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Controller Design using Pade approximation and mixed methods

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Abstract

This paper proposed a new approach of controller design. By using model matching method we design a PID controller for higher order system. A higher order model is reduced by Pade approximation and mixed method, then design a controller for reduced order model. We take a comparatively study between close loop response of higher order system and reduced order system. The model order reduction and controller design is illustrated through a numerical example.

Keywords- *Reduced order model, Pade approximation, Mixed methods, Model order reduction, PID controller, Model matching method*.

1. Introduction

The model order reduction is generally applicable almost in all the fields of science and engineering. But, the application in electrical engineering is classical problem; this is due the fact that electrical engineering problems most often involve large scale systems [1], or very fast processes that have to be controlled using low-order controllers. The analysis and synthesis of higher order systems are difficult and generally not desirable on economic and computational considerations. Thus, it is necessary to obtain a lower order system so that, it maintains the characteristics of the original system. Pade approximation [2] is the method of model order reduction of the higher order system. This gives the simplification of a model after converting it into a reduced order model. A different approach can be used to simplify a model which results are stable model. In this approach the numerator coefficients can be obtained by Pade approximation [3] and denominator coefficients can be obtained by other methods in frequency domain [4-9]. Several methods [2, 10, 11, and 12] have been developed for designing a PID controller. In this paper a simple algebraic scheme is proposed to design a PID controller for Linear Time Invariant Continuous System. The closed loop transfer function of the original plant and reduced order models with optimal controller are compared with the reference model transfer function in frequency domain.

2. Statement of problem

2.1. PID Controller *Transfer Function*

The standard block diagram of PID controller is shown in Fig.1. PID controller can be mathematically represented as[13],

$$u(t) = k_1\left[e(t) + \frac{1}{T_i}\int_0^t e(\tau)\,d + T_d\frac{de(t)}{dt}\right] \quad \ldots (1)$$

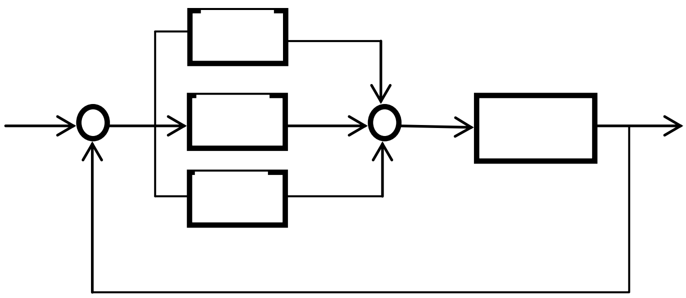

Fig. 1 General block diagram of PID controller

Where $u(t)$ and $e(t)$ denotes the control and error signals of the system. k_1 is the proportion gain, T_i and T_d represents the integral and derivative time constants respectively. The corresponding PID controller transfer function $G_c\ (s)$ is given as

$$G_c(s) = k_1\left[1 + \frac{1}{T_i s} + T_d s\right] \quad \ldots (2)$$

Equation (2) can be rewritten as

$$G_c(s) = k_1 + \frac{k_2}{s} + k_3 s \quad \ldots (3)$$

k_2 and k_3 are represents the integral and derivative gain values of the controller.

2.2. Higher Order Transfer Function

Let higher order system or process whose performance is unsatisfactory may be described by the transfer function

$$G(s) = \frac{N(s)}{D(s)} = \frac{A_{21} + A_{22}s + A_{23}s^2 + \cdots A_{2n}s^{n-1}}{A_{11} + A_{12}s + A_{13}s^2 + \cdots A_{1,n+1}s^n} \qquad \ldots (4)$$

and a reference model having the desired performance is given.

2.3. Lower Order Transfer Function

To find a *rth* lower order model for the above continuous system, where $r < n$ in the following form, such that the lower order model retains the characteristics of the original system and approximates its response as closely as possible for the same type of inputs.

$$R(s) = \frac{a_{21} + a_{22}s + a_{23}s^2 + \cdots a_{2r}s^{r-1}}{a_{11} + a_{12}s + a_{13}s^2 + \cdots a_{1,r+1}s^r} \qquad \ldots (5)$$

Where, $a_{2,j}$ and $a_{1,j}$ are scalar constants.

Objective is to derive a controller such that the performance of the augmented process matches with that of the reference model. To reduce the computational complexities and difficulties of implementation, the higher order of the system is reduced into lower second order system. And PID controller is also derived for reduced order system.

3. Reduction method

The r^{th}-order reduced approximant R(s) for G(s) is obtained by different methods as

3.1. Pade approximation

This approach stems from the theory of Pade [2] and was later used for model reduction by Shamash [4]. Before a formal presentation of the method is done, consider the following definition.

Consider a function

$$f(s) = c_0 + c_1 s + c_2 s^2 + \ldots \qquad \ldots (6)$$

and a rational function Um(s)/ Vn(s) where Um(s) and Vn(s) are m^{th} and n^{th} order polynomials in s respectively, and m <= n. The rational function Um(s)/ Vn(s) is said to be a Pade approximant of f(s) if and only if the first (m+n) terms of the power series expansions of f(s) and Um(s)/ Vn(s) are identical. For the function *f*(*s*) in Eqn. (6) to be approximated, let the following Pade approximant be defined.

$$\frac{Un(s)}{Vn(s)} = \frac{a_0 + a_1 s + a_2 s^2 + \ldots a_{n-1}s^{n-1}}{b_0 + b_1 s + b_2 s^2 + \ldots b_{n-1}s^{n-1} + s^n} \qquad \ldots (7)$$

For the first ($m + n$) terms of Eqn. (6) and Eqn. (7) to be equivalent, it becomes apparent that the following set of relations must hold:

$$\begin{aligned} a_0 &= b_0 c_o \\ a_1 &= b_0 c_1 + b_1 c_0 \\ a_{n-1} &= b_0 c_{n-1} + b_1 c_{n-2} + \ldots b_{n-1} c_0 \\ 0 &= b_0 c_n + b_1 c_{n-1} + \ldots b_n c_0 \\ &. \\ 0 &= b_0 c_{2n-1} + b_1 c_{2n-2} + \ldots b_{n-2} c_n + c_{n-1} \end{aligned} \qquad \ldots (8)$$

Once the coefficients c_i, i = 0, 1, 2... are find out [2] using Eqn. (9) and $c_j = (-1)^j a_{j+2,1}$

for the full model,

$$G(s) = \frac{d_0 + d_1 s + d_2 s^2 + \ldots d_{m-1}s^{m-1}}{e_0 + e_1 s + e_2 s^2 + \ldots e_{m-1}s^{m-1} + e_m s^m} \quad \ldots (9)$$

Eqns. (8) can be written in matrix form as

$$\begin{bmatrix} c_n & c_{n-1} & \ldots & c_1 \\ c_{n+1} & c_n & \ldots & c_2 \\ c_{n+2} & c_{n+1} & \ldots & c_3 \\ & & & \vdots \\ c_{2n-1} & c_{2n-2} & \ldots & c_n \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ . \\ b_{n-2} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} -c_0 \\ -c_1 \\ . \\ -c_{n-2} \\ -c_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} c_0 & 0 & 0 & 0 \\ c_1 & c_0 & 0 & 0 \\ . & . & . & .. \\ c_{n-2} & c_{n-1} & c_1 & c_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ . \\ b_{n-2} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ . \\ a_{n-2} \\ a_{n-1} \end{bmatrix} \qquad ..(10)$$

3.2. Mixed Routh Hurwitz array and Pade approximation

In this method numerator is obtained by Pade approximation and denominator is obtained by Routh Hurwitz array whose detail description is given by Shamash in [4]. After calculating the coefficient of denominator by Routh Hurwitz array, numerator coefficient calculated by equation (8)

3.3 Mixed Routh approximation and Pade approximation

In this method numerator is obtained by Pade approximation and denominator is obtained by Routh approximation.

Routh Approximation Algorithm

Step 1: Determine the reciprocal of the full model $\hat{G}(s)$

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Step 2: Construct the $\hat{\alpha}-\hat{\beta}$ tables [1] corresponding to $\hat{G}(s)$

Step 3: For a k^{th} order reduced model use recursive formulae in Eqn. (11 and 12) to find

$$\hat{R}_k(s)=\hat{P}_k(s)/\hat{Q}_k(s)$$

Step 4: Reverse the coefficients of $\hat{P}_k(s)$ and $\hat{Q}_k(s)$ back to find $R_k(s)=P_k(s)/Q_k(s)$

In step 3 $\hat{P}_k(s) and \hat{Q}_k(s)$ can be finding by the following formula

$$\hat{P}_k(s)=\alpha_k\,\hat{P}_{k-1}(s)+\hat{P}_{k-2}(s)+\beta_k \qquad ...(11)$$

$$\hat{Q}_k(s)=\alpha_k\,\hat{Q}_{k-1}(s)+\hat{Q}_{k-2}(s) \qquad ...(12)$$

Where k = 1,2,3,4...

3.4. Mixed Pade approximation and Truncation method.

In this method first denominator is obtained by Truncation method [9] and then Numerator is obtained by Pade approximation.

4. General Algorithm for designing the PID controller

Step 1 Construction of a specified model whose closed loop system must approximate to that of the original closed loop response. Let it be specified as:

$$T(s)=\frac{a_0^*+a_1^*s+a_2^*s^2+...a_m^*s^m}{b_0^*+b_1^*s+b_2^*s^2+...\ b_n^*s^n} \qquad ...(13)$$

$$= c_0+c_1s+c_2s^2+... \qquad ...(14)$$

Where equation (14) is the power series expansion of equation (13) about s=0.

Step 2 Specify the structure of the controller and express that in the form of transfer function as given below.

$$G_c(s)=k_1+\frac{k_2}{s}+k_3s \qquad ...(15)$$

Step 3 Determination of closed loop transfer function consisting of unknown controller parameters.

$$G_{CL}(s)=\frac{\left(k_1+\frac{k_2}{s}+k_3s\right)\left[\frac{h_0+h_1s+h_2s^2+..}{g_0+g_1s+g_2s^2+..}\right]}{1+\left(k_1+\frac{k_2}{s}+k_3s\right)\left[\frac{h_0+h_1s+h_2s^2+..}{g_0+g_1s+g_2s^2+..}\right]} \qquad ...(16)$$

This can be written as

$$G_{CL}(s)=\frac{a_0+a_1s+a_2s^2+a_3s^3+.....}{b_0+b_1s+b_2s^2+b_3s^3+.....} \qquad ...(17)$$

Step 4 comparing the coefficients of both transfer function (16) and (17) we get

$a_0=h_0.k_0$

$a_1=h_0.k_1+h_1.k_2$

$a_2=h_1k_1+h_2k_2+h_0k_3$

$a_3=h_2k_1+h_3k_2+h_1k_3$

$a_4=h_3k_1+h_4k_2+h_2k_3$

$a_5=h_4k_1+h_5k_2+h_3k_3$

$a_6=h_5k_1+h_4k_3$

$a_7=h_5k_3$ ... (18)

And

$b_0= h_0.k_0$

$b_1= g_0+h_0.k_1+h_1.k_2$

$b_2= g_1+h_1.k_1+h_2.k_2+h_0.k_3$

$b_3= g_2+h_2.k_1+h_3.k_2+h_1.k_3$

$b_4= g_3+h_3.k_1+h_4.k_2+h_2.k_3$

$b_5= g_4+h_4.k_1+h_5.k_2+h_3.k_3$

$b_6= g_5+h_5.k_1+h_4.k_3$

$b_7= g_6+h_5.k_3$ … (19)

Step 5 Now comparing the closed loop transfer function of the plant and controller with model transfer function we get:

$$a_0=b_0.c_0$$

$$a_1=b_0.c_1+b_1.c_0$$

$$a_2=b_0.c_2+b_1.c_1+b_2.c_0$$

$$a_3=b_0.c_3+b_1.c_2+b_2.c_1+b_3.c_0$$

$$a_4=b_0.c_4+b_1.c_3+b_2.c_2+b_3.c_1+b_4.c_0$$

:

… (20)

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Step 6 Now substituting the values of a's, b's and c's and solving linear simultaneous equations. We can find out the values of k_1, k_2, k_3.

6. Example

Consider the synchronous machine excitation control problem from [12], where

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & -33.6 & -155.94 & -209.46 & -102.42 & -18.3 \end{bmatrix}$$

$B= [0\ 0\ 0\ 0\ 0\ 1]^T$

$C= [2\ 3\ 16\ 20\ 8\ 1]$

$D= 0$

Choosing $r = 1$ and

$Q = \text{dig}\ ([1, 1, 1, 1, 1, 1])$

By solving Riccati equation [14-16] we get the values of K's feedback gains as follows.

$K^T= [0.2361\ 1.1981\ 1.6374\ 0.8591\ 0.2232\ 0.0395]$

Using feedback gain matrix the reference model is obtained

$$T^*(s)$$

$$= \left[.8944 - 12.5772\, s + 139.8524\, s^2 - 1365.2\, s^3 + 12536 s^4 \ldots\right]$$

$c_0= 0.8944$

$c_1= -12.5772$

$c_2=139.8524$

$c_3= -1365.2$

$c_4=12536$

The matrix A, B and C gives the plant transfer function as

$$G(s) = [SI - A]^{-1}B$$

$$G_{(s)} = \frac{2+3s+16s^2+20s^3+8s^4+s^5}{2+33.600s+155.9400s^2+209.4600s^3+102.4200s^4+18.300s^5+s^6}$$

By using plant transfer function and PID controller transfer function we get the closed loop transfer function of the plant as

$$G_{CL}(s) = \frac{\left(k_1 + \frac{k_2}{s} + k_3 s\right) G(s)}{1 + \left(k_1 + \frac{k_2}{s} + k_3 s\right) G(s)}$$

Using equation (18), (19) and (20) and solving linear simultaneous equations we get the values of k_1, k_2 , and k_3 as

$k_1=0.5405$

$k_2=0.0380$

$k_3=1.2156$

On substituting values of k_1, k_2 and k_3 in equation (18) and (19) we get values of a's and b's closed loop transfer function as

$$G_{CL(s)} = \frac{0.0760+1.1950s+4.6607s^2+13.0548s^3+30.5636s^4+28.6740s^5+10.2653s^6+1.2156s^7}{0.0760+3.1950s+38.2607s^2+168.9948s^3+240.0236s^4+131.0940s^5+28.5653s^6+2.2156s^7}$$

PID Controller Design Using Reduced Order Model

The original plant transfer function is first reduced by different methods of model order reduction and obtained a second order reduced models as below.

A: Pade approximation

$$R_{2PA}(s) = \frac{0.02645s+0.01266}{s^2+0.2202s+0.01266}$$

B: Mixed method by Pade approximation with Routh Hurwitz array

$$R_{2PRH}(s) = \frac{-0.0043s+2}{137.1807s^2+30.5957s+2}$$

C: Mixed method by Pade approximation with Routh approximation

$$R_{2PRA}(s) = \frac{0.0214s+0.0139}{s^2+0.3241s+0.0139}$$

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D: Mixed method by Pade approximation with Truncation method

$$R_{2PT}(s) = \frac{3s+2}{155.94s^2+33.6s+2}$$

A comparison of step responses is shown in Fig2.

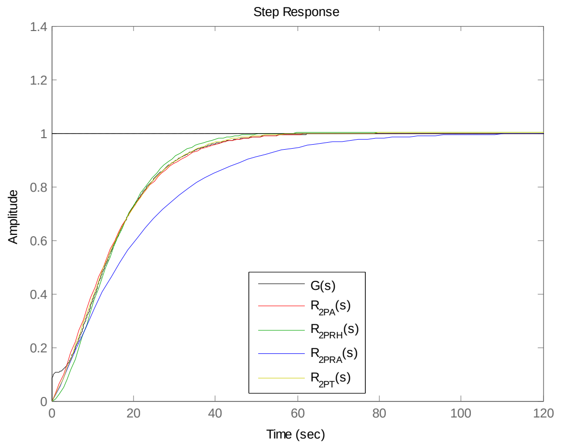

Fig-2 Comparison of step responses of original system and reduced models obtained by Pade approximation and Mixed methods

Now the compensator $(k_1 + \frac{k_2}{s} + k_3 s)$ is introduced with reduced plant models and we get closed loop transfer functions as.

E: Close loop response of Pade approximation

$G_{CLPA}(s)=$

$$\frac{0.0026864+0.03236s+0.05991s^2+0.008384s^3}{0.0026864+0.04502s+0.28011s^2+1.008384s^3}$$

F: Close loop response of mixed Pade approximation and Routh Hurwitz array

$G_{CLPRH}(s)=$

$$\frac{0.129+1.5699s+4.8484s^2-0.0104s^3}{0.129+3.5699s+35.4441s^2+137.1702s^3}$$

G: Close loop response of mixed Pade approximation and Routh approximation

$G_{CLPRA}(s)=$

$$\frac{0.00088+0.01227s+0.03220s^2+0.02371s^3}{0.00088+0.02617s+0.26630s^2+1.02371s^3}$$

Table 1: Comparison of step response of open loop systems

H: Close loop response of mixed Pade approximation and Truncation method

$G_{CLPT}(s)=$

$$\frac{0.1274+1.7553s+6.1441s^2+5.6967s^3}{0.1274+3.7553s+39.7441s^2+161.6367s^3}$$

A comparison of step responses is shown in Fig 3.

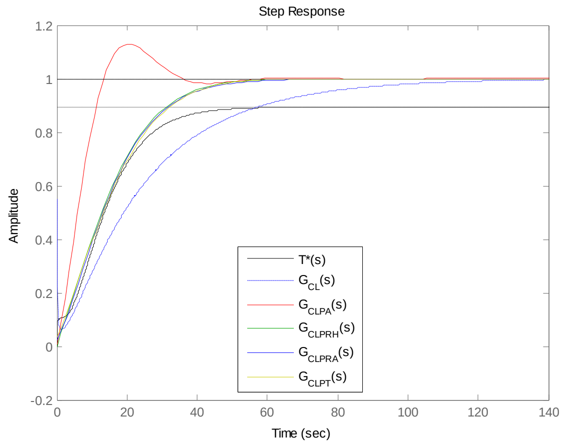

Fig-3 give the comparison of close loop responses of original plant and reduced order models which give best responses by different order reduction techniques with reference model transfer function

7. Conclusion

It is shown that the close loop response by Pade Approximation, $G_{CLPA}(s)$ shows the best overall performance specifications as comparison to original system and reduced order model. It is better to get close loop response of a reduced order model because these gives the better performance specifications as compare to original model close loop response specification. Fig-2 shows that open loop response of reduced order models have nearly same specifications except the mixed Pade and Routh approximation. Fig-3 show that the except Pade approximation all close loop response of reduced models have nearly same specifications. Table-1 gives the comparison on the basis of various parameters of original plant and second order reduced models. Table-2 gives the comparison on the basis of various parameters of closed loop responses.

Specification\Models	G(s)	$R_{2PA}(s)$	$R_{2PRH}(s)$	$R_{2PRA}(s)$	$R_{2PT}(s)$
Rise time(sec)	30.2	28.4	24.8	44.3	27.3
Settling time(sec)	47	47.3	41.1	79.4	45
Peak amplitude Overshoot (%) At time(sec)	>=0.999 0 >=80	>=0.998 0 >=70	>=1 0.0517 >=60	>=.997 0 >=120	>=1 0.00627 >=80
Final value	1	1	1	1	1

Table 2: Comparison of step response of closed loop systems

Specification\Models	T*(s)	$G_{CL}(s)$	$G_{CLPA}(s)$	$G_{CLPRH}(s)$	$G_{CLPRA}(s)$	$G_{CLPT}(s)$
Rise time(sec)	27.8	54.9	9.9	29.2	29.5	30.2
Settling time(sec)	42.3	101	32.9	46.4	48.8	47.6
Peak amplitude Overshoot (%) At time(sec)	>=0.892 0 >60	>=.996 0 >=140	1.13 12.9 20.1	>=1 0 >=120	>=.997 0 >=70	>=1 0 >=140
Final value	0.894	1	1	1	1	1

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