' \left[\mathbf{M}_{\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}} \begin{pmatrix} \mathbf{V}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{V}_2 \end{pmatrix} \mathbf{M}_{\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}} \right]^+ \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{Z}_1\mathbf{Y}_1 \\ \mathbf{Z}_2\mathbf{Y}_2 \end{pmatrix}' \mathbf{M}_{\begin{pmatrix} \mathbf{Z}_1\mathbf{X}_1 \\ \mathbf{Z}_2\mathbf{X}_2 \end{pmatrix}} \begin{pmatrix} \mathbf{Z}_1\mathbf{Y}_1 \\ \mathbf{Z}_2\mathbf{Y}_2 \end{pmatrix}. \end{aligned}$$

Here

$$\mathbf{M}_{\begin{pmatrix} \mathbf{Z}_1\mathbf{X}_1 \\ \mathbf{Z}_2\mathbf{X}_2 \end{pmatrix}} = \begin{pmatrix} \mathbf{M}_{\mathbf{Z}_1\mathbf{X}_1}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{pmatrix} + \mathbf{S},$$

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{1,1}, & \mathbf{S}_{1,2} \\ \mathbf{S}_{2,1}, & \mathbf{S}_{2,2} \end{pmatrix},$$

$$\begin{aligned} \mathbf{S}_{1,1} &= \mathbf{Z}_1\mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_2'\left[\mathbf{V}_2 + \mathbf{X}_2(\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_2'\right]^{-1} \times \\ &\quad \times \mathbf{X}_2(\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{Z}_1', \\ \mathbf{S}_{1,2} &= -\mathbf{Z}_1\mathbf{X}_1(\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{X}_1 + \mathbf{X}_2'\mathbf{V}_2^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{Z}_2' = \mathbf{S}_{2,1}', \\ \mathbf{S}_{2,2} &= \mathbf{I} - \mathbf{Z}_2\mathbf{X}_2(\mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{X}_1 + \mathbf{X}_2'\mathbf{V}_2^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\mathbf{Z}_2' \end{aligned}$$

and

$$\begin{pmatrix} \mathbf{M}_{\mathbf{Z}_1\mathbf{X}_1}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{pmatrix}\mathbf{S} = \mathbf{0}, \qquad \mathbf{S}^2 = \mathbf{S}.$$

Thus

$$Q_2(\mathbf{Y}_1, \mathbf{Y}_2) = \chi^2_{n_1-k}(0) + \chi^2_{n_2}(0),$$

since $r(\mathbf{M}_{\mathbf{Z}_1\mathbf{X}_1}) = r(\mathbf{M}_{\mathbf{X}_1}) = \mathrm{Tr}(\mathbf{M}_{\mathbf{X}_1}) = n_1 - k$ and $r\left(\mathbf{M}_{\begin{pmatrix}\mathbf{Z}_1\mathbf{X}_1\\ \mathbf{Z}_2\mathbf{X}_2\end{pmatrix}}\right) = \mathrm{Tr}\left(\mathbf{M}_{\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}}\right) =$ $n_1 + n_2 - k$, $\mathrm{Tr}\left(\mathbf{M}_{\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}}\right) = \mathrm{Tr}(\mathbf{M}_{\mathbf{X}_1}) + \mathrm{Tr}(\mathbf{S})$. The random variable $\chi^2_{n_1-k}(0)$ and $\chi^2_{n_2}(0)$ are stochastically independent and thus it is sufficient to utilize the relationship

$$\frac{\chi^2_{f_1}}{\chi^2_{f_1} + \chi^2_{f_2}} \sim B\left(\tfrac{f_1}{2}, \tfrac{f_2}{2}\right).$$

□

If

$$\mathbf{Y}_1 \sim N_{n_1}(\mathbf{X}_1\boldsymbol{\beta}, \vartheta_1\mathbf{V}_1), \qquad r(\mathbf{X}_1) = k_1 < n_1, \quad \mathbf{V}_1 \text{ p.d.}$$

and

$$\mathbf{Y}_2 \sim N_{n_2}(\mathbf{X}_2\boldsymbol{\beta}, \vartheta_2\mathbf{V}_2), \qquad \mathbf{V}_2 \text{ p.d.}$$

and $\vartheta_1 \neq \vartheta_2$, then the situation is a little more complicated. The estimator of ϑ_1 in the original model is

$$\begin{aligned}\hat{\vartheta}_1(\mathbf{Y}_1) &= \left[\mathbf{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}(\mathbf{Y}_1)\right]'\mathbf{V}_1^{-1}\left[\mathbf{Y}_1 - \mathbf{X}_1\hat{\boldsymbol{\beta}}(\mathbf{Y}_1)\right]/(n_1-k)\\ &\sim \vartheta_1\chi^2_{n_1-k}(0)/(n_1-k) \sim_1 \left[\vartheta_1, 2\vartheta_1^2/(n_1-k)\right]\end{aligned}$$

and analogously

$$\begin{aligned}\hat{\vartheta}_2(\mathbf{Y}_2) &= \left[\mathbf{Y}_2 - \widehat{\mathbf{X}_2\boldsymbol{\beta}}(\mathbf{Y}_2)\right]'\mathbf{V}_2^{-1}\left[\mathbf{Y}_2 - \widehat{\mathbf{X}_2\boldsymbol{\beta}}(\mathbf{Y}_2)\right]/(n_2-k)\\ &\sim \vartheta_2\chi^2_{n_2-k}(0)/(n_2-k) \sim_2 \left[\vartheta_2, 2\vartheta_2^2/(n_2-k)\right]\end{aligned}$$

if $n_2 > k$. If $n_2 \leq k$, the parameter ϑ_2 cannot be estimated.

Let in the following theorem the model

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim_{n_1+n_2} \left[\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}\boldsymbol{\beta}, \vartheta_1\begin{pmatrix}\mathbf{V}_1, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix} + \vartheta_2\begin{pmatrix}\mathbf{0}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{V}_2\end{pmatrix}\right], \tag{5}$$

where $r(\mathbf{X}_1) = k < n_1$, $r(\mathbf{X}_2) = k < n_2$, $\mathbf{V}_1$, $\mathbf{V}_2$ are p.d., be considered.

Theorem 3.7. *In the model* (5) *the estimator of both variance components exists. It means that the matrix* $\mathbf{S}_*$, *where*

$$* = \left[\mathbf{M}_{\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}}\begin{pmatrix}\boldsymbol{\Sigma}_{1,0}, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_{2,0}\end{pmatrix}\mathbf{M}_{\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}}\right]^+,$$

is nonsingular and the MINQUE can be written in the form

$$\hat{\boldsymbol{\vartheta}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2)=\mathbf{S}_*^{-1}\begin{pmatrix}\boldsymbol{v}_1'(\boldsymbol{Y}_1)\frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}^2}\boldsymbol{v}_1(\boldsymbol{Y}_1)+\boldsymbol{w}_2'\mathbf{A}_1\boldsymbol{w}_2\\ \boldsymbol{w}_2'\left[\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}-\frac{2}{\vartheta_{2,0}^3}\mathbf{V}_2^{-1}\mathbf{X}_2\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)^{-1}\mathbf{X}'\mathbf{V}_2^{-1}+\mathbf{A}_2\right]\boldsymbol{w}_2\end{pmatrix}.$$

The matrix $\mathbf{S}_*$ *can be written in the form*

$$\mathbf{S}_*=\begin{pmatrix}n_1/\vartheta_{1,0}^2, & 0\\ 0, & n_2/\vartheta_{2,0}^2\end{pmatrix}+\begin{pmatrix}(1/\vartheta_{1,0}^2)c_{1,1}, & \big(1/(\vartheta_{1,0}\vartheta_{2,0})\big)c_{1,2}\\ \big(1/(\vartheta_{2,0}\vartheta_{1,0})\big)c_{2,1}, & (1/\vartheta_{2,0}^2)c_{2,2}\end{pmatrix},$$

where

$$\begin{aligned}
c_{1,1} &= -2\,\mathrm{Tr}\left[\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)^{-1}\frac{\mathbf{H}_1}{\vartheta_{1,0}}\right]\\
&\quad+\mathrm{Tr}\left[\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)^{-1}\frac{\mathbf{H}_1}{\vartheta_{1,0}}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)^{-1}\frac{\mathbf{H}_1}{\vartheta_{1,0}}\right],\\
c_{1,2} &= \mathrm{Tr}\left[\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)^{-1}\frac{\mathbf{H}_1}{\vartheta_{1,0}}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)^{-1}\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right]=c_{2,1},\\
c_{2,2} &= -2\,\mathrm{Tr}\left[\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)^{-1}\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right]\\
&\quad+\mathrm{Tr}\left[\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)^{-1}\frac{\mathbf{H}_2}{\vartheta_{2,0}}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)^{-1}\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right],
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{v}_1(\boldsymbol{Y}_1) &= \boldsymbol{Y}_1-\mathbf{X}_1\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1),\\
\mathbf{A}_i &= \frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}^4}\mathbf{X}_2\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)^{-1}\mathbf{H}_i\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1}, \qquad i=1,2.
\end{aligned}$$

If n_1-k *and* n_2-k *are sufficiently large, then*

$$\mathbf{S}_*^{-1}=\begin{pmatrix}\frac{\vartheta_{1,0}^2}{n_1}, & 0\\ 0, & \frac{\vartheta_{2,0}^2}{n_2}\end{pmatrix}-\begin{pmatrix}\frac{\vartheta_{1,0}^2}{n_1^2}c_{1,1}, & \frac{\vartheta_{1,0}\vartheta_{2,0}}{n_1n_2}c_{1,2}\\ \frac{\vartheta_{2,0}\vartheta_{1,0}}{n_2n_1}c_{2,1}, & \frac{\vartheta_{2,0}^2}{n_2^2}c_{2,2}\end{pmatrix}.$$

Thus

$$\hat{\boldsymbol{\vartheta}}(\mathbf{Y}_1, \mathbf{Y}_2) = \begin{pmatrix} \frac{1}{n_1} - \frac{c_{1,1}}{n_1^2}, & -\frac{\vartheta_{1,0}}{\vartheta_{2,0}} \frac{c_{1,2}}{n_1 n_2} \\ -\frac{\vartheta_{1,0}}{\vartheta_{2,0}} \frac{c_{2,1}}{n_2 n_1}, & \frac{1}{n_2} - \frac{c_{2,2}}{n_2^2} \end{pmatrix} \times$$
$$\times \begin{pmatrix} \mathbf{v}_1'(\mathbf{Y}_1) \frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}^2} \mathbf{v}_1(\mathbf{Y}_1) + \mathbf{w}_2' \mathbf{A}_1 \mathbf{w}_2 \\ \mathbf{w}_2' \left[\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}} - \frac{2}{\vartheta_{2,0}^3} \mathbf{V}_2^{-1} \mathbf{X}_2 \left(\frac{\mathbf{H}_1}{\vartheta_{1,0}} + \frac{\mathbf{H}_2}{\vartheta_{2,0}} \right)^{-1} \mathbf{X}' \mathbf{V}_2^{-1} + \mathbf{A}_2 \right] \mathbf{w}_2 \end{pmatrix}.$$

The correction γ *of the estimator* $\hat{\vartheta}_1(\mathbf{Y}_1) = \big[\mathbf{Y}_1 - \mathbf{X}_1 \hat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big]' \mathbf{V}_1^{-1} \big[\mathbf{Y}_1 - \mathbf{X}_1 \hat{\boldsymbol{\beta}}(\mathbf{Y}_1)\big] / (n_1 - k)$, *which is based on the first experiment only can be expressed, for sufficiently large* n_1 *and* n_2, *respectively, as* $\hat{\vartheta}_1(\mathbf{Y}_1, \mathbf{Y}_2) = \big[\mathbf{v}'(\mathbf{Y}_1) \mathbf{V}_1^{-1} \mathbf{v}(\mathbf{Y}_1) + \vartheta_{1,0}^2 \mathbf{w}_2' \mathbf{A}_1 \mathbf{w}_2\big] / n_1 = \hat{\vartheta}_1(\mathbf{Y}_1) + \gamma$, *where*

$$\gamma = -\frac{k}{n_1} \hat{\vartheta}_1(\mathbf{Y}_1) + \frac{\vartheta_{1,0}^2}{n_1} \mathbf{w}_2' \mathbf{A}_1 \mathbf{w}_2 .$$

P r o o f. With respect to definition of $\{\mathbf{S}_*\}_{i,j}$ (cf. Lemma 2.2) we obtain after some simple however rather tedious calculation the expression

$$\{\mathbf{S}_*\}_{1,1} = \frac{n_1}{\vartheta_{1,0}^2} + \frac{1}{\vartheta_{1,0}^2} \operatorname{Tr} \left[\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}} + \frac{\mathbf{H}_2}{\vartheta_{2,0}} \right)^{-1} \frac{\mathbf{H}_1}{\vartheta_{1,0}} \left(\frac{\mathbf{H}_1}{\vartheta_{1,0}} + \frac{\mathbf{H}_2}{\vartheta_{2,0}} \right)^{-1} \frac{\mathbf{H}_1}{\vartheta_{1,0}} \right]$$
$$- \frac{1}{\vartheta_{1,0}^2} 2 \operatorname{Tr} \left[\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}} + \frac{\mathbf{H}_2}{\vartheta_{2,0}} \right)^{-1} \frac{\mathbf{H}_1}{\vartheta_{1,0}} \right].$$

Analogously other elements of the matrix $\mathbf{S}_*$ can be obtained. Thus

$$\mathbf{S}_* = \begin{pmatrix} n_1/\vartheta_{1,0}^2, & 0 \\ 0, & n_2/\vartheta_{2,0}^2 \end{pmatrix} + \begin{pmatrix} (1/\vartheta_{1,0}^2) c_{1,1}, & \big(1/(\vartheta_{1,0}\vartheta_{2,0})\big) c_{1,2} \\ \big(1/(\vartheta_{2,0}\vartheta_{1,0})\big) c_{2,1}, & (1/\vartheta_{2,0}^2) c_{2,2} \end{pmatrix}.$$

If the matrix $\begin{pmatrix} n_1/\vartheta_{1,0}^2, & 0 \\ 0, & n_2/\vartheta_{2,0}^2 \end{pmatrix}$ is sufficiently larger in Loevner sense than the matrix $\begin{pmatrix} (1/\vartheta_{1,0}^2) c_{1,1}, & \big(1/(\vartheta_{1,0}\vartheta_{2,0})\big) c_{1,2} \\ \big(1/(\vartheta_{2,0}\vartheta_{1,0})\big) c_{2,1}, & (1/\vartheta_{2,0}^2) c_{2,2} \end{pmatrix}$, then

$$\mathbf{S}_*^{-1} =$$
$$= \begin{pmatrix} \frac{\vartheta_{1,0}^2}{n_1}, & 0 \\ 0, & \frac{\vartheta_{2,0}^2}{n_2} \end{pmatrix} - \begin{pmatrix} \frac{\vartheta_{1,0}^2}{n_1}, & 0 \\ 0, & \frac{\vartheta_{2,0}^2}{n_2} \end{pmatrix} \begin{pmatrix} \frac{1}{\vartheta_{1,0}^2} c_{1,1}, & \frac{1}{\vartheta_{1,0}\vartheta_{2,0}} c_{1,2} \\ \frac{1}{\vartheta_{2,0}\vartheta_{1,0}} c_{2,1}, & \frac{1}{\vartheta_{2,0}^2} c_{2,2} \end{pmatrix} \begin{pmatrix} \frac{\vartheta_{1,0}^2}{n_1}, & 0 \\ 0, & \frac{\vartheta_{2,0}^2}{n_2} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\vartheta_{1,0}^2}{n_1}, & 0 \\ 0, & \frac{\vartheta_{2,0}^2}{n_2} \end{pmatrix} - \begin{pmatrix} \frac{\vartheta_{1,0}^2}{n_1^2} c_{1,1}, & \frac{\vartheta_{1,0}\vartheta_{2,0}}{n_1 n_2} c_{1,2} \\ \frac{\vartheta_{2,0}\vartheta_{1,0}}{n_2 n_1} c_{2,1}, & \frac{\vartheta_{2,0}^2}{n_2^2} c_{2,2} \end{pmatrix}.$$

Further

$$\mathbf{v}_1'(\mathbf{Y}_1,\mathbf{Y}_2)\frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}^2}\mathbf{v}_1(\mathbf{Y}_1,\mathbf{Y}_2) = \mathbf{v}_1'(\mathbf{Y}_1)\frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}^2}\mathbf{v}_1(\mathbf{Y}_1) + \mathbf{w}_2'\mathbf{A}_1\mathbf{w}_2\,,$$

$$\mathbf{v}_2'(\mathbf{Y}_1,\mathbf{Y}_2)\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}^2}\mathbf{v}_2(\mathbf{Y}_1,\mathbf{Y}_2) =$$

$$= \frac{1}{\vartheta_{2,0}^2}\mathbf{w}_2'\left[\mathbf{V}_2^{-1} - \frac{1}{\vartheta_{2,0}}\mathbf{V}_2^{-1}\mathbf{X}_2\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}} + \frac{\mathbf{H}_2^{-1}}{\vartheta_{2,0}}\right)^{-1}\mathbf{X}_2'\mathbf{V}_2^{-1} + \mathbf{A}_2\right]\mathbf{w}_2\,.$$

Now the statement is obvious. □

4. Sensitivity approach

In the case of the model (5) with unknown ϑ_1 and ϑ_2, the $\boldsymbol{\vartheta}_0$-LBLUE

$$\hat{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2,\boldsymbol{\vartheta}_0) = \hat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \left(\frac{\mathbf{H}_1}{\vartheta_{1,0}} + \frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}\left[\mathbf{Y}_2 - \mathbf{X}_2\hat{\boldsymbol{\beta}}(\mathbf{Y}_1)\right]$$

is one of possible estimators. Another possibility is to use the $\boldsymbol{\vartheta}_0$-MINQUE or replicated REML of $\boldsymbol{\vartheta}$ in the plug-in estimator of $\boldsymbol{\beta}$, i.e.

$$\tilde{\boldsymbol{\beta}}(\mathbf{Y}_1,\mathbf{Y}_2,\hat{\boldsymbol{\vartheta}}) = \hat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \left(\frac{\mathbf{H}_1}{\hat{\vartheta}_1} + \frac{\mathbf{H}_2}{\hat{\vartheta}_2}\right)^{-1}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\hat{\vartheta}_2}\mathbf{w}_2\,.$$

The problem is to find statistical properties of such estimator. If the simulation approach is not taken into account, it is a difficult problem and for many situations it seems more suitable to investigate whether uncertainty in $\hat{\boldsymbol{\vartheta}}$ deteriorates properties of the $\boldsymbol{\vartheta}^*$-LBLUE of the estimator $\boldsymbol{\beta}$ or not, i.e. to find a insensitivity region.

The insensitivity region at the value ϑ_0 is a set of values $\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta}$ ($\delta\boldsymbol{\vartheta}$ is an infinitesimal shift of the vector value $\boldsymbol{\vartheta}$) with following property. If $\boldsymbol{\vartheta}^*$ (the actual value of the parameter $\boldsymbol{\vartheta}$) is an element of this set, then a deterioration of a statistical inference at the point $\boldsymbol{\vartheta}_0$ is smaller than a prescribed value. E.g. in the case of an estimator of a linear function $\mathbf{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{R}^k$, calculated at the point $\boldsymbol{\vartheta}_0$

$$\sqrt{\mathrm{Var}_{\vartheta^*}\left[\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}_1,\boldsymbol{\vartheta}^*)\right]}(1+\varepsilon) \ge \sqrt{\mathrm{Var}_{\vartheta_0}\left[\mathbf{h}'\hat{\boldsymbol{\beta}}(\mathbf{Y}_1,\boldsymbol{\vartheta}_0)\right]}\,,$$

is valid, where $\varepsilon > 0$ is a prescribed sufficiently small number.

Insensitivity regions can be found for many other statistical problems, where estimated variance components must be used. Some examples are given in [2], [5], [7], [8], [9], [10], [12].

Lemma 4.1. *Let $\delta\vartheta_1$ and $\delta\vartheta_2$ be infinitesimal shifts of the parameters ϑ_1 and ϑ_2, respectively. Then in the model* (5)

$$\begin{aligned}\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2,\vartheta_1+\delta\vartheta_1,\vartheta_2+\delta\vartheta_2)&\\ \approx \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2,\vartheta_1,\vartheta_2)&-\left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}\left[\mathbf{X}_1'\frac{\mathbf{V}_1^{-1}}{\vartheta_1^2}\boldsymbol{v}_1(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\delta\vartheta_1\right.\\ &\left.+\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2^2}\boldsymbol{v}_2(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\delta\vartheta_2\right],\end{aligned}$$

$$\begin{aligned}(\forall \boldsymbol{h}\in\mathbb{R}^k)\Big(\mathrm{Var}\big[\boldsymbol{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2,\vartheta_1+\delta\vartheta_1,\vartheta_2+\delta\vartheta_2)\big]&\\ \approx \boldsymbol{h}'\left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}\boldsymbol{h}+(\delta\vartheta_1,\delta\vartheta_2)\mathbf{W}_h\begin{pmatrix}\delta\vartheta_1\\ \delta\vartheta_2\end{pmatrix}\Big),&\end{aligned}$$

is valid, where

$$\begin{aligned}\mathbf{W}_h &= \begin{pmatrix}\boldsymbol{h}'\mathbf{U}_1\boldsymbol{h}, & 0\\ 0, & \boldsymbol{h}'\mathbf{U}_2\boldsymbol{h}\end{pmatrix}-\begin{pmatrix}\boldsymbol{h}'\mathbf{T}_{1,1}\boldsymbol{h}, & \boldsymbol{h}'\mathbf{T}_{1,2}\boldsymbol{h}\\ \boldsymbol{h}'\mathbf{T}_{2,1}\boldsymbol{h}, & \boldsymbol{h}'\mathbf{T}_{2,2}\boldsymbol{h}\end{pmatrix},\\ \mathbf{U}_1 &= \left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}\frac{\mathbf{H}_1}{\vartheta_1^3}\left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1},\\ \mathbf{U}_2 &= \left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}\frac{\mathbf{H}_2}{\vartheta_2^3}\left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1},\\ \mathbf{T}_{1,1} &= \left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}\frac{\mathbf{H}_1}{\vartheta_1^2}\left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}\frac{\mathbf{H}_1}{\vartheta_1^2}\left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1},\\ \mathbf{T}_{1,2} &= \left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}\frac{\mathbf{H}_1}{\vartheta_1^2}\left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}\frac{\mathbf{H}_2}{\vartheta_2^2}\left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}=\mathbf{T}_{2,1}',\\ \mathbf{T}_{2,2} &= \left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}\frac{\mathbf{H}_2}{\vartheta_2^2}\left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}\frac{\mathbf{H}_2}{\vartheta_2^2}\left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}.\end{aligned}$$

Proof. Since

$$\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2,\vartheta_1,\vartheta_2)=\left(\frac{\mathbf{H}_1}{\vartheta_1}+\frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}\left(\mathbf{X}_1'\frac{\mathbf{V}_1^{-1}}{\vartheta_1}\boldsymbol{Y}_1+\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_2}\boldsymbol{Y}_2\right),$$

we have

$$\frac{\partial \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1, \vartheta_2)}{\partial \vartheta_1} = -\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1} \frac{1}{\vartheta_1^2} \mathbf{X}_1' \mathbf{V}_1^{-1} \boldsymbol{v}_1(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\,.$$

Analogously

$$\frac{\partial \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1, \vartheta_2)}{\partial \vartheta_2} = -\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1} \frac{1}{\vartheta_2^2} \mathbf{X}_2' \mathbf{V}_2^{-1} \boldsymbol{v}_2(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\,.$$

Since

$$\begin{aligned}
&\operatorname{Var}_\vartheta \begin{pmatrix} \boldsymbol{v}_1(\boldsymbol{Y}_1, \boldsymbol{Y}_1) \\ \boldsymbol{v}_2(\boldsymbol{Y}_1, \boldsymbol{Y}_2) \end{pmatrix} \\
&= \begin{pmatrix} \vartheta_1 \mathbf{V}_1 - \mathbf{X}_1 \left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1} \mathbf{X}_1', & -\mathbf{X}_1 \left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1} \mathbf{X}_2' \\ -\mathbf{X}_2 \left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1} \mathbf{X}_1', & \vartheta_2 \mathbf{V}_2 - \mathbf{X}_2 \left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1} \mathbf{X}_2' \end{pmatrix},
\end{aligned}$$

we have

$$\begin{aligned}
&\operatorname{Var}_\vartheta \begin{pmatrix} \boldsymbol{h}' \partial \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1, \vartheta_2)/\partial \vartheta_1 \\ \boldsymbol{h}' \partial \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1, \vartheta_2)/\partial \vartheta_2 \end{pmatrix} \\
&\qquad = \begin{pmatrix} \boldsymbol{h}' \mathbf{U}_1 \boldsymbol{h}, & 0 \\ 0 & \boldsymbol{h}' \mathbf{U}_2 \boldsymbol{h} \end{pmatrix} - \begin{pmatrix} \boldsymbol{h}' \mathbf{T}_{1,1} \boldsymbol{h}, & \boldsymbol{h}' \mathbf{T}_{1,2} \boldsymbol{h} \\ \boldsymbol{h}' \mathbf{T}_{2,1} \boldsymbol{h}, & \boldsymbol{h}' \mathbf{T}_{2,2} \boldsymbol{h} \end{pmatrix}.
\end{aligned}$$

□

Since

$$\operatorname{cov}_\vartheta \left[\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2), \begin{pmatrix} \boldsymbol{v}_1(\boldsymbol{Y}_1, \boldsymbol{Y}_2) \\ \boldsymbol{v}_2(\boldsymbol{Y}_1, \boldsymbol{Y}_2) \end{pmatrix}\right] = \mathbf{0}\,,$$

Lemma 4.1 implies

$$\begin{aligned}
&\operatorname{Var}_\vartheta \big[\boldsymbol{h}' \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1 + \delta\vartheta_1, \vartheta_2 + \delta\vartheta_2)\big] \\
&\qquad \approx \operatorname{Var}_\vartheta \big[\boldsymbol{h}' \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1, \vartheta_2)\big] + \delta\boldsymbol{\vartheta}' \mathbf{W}_h \delta\boldsymbol{\vartheta} \\
\Longrightarrow &\sqrt{\operatorname{Var}_\vartheta \big[\boldsymbol{h}' \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1 + \delta\vartheta_1, \vartheta_2 + \delta\vartheta_2)\big]} \\
&\qquad \approx \sqrt{\operatorname{Var}_\vartheta \big[\boldsymbol{h}' \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1, \vartheta_2)\big]} \sqrt{1 + \frac{\delta\boldsymbol{\vartheta}' \mathbf{W}_h \delta\boldsymbol{\vartheta}}{\operatorname{Var}_\vartheta \big[\boldsymbol{h}' \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1, \vartheta_2)\big]}}\,.
\end{aligned}$$

Let $\varepsilon > 0$ be such small number that the enlargement of the variance of the estimator $\sqrt{\operatorname{Var}_{\vartheta^*}(\boldsymbol{h}'\hat{\boldsymbol{\beta}})}$ by the factor $(1+\varepsilon)$ can be tolerated. Let

$$\sqrt{1 + \frac{\delta\boldsymbol{\vartheta}' \mathbf{W}_h \delta\boldsymbol{\vartheta}}{\operatorname{Var}_\vartheta \big[\boldsymbol{h}' \hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1, \vartheta_2)\big]}} \le 1 + \varepsilon\,.$$

Then the insensitivity region $\mathcal{N}_\vartheta$ at the point $\boldsymbol{\vartheta}$ with respect to enlargement of the standard deviation of the estimator of the function $\boldsymbol{h}'\boldsymbol{\beta}, \boldsymbol{\beta} \in \mathbb{R}^k$, is given in the following theorem.

Theorem 4.2. *The insensitivity region $\mathcal{N}_\vartheta$ is*

$$\mathcal{N}_\vartheta = \left\{\delta\boldsymbol{\vartheta} :\ \delta\boldsymbol{\vartheta}'\mathbf{W}_h\delta\boldsymbol{\vartheta} \leq 2\varepsilon\boldsymbol{h}'\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)^{-1}\boldsymbol{h}\right\},$$

i.e.

$$\delta\boldsymbol{\vartheta} \in \mathcal{N}_\vartheta \implies \sqrt{\mathrm{Var}_\vartheta\big[\boldsymbol{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1 + \delta\vartheta_1, \vartheta_2 + \delta\vartheta_2)\big]} \leq (1+\varepsilon)\sqrt{\mathrm{Var}_\vartheta\big[\boldsymbol{h}'\hat{\boldsymbol{\beta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1, \vartheta_2)\big]}\,.$$

P r o o f. It is implied by Lemma 4.1. □

Remark 4.3. In order to utilize information on $\mathcal{N}_\vartheta$ at the point $\boldsymbol{\vartheta}$, it must be known that the actual value $\boldsymbol{\vartheta}^*$ of $\boldsymbol{\vartheta}$ is sufficiently near to $\boldsymbol{\vartheta}$. The $(1-\alpha)$-confidence region (for sufficiently small α) can help to check it.

With respect to Theorem 3.7

$$\mathrm{Var}_{\vartheta_0}\big[\hat{\boldsymbol{\vartheta}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\big] = 2\mathbf{S}_*^{-1} = 2\begin{pmatrix} \frac{\vartheta_{1,0}^2}{n_1} - \frac{\vartheta_{1,0}^2}{n_1^2}c_{1,1}, & -\frac{\vartheta_{1,0}\vartheta_{2,0}}{n_1 n_2}c_{1,2} \\ -\frac{\vartheta_{2,0}\vartheta_{1,0}}{n_2 n_1}c_{2,1}, & \frac{\vartheta_{2,0}^2}{n_2} - \frac{\vartheta_{2,0}^2}{n_2^2}c_{2,2} \end{pmatrix}$$

and regarding the Scheffé theorem [16] we have

$$P\left\{(\boldsymbol{\vartheta}^* - \hat{\boldsymbol{\vartheta}})'(2\mathbf{S}_*^{-1})^{-1}(\boldsymbol{\vartheta}^* - \hat{\boldsymbol{\vartheta}}) \leq \tfrac{2}{\alpha}\right\} = P\left\{(\forall \boldsymbol{h} \in \mathbb{R}^2)\left(|\boldsymbol{h}'(\boldsymbol{\vartheta}^* - \hat{\boldsymbol{\vartheta}})| \leq \sqrt{\tfrac{2}{\alpha}}\sqrt{\boldsymbol{h}'2\mathbf{S}_*^{-1}\boldsymbol{h}}\right)\right\}.$$

With respect to the Bonferroni rule [3, p. 492],

$$P\left\{(\forall i \in \{1,2\})\left(|\boldsymbol{e}_i'(\boldsymbol{\vartheta}^* - \hat{\boldsymbol{\vartheta}})| \leq \sqrt{\tfrac{2}{\alpha}}\sqrt{\boldsymbol{e}_i'2\mathbf{S}_*^{-1}\boldsymbol{e}_i}\right)\right\} \approx 1 - \alpha\,.$$

Thus the containment

$$\mathcal{C}(\boldsymbol{\vartheta}^*) = \left\{\boldsymbol{\vartheta} :\ \tfrac{1}{2}(\boldsymbol{\vartheta} - \hat{\boldsymbol{\vartheta}})'\mathbf{S}_*(\boldsymbol{\vartheta} - \hat{\boldsymbol{\vartheta}}) \leq \tfrac{2}{\alpha}\right\} \subset \{\boldsymbol{\vartheta}_0 + \delta\boldsymbol{\vartheta} :\ \delta\boldsymbol{\vartheta} \in \mathcal{N}_{\vartheta_0}\},$$

can serve as a guaranty that $\boldsymbol{\vartheta}_0$ is sufficiently near to the actual value $\boldsymbol{\vartheta}^*$ and that $\boldsymbol{\vartheta}^*$ is in the insensitivity region $\mathcal{N}_{\vartheta_0}$. In [11] it is shown that the requirement of the containment may be in some situation too rigorous.

5. Models with constraints

In this section the model

$$\boldsymbol{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}), \qquad \boldsymbol{\beta} \in \underline{\beta} = \{\boldsymbol{\beta}:\ \boldsymbol{b} + \mathbf{B}\boldsymbol{\beta} = \boldsymbol{0}\}, \tag{6}$$

will be under consideration.

Lemma 5.1. *Let in* (6) $r(\mathbf{X}_{n,k}) = k < n$, $r(\mathbf{B}_{q,k}) = q < k$, $\boldsymbol{\Sigma}$ *p.d be valid. Then the BLUE of* $\boldsymbol{\beta}$ *is*

$$\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}) = \hat{\boldsymbol{\beta}}(\boldsymbol{Y}) - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\big[\mathbf{B}\hat{\boldsymbol{\beta}}(\boldsymbol{Y}) + \boldsymbol{b}\big],$$

where $\mathbf{C} = \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}, \hat{\boldsymbol{\beta}}(\boldsymbol{Y}) = \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}$ *(the BLUE in the model without constraints). Further*

$$\operatorname{Var}\big[\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y})\big] = \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1} = \big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}\mathbf{M}_{\mathbf{B}'}\big]^{+}.$$

P r o o f. Cf. e.g. [4, Chap. 2]. □

Let the first experiment be (6), where $r\big(\mathbf{X}_{1,(n_1,k)}\big) = k < n_1$, $r(\mathbf{B}_{q,k}) = q < k$, $\boldsymbol{\Sigma}_1$ p.d.. The additional experiment is $\boldsymbol{Y}_2 \sim_{n_2} (\mathbf{X}_2\boldsymbol{\beta}, \boldsymbol{\Sigma}_2)$, $\boldsymbol{\Sigma}_2$ p.d. (also in the additional experiment the parameter $\boldsymbol{\beta}$ must satisfy the constraints $\boldsymbol{b}+\mathbf{B}\boldsymbol{\beta} = \boldsymbol{0}$; the matrix $\mathbf{X}_2$ need not have the full rank in columns).

Theorem 5.2. *In the given situation the BLUE of* $\boldsymbol{\beta}$ *is*

$$\begin{aligned}\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) &= \hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1) + \boldsymbol{k}_I,\\ \boldsymbol{k}_I &= \big[\mathbf{M}_{\mathbf{B}'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\boldsymbol{w}_{I,2},\\ \boldsymbol{w}_{I,2} &= \boldsymbol{Y}_2 - \mathbf{X}_2\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1),\end{aligned}$$

where $\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1)$ *is the BLUE based on the first experiment with constraints.*

P r o o f. The experiment with constraints

$$\begin{pmatrix}\boldsymbol{Y}_1\\ \boldsymbol{Y}_2\end{pmatrix} \sim_{n_1+n_2} \left[\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix}\boldsymbol{\Sigma}_1, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_2\end{pmatrix}\right], \qquad \boldsymbol{\beta} \in \underline{\beta},$$

is equivalent to experiment without constraints

$$\begin{pmatrix}\boldsymbol{Y}_1 - \mathbf{X}_1\boldsymbol{\beta}_0\\ \boldsymbol{Y}_2 - \mathbf{X}_2\boldsymbol{\beta}_0\end{pmatrix} \sim_{n_1+n_2} \left[\begin{pmatrix}\mathbf{X}_1\mathbf{K}_{\mathbf{B}}\\ \mathbf{X}_2\mathbf{K}_{\mathbf{B}}\end{pmatrix}\gamma, \begin{pmatrix}\boldsymbol{\Sigma}_1, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}_2\end{pmatrix}\right], \qquad \gamma \in \mathbb{R}^{k-q},$$

where $\boldsymbol{\beta}_0$ is any vector satisfying the constraints $\boldsymbol{b} + \mathbf{B}\boldsymbol{\beta}_0 = \boldsymbol{0}$ and $\mathbf{K_B}$ is a $k \times (k-q)$ matrix satisfying the equality $\mathcal{K}\rceil\nabla(\mathbf{B}) = \{\boldsymbol{u} :\ \mathbf{B}\boldsymbol{u} = \boldsymbol{0}\} = \mathcal{M}(\mathbf{K_B})$. With respect to Lemma 3.1

$$\begin{aligned}&\widehat{\mathbf{K_B}\gamma}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\\ &= \widehat{\mathbf{K_B}\gamma}(\boldsymbol{Y}_1) + \mathbf{K_B}\big[\mathbf{K}'_{\mathbf{B}}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{K_B}\big]^{-1}\mathbf{K}'_{\mathbf{B}}\mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}\big[\boldsymbol{Y}_2 - \mathbf{X}_2\beta_0 - \mathbf{X}_2\widehat{\mathbf{K_B}\gamma}(\boldsymbol{Y}_1)\big]\,,\end{aligned}$$

is valid, i.e.

$$\begin{aligned}\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) &= \beta_0 + \mathbf{K_B}\hat{\gamma}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = \hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1) + \boldsymbol{k}\,,\\ \boldsymbol{k} &= \big[\mathbf{M}_{\mathbf{B}'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}\big[\boldsymbol{Y}_2 - \mathbf{X}_2\hat{\hat{\beta}}(\boldsymbol{Y}_1)\big]\,.\end{aligned}$$

The last equality is implied by the following relationships.

$$\begin{aligned}\mathbf{K_B}\big[\mathbf{K}'_{\mathbf{B}}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{K_B}\big]^{-1}\mathbf{K}'_{\mathbf{B}}(\mathbf{C}_1 + \mathbf{C}_2) &= \mathbf{P}_{\mathbf{K_B}}^{(\mathbf{C}_1+\mathbf{C}_2)} = \mathbf{P}_{\mathbf{M}_{\mathbf{B}'}}^{(\mathbf{C}_1+\mathbf{C}_2)}\\ &= \mathbf{M}_{\mathbf{B}'}\big[\mathbf{M}_{\mathbf{B}'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{M}_{\mathbf{B}'}(\mathbf{C}_1 + \mathbf{C}_2)\\ \Longrightarrow \mathbf{K_B}\big[\mathbf{K}'_{\mathbf{B}}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{K_B}\big]^{-1}\mathbf{K}'_{\mathbf{B}} &= \mathbf{M}_{\mathbf{B}'}\big[\mathbf{M}_{\mathbf{B}'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{M}_{\mathbf{B}'}\,.\end{aligned}$$

Further

$$\mathbf{M}_{\mathbf{B}'}\big[\mathbf{M}_{\mathbf{B}'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{\mathbf{B}'}\big]^{+}\mathbf{M}_{\mathbf{B}'} = [\mathbf{M}_{\mathbf{B}'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{\mathbf{B}'}]^{+}\,.$$

□

THEOREM 5.3. *The covariance matrix* $\mathrm{Var}\big[\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\big]$ *of the estimator from* Theorem 5.2 *is*

$$\begin{aligned}\mathrm{Var}\big[\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\big] &= \mathrm{Var}\big[\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1)\big] - \mathbf{K}_I\,,\\ \mathbf{K}_I &= \mathrm{Var}\big[\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1)\big]\mathbf{X}'_2\big[\mathrm{Var}(\boldsymbol{w}_{I,2})\big]^{-1}\mathbf{X}_2\,\mathrm{Var}\big[\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1)\big]\,.\end{aligned}$$

P r o o f. Analogously as in Lemma 3.3 (cf. also Lemma 5.1)

$$\begin{aligned}&\mathrm{Var}\big[\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\big]\\ &= \big[\mathbf{M}_{\mathbf{B}'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{\mathbf{B}'}\big]^{+} = \big[\mathbf{M}_{\mathbf{B}'}\mathbf{C}_1\mathbf{M}_{\mathbf{B}'} + \mathbf{M}_{\mathbf{B}'}\mathbf{C}_2\mathbf{M}_{\mathbf{B}'}\big]^{+}\\ &= (\mathbf{M}_{\mathbf{B}'}\mathbf{C}_1\mathbf{M}_{\mathbf{B}'})^{+} - (\mathbf{M}_{\mathbf{B}'}\mathbf{C}_1\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'_2\big[\boldsymbol{\Sigma}_2 + \mathbf{X}_2(\mathbf{M}_{\mathbf{B}'}\mathbf{C}_1\mathbf{M}_{\mathbf{B}'})^{+}\mathbf{X}'_2\big]^{-1} \times\\ &\quad \times \mathbf{X}_2\mathbf{M}_{\mathbf{B}'}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}_1\mathbf{M}_{\mathbf{B}'})^{+}\\ &= \mathrm{Var}\big[\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1)\big] - \mathbf{K}_I\,,\end{aligned}$$

since

$$\mathrm{Var}\big[\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1)\big] = (\mathbf{M}_{\mathbf{B}'}\mathbf{C}_1\mathbf{M}_{\mathbf{B}'})^+ \,,$$
$$\boldsymbol{\Sigma}_2 + \mathbf{X}_2(\mathbf{M}_{\mathbf{B}'}\mathbf{C}_1\mathbf{M}_{\mathbf{B}'})^+\mathbf{X}_2' = \mathrm{Var}(\boldsymbol{w}_{I,2}) \,,$$
$$\mathbf{M}_{\mathbf{B}'}(\mathbf{M}_{\mathbf{B}'}\mathbf{C}_1\mathbf{M}_{\mathbf{B}'})^+ = (\mathbf{M}_{\mathbf{B}'}\mathbf{C}_1\mathbf{M}_{\mathbf{B}'})^+\mathbf{M}_{\mathbf{B}'} = (\mathbf{M}_{\mathbf{B}'}\mathbf{C}_1\mathbf{M}_{\mathbf{B}'})^+ \,.$$

□

If $\boldsymbol{\Sigma}_1 = \vartheta\mathbf{V}_1$, $\boldsymbol{\Sigma}_2 = \vartheta\mathbf{V}_2$, then the BLUE $\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ can be calculated by the help of $\mathbf{V}_1$ and $\mathbf{V}_2$ instead of $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$, respectively. The best estimator (in the case of normality) of ϑ is given by the following theorem.

THEOREM 5.4. *The estimator $\hat{\vartheta}_I(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ of ϑ based on both experiment is*

$$\hat{\vartheta}_I(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = \hat{\vartheta}_I(\boldsymbol{Y}_1) + k_I \,,$$

where

$$\begin{aligned} k_I = {}& \frac{1}{n_1+n_2+q-k}\Big(-n_2\hat{\vartheta}_I(\boldsymbol{Y}_1) + \boldsymbol{w}_{I,2}'\big\{\mathbf{V}_2^{-1} \\ & - \mathbf{V}_2^{-1}\mathbf{X}_2\big[\mathbf{M}_{\mathbf{B}'}(\mathbf{H}_1+\mathbf{H}_2)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}_2'\mathbf{V}_2\big\}\boldsymbol{w}_{I,2}\Big) \,, \\ \hat{\vartheta}_I(\boldsymbol{Y}_1) = {}& \frac{1}{n_1+q-k}\big[\boldsymbol{Y}_1 - \mathbf{X}_1\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1)\big]'\mathbf{V}^{-1}\big[\boldsymbol{Y}_1 - \mathbf{X}_1\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1)\big] \,, \\ \hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1) = {}& \mathbf{H}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\boldsymbol{Y}_1 - \mathbf{H}_1^{-1}\mathbf{B}'(\mathbf{B}\mathbf{H}_1^{-1}\mathbf{B}')^{-1}[\mathbf{B}\mathbf{H}_1^{-1}\mathbf{X}_1'\mathbf{V}_1^{-1}\boldsymbol{Y}_1 + \boldsymbol{b}] \,, \\ & \mathbf{H}_1 = \mathbf{X}_1'\mathbf{V}_1^{-1}\mathbf{X}_1 \,, \qquad \boldsymbol{w}_{I,2} = \boldsymbol{Y}_2 - \mathbf{X}_2\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1) \,. \end{aligned}$$

P r o o f. With respect to the proof of Theorem 5.2 both experiments can be rewritten as (5), where

$$\begin{pmatrix} \boldsymbol{\Sigma}_1, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_2 \end{pmatrix} = \vartheta \begin{pmatrix} \mathbf{V}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{V}_2 \end{pmatrix} .$$

Now it can be proceeded as in the proof of Theorem 3.7.

$$\begin{aligned} \hat{\vartheta}_I(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = \frac{1}{n_1+n_2-r(\mathbf{X}_1\mathbf{K}_{\mathbf{B}})}\big[&\boldsymbol{v}_{I,1}'(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\mathbf{V}_1^{-1}\boldsymbol{v}_{I,1}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) \\ &+\boldsymbol{v}_{I,2}'(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\mathbf{V}_2^{-1}\boldsymbol{v}_{I,2}(\boldsymbol{Y}_1, \boldsymbol{Y}_2)\big] \,, \end{aligned}$$

where (cf. Theorem 5.2)

$$\boldsymbol{v}_{I,1}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) = \boldsymbol{Y}_1 - \mathbf{X}_1\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1) - \mathbf{X}_1\big[\mathbf{M}_{\mathbf{B}'}(\mathbf{H}_1+\mathbf{H}_2)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}_2'\mathbf{V}_2^{-1}\big[\boldsymbol{Y}_2 - \mathbf{X}_2\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1)\big]\,,$$

$$\boldsymbol{v}_{I,2}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) = \boldsymbol{Y}_2 - \mathbf{X}_2\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1) - \mathbf{X}_2\big[\mathbf{M}_{\mathbf{B}'}(\mathbf{H}_1+\mathbf{H}_2)\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{X}_2'\mathbf{V}_2^{-1}\big[\boldsymbol{Y}_2 - \mathbf{X}_2\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1)\big]\,.$$

Since

$$\begin{aligned}\boldsymbol{v}_{I,1}(\boldsymbol{Y}_1) &= \boldsymbol{Y}_1 - \mathbf{X}_1\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1)\,,\\ \boldsymbol{v}_{I,2}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) &= \boldsymbol{w}_{I,2} - \mathbf{X}_2[\mathbf{M}_{\mathbf{B}'}(\mathbf{H}_1+\mathbf{H}_2)\mathbf{M}_{\mathbf{B}'}]^+\mathbf{X}_2'\mathbf{V}_2^{-1}\boldsymbol{w}_{I,2}\,,\\ \boldsymbol{v}_{I,1}'\mathbf{V}_1^{-1}\mathbf{X}_1 &= \boldsymbol{0}\end{aligned}$$

and

$$\hat{\vartheta}_1(\boldsymbol{Y}_1) = \boldsymbol{v}_{I,1}'(\boldsymbol{Y}_1)\mathbf{V}_1^{-1}\boldsymbol{v}_{I,1}(\boldsymbol{Y}_1)/(n_1+q-k)\,,$$

the proof can be easily finished. □

If $\boldsymbol{\Sigma}_1 = \vartheta_1\mathbf{V}_1$ and $\boldsymbol{\Sigma}_2 = \vartheta_2\mathbf{V}_2$, respectively, and at the same time $\vartheta_1 \neq \vartheta_2$, then the estimators $\hat{\vartheta}_1(\boldsymbol{Y}_1,\boldsymbol{Y}_2)$ and $\hat{\vartheta}_2(\boldsymbol{Y}_1,\boldsymbol{Y}_2)$ $(n_2+q>k)$ are given by the following theorem.

Theorem 5.5. *The $\boldsymbol{\vartheta}_0$-MINQUE of ϑ_1 and ϑ_2, respectively, in the model*

$$\begin{pmatrix}\boldsymbol{Y}_1\\ \boldsymbol{Y}_2\end{pmatrix} \sim_{n_1+n_2}\left[\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}\boldsymbol{\beta}, \vartheta_1\begin{pmatrix}\mathbf{V}_1, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix} + \vartheta_2\begin{pmatrix}\mathbf{0}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{V}_2\end{pmatrix}\right], \qquad \boldsymbol{\beta}\in\underline{\beta}\,,$$

where $r(\mathbf{X}_1) = k < n_1$, $\mathbf{V}_1, \mathbf{V}_2$ are p.d. and $r(\mathbf{B}_{q,k}) = q < k$, is

$$\begin{pmatrix}\hat{\vartheta}_1(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\\ \hat{\vartheta}_2(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\end{pmatrix} = \mathbf{S}_{I,*}^{-1}\begin{pmatrix}\boldsymbol{v}_{I,1}'(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\frac{1}{\vartheta_{1,0}^2}\mathbf{V}_1^{-1}\boldsymbol{v}_{I,1}(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\\ \boldsymbol{v}_{I,2}'(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\frac{1}{\vartheta_{2,0}^2}\mathbf{V}_2^{-1}\boldsymbol{v}_{I,2}(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\end{pmatrix},$$

where

$$\mathbf{S}_{I,*} = \begin{pmatrix}\frac{n_1}{\vartheta_{1,0}^2}, & 0\\ 0, & \frac{n_2}{\vartheta_{2,0}^2}\end{pmatrix} + \begin{pmatrix}\frac{c_{1,1}}{\vartheta_{1,0}^2}, & \frac{c_{1,2}}{\vartheta_{1,0}\vartheta_{2,0}}\\ \frac{c_{2,1}}{\vartheta_{2,0}\vartheta_{1,0}}, & \frac{c_{2,2}}{\vartheta_{2,0}^2}\end{pmatrix},$$

$$
\begin{aligned}
c_{1,1} &= -2\,\mathrm{Tr}\left\{\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)\mathbf{M}_{\mathbf{B}'}\right]^{+}\frac{\mathbf{H}_1}{\vartheta_{1,0}}\right\} \\
&\quad + \mathrm{Tr}\left\{\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)\mathbf{M}_{\mathbf{B}'}\right]^{+}\frac{\mathbf{H}_1}{\vartheta_{1,0}}\times\right. \\
&\quad \left.\times\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)\mathbf{M}_{\mathbf{B}'}\right]^{+}\frac{\mathbf{H}_1}{\vartheta_{1,0}}\right\}, \\
c_{1,2} &= \mathrm{Tr}\left\{\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)\mathbf{M}_{\mathbf{B}'}\right]^{+}\frac{\mathbf{H}_1}{\vartheta_{1,0}}\times\right. \\
&\quad \left.\times\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)\mathbf{M}_{\mathbf{B}'}\right]^{+}\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right\} = c_{2,1}\,, \\
c_{2,2} &= -2\,\mathrm{Tr}\left\{\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)\mathbf{M}_{\mathbf{B}'}\right]^{+}\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right\} \\
&\quad + \mathrm{Tr}\left\{\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)\mathbf{M}_{\mathbf{B}'}\right]^{+}\frac{\mathbf{H}_2}{\vartheta_{2,0}}\times\right. \\
&\quad \left.\times\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)\mathbf{M}_{\mathbf{B}'}\right]^{+}\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right\}.
\end{aligned}
$$

P r o o f. In the first step we reparametrize the model as in the proof of Theorem 5.2 Now with respect to Lemma 2.3 it is sufficient to use the following substitution scheme

$$
\boldsymbol{Y} \mapsto \begin{pmatrix} \boldsymbol{Y}_1 - \mathbf{X}_1\boldsymbol{\beta}_0 \\ \boldsymbol{Y}_2 - \mathbf{X}_2\boldsymbol{\beta}_0 \end{pmatrix}, \qquad \mathbf{X} \mapsto \begin{pmatrix} \mathbf{X}_1\mathbf{K}_{\mathbf{B}} \\ \mathbf{X}_2\mathbf{K}_{\mathbf{B}} \end{pmatrix}, \qquad \boldsymbol{\Sigma}_0 \mapsto \begin{pmatrix} \vartheta_{1,0}\mathbf{V}_1, & \mathbf{0} \\ \mathbf{0}, & \vartheta_{2,0}\mathbf{V}_2 \end{pmatrix}.
$$

Thus it can be obtained

$$
\begin{aligned}
\boldsymbol{v}(\boldsymbol{Y}) \mapsto \begin{pmatrix} \boldsymbol{Y}_1 - \mathbf{X}_1\boldsymbol{\beta}_0 \\ \boldsymbol{Y}_2 - \mathbf{X}_2\boldsymbol{\beta}_0 \end{pmatrix} - \begin{pmatrix} \mathbf{X}_1\mathbf{K}_{\mathbf{B}}\hat{\gamma}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) \\ \mathbf{X}_2\mathbf{K}_{\mathbf{B}}\hat{\gamma}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) \end{pmatrix} &= \begin{pmatrix} \boldsymbol{Y}_1 - \mathbf{X}_1\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) \\ \boldsymbol{Y}_2 - \mathbf{X}_2\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{v}_{I,1}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) \\ \boldsymbol{v}_{I,2}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) \end{pmatrix},
\end{aligned}
$$

$$
\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_1\boldsymbol{\Sigma}_0^{-1} \mapsto \begin{pmatrix} \frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}^2}, & \boldsymbol{0} \\ \boldsymbol{0}, & \boldsymbol{0} \end{pmatrix}, \qquad \boldsymbol{\Sigma}_0^{-1}\mathbf{V}_2\boldsymbol{\Sigma}_0^{-1} \mapsto \begin{pmatrix} \boldsymbol{0}, & \boldsymbol{0} \\ \boldsymbol{0}, & \frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}^2} \end{pmatrix},
$$

$$\begin{aligned}
&\mathrm{Tr}\big[(\mathbf{M_X}\boldsymbol{\Sigma}_0\mathbf{M_X})^+\mathbf{V}_1(\mathbf{M_X}\boldsymbol{\Sigma}_0\mathbf{M_X})^+\mathbf{V}_1\big]\\
&\quad\mapsto \mathrm{Tr}\left\{\left[\mathbf{M}_{\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}}\begin{pmatrix}\vartheta_{1,0}\mathbf{V}_1, & \mathbf{0}\\ \mathbf{0}, & \vartheta_{2,0}\mathbf{V}_2\end{pmatrix}\mathbf{M}_{\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}}\right]^+\begin{pmatrix}\mathbf{V}_1, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\times\right.\\
&\qquad\left.\times\left[\mathbf{M}_{\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}}\begin{pmatrix}\vartheta_{1,0}\mathbf{V}_1, & \mathbf{0}\\ \mathbf{0}, & \vartheta_{2,0}\mathbf{V}_2\end{pmatrix}\mathbf{M}_{\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}}\right]^+\begin{pmatrix}\mathbf{V}_1, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\right\}\\
&= \frac{n_1}{\vartheta_{1,0}^2}+\frac{c_{1,1}}{\vartheta_{1,0}^2}.
\end{aligned}$$

Analogously

$$\begin{aligned}
\mathrm{Tr}\big[(\mathbf{M_X}\boldsymbol{\Sigma}_0\mathbf{M_X})^+\mathbf{V}_1(\mathbf{M_X}\boldsymbol{\Sigma}_0\mathbf{M_X})^+\mathbf{V}_2\big] &\to \frac{c_{1,2}}{\vartheta_{1,0}\vartheta_{2,0}}\,,\\
\mathrm{Tr}\big[(\mathbf{M_X}\boldsymbol{\Sigma}_0\mathbf{M_X})^+\mathbf{V}_2(\mathbf{M_X}\boldsymbol{\Sigma}_0\mathbf{M_X})^+\mathbf{V}_2\big] &\to \frac{n_2}{\vartheta_{2,0}^2}+\frac{c_{2,2}}{\vartheta_{2,0}^2}\,.
\end{aligned}$$

The rest of the proof is obvious. □

Corollary 5.6. *If in* Theorem 5.5 n_1 *and* n_2 *are sufficiently large, then it is valid*

$$\mathbf{S}_{I,*}^{-1}=\begin{pmatrix}\frac{\vartheta_{1,0}^2}{n_1}-\frac{\vartheta_{1,0}^2c_{1,1}}{n_1^2}, & -\frac{\vartheta_{1,0}\vartheta_{2,0}c_{1,2}}{n_1n_2}\\ -\frac{\vartheta_{2,0}\vartheta_{1,0}c_{2,1}}{n_2n_1}, & \frac{\vartheta_{2,0}^2}{n_2}-\frac{\vartheta_{2,0}^2c_{2,2}}{n_2^2}\end{pmatrix}$$

and thus

$$\begin{pmatrix}\hat\vartheta_1(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\\ \hat\vartheta_2(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\end{pmatrix}\approx\mathbf{S}_{I,*}^{-1}\begin{pmatrix}\boldsymbol{v}_{I,1}'(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\frac{\mathbf{V}_1^{-1}}{\vartheta_{1,0}^2}\boldsymbol{v}_{I,1}(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\\ \boldsymbol{v}_{I,2}'(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}^2}\boldsymbol{v}_{I,2}(\boldsymbol{Y}_1,\boldsymbol{Y}_2)\end{pmatrix}.$$

Since

$$\begin{aligned}
\boldsymbol{v}_{I,1}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) &= \boldsymbol{v}_{I,1}(\boldsymbol{Y}_1)-\mathbf{X}_1\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)\mathbf{M}_{\mathbf{B}'}\right]^+\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}\boldsymbol{w}_{I,2}\,,\\
\boldsymbol{v}_{I,2}(\boldsymbol{Y}_1,\boldsymbol{Y}_2) &= \left\{\mathbf{I}-\mathbf{X}_2\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}}+\frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)\mathbf{M}_{\mathbf{B}'}\right]^+\right\}\mathbf{X}_2'\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}\boldsymbol{w}_{I,2}\,,
\end{aligned}$$

the estimator $\hat\vartheta_1(\boldsymbol{Y}_1,\boldsymbol{Y}_2)$ *of the parameter* ϑ_1 *can be expressed as follows*

$$\hat\vartheta_1(\boldsymbol{Y}_1,\boldsymbol{Y}_2)=\hat\vartheta_1(\boldsymbol{Y}_1)+\gamma_{I,1}\,,$$

where

$$\gamma_{I,1} = \frac{1}{n_1}\Bigg\{ -k\hat{\vartheta}_1(\boldsymbol{Y}_1) + \boldsymbol{w}'_{I,2}\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}\mathbf{X}_2\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}} + \frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)\mathbf{M}_{\mathbf{B}'}\right]^+ \times$$
$$\times\, \mathbf{H}_1\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_{1,0}} + \frac{\mathbf{H}_2}{\vartheta_{2,0}}\right)\mathbf{M}_{\mathbf{B}'}\right]^+ \mathbf{X}'_2\frac{\mathbf{V}_2^{-1}}{\vartheta_{2,0}}\boldsymbol{w}_{I,2}\Bigg\}.$$

Remark 5.7. Analogously as in the model without constraints (cf. Section 4) a consideration on the insensitivity region can be proceeded. Since the methodology is the same as in Section 4, it is sufficient to state the resulting theorem without proofs only. Let

$$\begin{aligned}
\mathbf{W}_{I,h} &= \begin{pmatrix} \boldsymbol{h}'\mathbf{U}_{I,1}\boldsymbol{h}, & \boldsymbol{0} \\ \boldsymbol{0}, & \boldsymbol{h}'\mathbf{U}_{I,2}\boldsymbol{h} \end{pmatrix} - \begin{pmatrix} \boldsymbol{h}'\mathbf{T}_{I,(1,1)}\boldsymbol{h}, & \boldsymbol{h}'\mathbf{T}_{I,(1,2)}\boldsymbol{h} \\ \boldsymbol{h}'\mathbf{T}_{I,(2,1)}\boldsymbol{h}, & \boldsymbol{h}'\mathbf{T}_{I,(2,2)}\boldsymbol{h} \end{pmatrix}, \\
\mathbf{U}_{I,1} &= \left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+ \frac{\mathbf{H}_1}{\vartheta_1^3}\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+, \\
\mathbf{U}_{I,2} &= \left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+ \frac{\mathbf{H}_2}{\vartheta_2^3}\left[\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+, \\
\mathbf{T}_{I,(1,1)} &= \left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+ \frac{\mathbf{H}_1}{\vartheta_1^2}\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+ \times \\
&\quad \times \frac{\mathbf{H}_1}{\vartheta_1^2}\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+, \\
\mathbf{T}_{I,(1,2)} &= \left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+ \frac{\mathbf{H}_1}{\vartheta_1^2}\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+ \times \\
&\quad \times \frac{\mathbf{H}_2}{\vartheta_2^2}\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+ = \mathbf{T}'_{I,(2,1)}, \\
\mathbf{T}_{I,(2,2)} &= \left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+ \frac{\mathbf{H}_2}{\vartheta_2^2}\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+ \times \\
&\quad \times \frac{\mathbf{H}_2}{\vartheta_2^2}\left[\mathbf{M}_{\mathbf{B}'}\left(\frac{\mathbf{H}_1}{\vartheta_1} + \frac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+.
\end{aligned}$$

Theorem 5.8. *The insensitivity region $\mathcal{N}_{I,\vartheta}$ is*

$$\mathcal{N}_{I,\vartheta} = \left\{\delta\boldsymbol{\vartheta}:\ \delta\boldsymbol{\vartheta}'\mathbf{W}_{I,h}\delta\boldsymbol{\vartheta} \le 2\varepsilon\boldsymbol{h}'\left[\mathbf{M}_{\mathbf{B}'}\left(\tfrac{\mathbf{H}_1}{\vartheta_1} + \tfrac{\mathbf{H}_2}{\vartheta_2}\right)\mathbf{M}_{\mathbf{B}'}\right]^+\boldsymbol{h}\right\},$$

i.e.

$$\delta\boldsymbol{\vartheta} \in \mathcal{N}_{I,\vartheta} \implies \sqrt{\mathrm{Var}_{\vartheta}\big[\boldsymbol{h}'\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1 + \delta\vartheta_1, \vartheta_2 + \delta\vartheta_2)\big]} \\ \leq (1+\varepsilon)\sqrt{\mathrm{Var}_{\vartheta}\big[\boldsymbol{h}'\hat{\hat{\boldsymbol{\beta}}}(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \vartheta_1, \vartheta_2)\big]}\,.$$

REMARK 5.9. In order to utilize information on $\mathcal{N}_{I,\vartheta}$ at the point $\boldsymbol{\vartheta}$, it must be known that the actual value $\boldsymbol{\vartheta}^*$ of $\boldsymbol{\vartheta}$ is sufficiently near to $\boldsymbol{\vartheta}$. A similar consideration as in Remark 4.3 is to be made.

6. Conclusion

An additional experiment (updating in regression estimation) is relatively frequent in practice of many research domains (geodesy, physics, chemistry, technical science, biology, etc.). It influences estimators of model parameters, a determination of confidence regions, testing statistical hypotheses, etc.. Thus many statistical problems arise and even many of them are solved, still new problems occur. Since a class of regression model structures is rich it seems to be difficult to develop a universal algorithms in order to find corrections of original experiment results for all situations. Partial problems must be solved first.

In the preceding sections problems connecting with estimation of model parameters in linear nonsingular regression models are solved. It was found out that in the case of normality it is possible to find an explicit expression for corrections of the estimators from the original experiment and to accept/reject the decision that the estimators of the variance components can be used in plug-in estimators of the parameters of the mean value of the observation vector.

REFERENCES

[1] DIGGLE, P. J.—LIANG, K.-Y.—ZEGER, S. L.: *Analysis of Longitudinal Data*, Oxford Science Publications, Oxford, 1994.

[2] FIŠEROVÁ, E.—KUBÁČEK, L.: *Sensitivity analysis in singular mixed linear models with constraints*, Kybernetika (Prague) **39** (2003), 317–332.

[3] HUMAK, K. M. S.: *Statistische Methoden der Modellbildung. Band I*, Akademie-Verlag, Berlin, 1977.

[4] KUBÁČEK, L.—KUBÁČKOVÁ, L.—VOLAUFOVÁ, J.: *Statistical Models with Linear Structures*, Veda, Bratislava, 1995.

[5] KUBÁČEK, L.—KUBÁČKOVÁ, L.: *Nonsensitiveness regions in universal models*, Math. Slovaca **50** (2000), 219–240.

[6] KUBÁČEK, L.—KUBÁČKOVÁ, L.: *Statistics and Metrology*, Publishing House of Palacký University Olomouc, Olomouc 2000 (Czech).
[7] KUBÁČEK, L.—FIŠEROVÁ, E.: *Problems of sensitiveness and linearization in a determination of isobestic points*, Math. Slovaca **53** (2003), 407–426.
[8] KUBÁČEK, L.—FIŠEROVÁ, E.: *Isobestic points: sensitiveness and linearization*, Tatra Mt. Math. Publ. **26** (2003), 1–10.
[9] LEŠANSKÁ, E.: *Insensitivity regions for estimators of mean value parameters in mixed models with constraints*, Tatra Mt. Math. Publ. **22** (2001), 37–49.
[10] LEŠANSKÁ, E.: *Insensitivity regions for testing hypotheses in mixed models with constraints*, Tatra Mt. Math. Publ. **22** (2001), 209–222.
[11] LEŠANSKÁ, E.: *Optimization of the size of nonsensitiveness regions*, Appl. Math. **47** (2002), 9–23.
[12] LEŠANSKÁ, E.: *Nonsensitiveness regions for threshold ellipsoids*, Appl. Math. **47** (2002), 411–426.
[13] RAO, C. R.: *Linear Statistical Inference and Its Applications*, J. Wiley & Sons, New York-London-Sydney, 1965.
[14] RAO, C. R.—MITRA, S. K.: *Generalized Inverse of Matrices and Its Applications*, John Wiley & Sons, New York-London-Sydney-Toronto, 1971.
[15] RAO, C. R.—KLEFFE, J.: *Estimation of Variance Components and Applications*, North-Holland, Amsterdam, 1988.
[16] SCHEFFÉ, H.: *The Analysis of Variance*, J. Wiley, New York, 1959.

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