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# GENERATION OF NEW FRACTALS FOR SINE FUNCTION

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## Abstract

*The generation of fractals and study of the dynamics of transcendental function is one of emerging and interesting field of research nowadays. We introduce in this paper the complex dynamics of sine function of the type* $\{\sin(z^n)+c\}$, *where* $n \geq 2$ *and applied Ishikawa iteration to generate new Relative Superior Mandelbrot set and Relative Superior Julia set. Our results are different from those existing in the literature.*

## Keywords

Complex dynamics, Relative Superior Julia set, Relative Superior Mandelbrot set.

## 1. Introduction

Extracting qualitative information from data is a central goal of experimental science. In dynamical systems, for example, the data typically approximate an attractor or other invariant set and knowledge of the structure of these sets increases our understanding of the dynamics. The most qualitative description of an object is in terms of its topology — whether or not it is connected? Based on this objective, this paper studies the dynamical behavior of sine function.

The study of transcendental function has emerged out as discrete dynamical systems in numerical and complex analysis. It forms a rich dynamics for well known Julia sets and Mandelbrot sets[8]. On the other hand, the dynamics of iterated polynomials are one of the greatest pioneering work of Doaudy and Hubbard[10]. Given a polynomial of degree $n \geq 2$, the most important set is the Julia set J consisting of the points $z \in C$ which have no neighborhood in family of iterates, forms a normal family. Specially for the polynomials, one can start with the set of points $I$ which converge to infinity under iteration (escaping points) and its complement $K = C / I$ is known as filled in Julia sets and it consists of points with bounded orbits.

In other words, the Julia set $J_c$ of the function $Q_c$ where $Q_c = z^2 + c$ is either totally disconnected or connected. Its counterpart, Mandelbrot set for a family $Q_c$ is defined as $M_c = \{c \in C : \textit{orbit of } 0 \textit{ under iteration by } Q_c \textit{ is bounded}\}$. For $|c| > 2$, orbit of 0 escapes to $\infty$ so only $|c| \leq 2$ is considered. For any n, $|Q_c^{(n)}(0)| > 2$, then the orbit of 0 tends to infinity[8].

The key feature of this paper is to show that the sine function, which falls under category of transcendental function, is an example, where Julia set is all of $C$. There is a great difference between the dynamics of polynomials and transcendental functions. Picard's Theorem [21] tells us that for a transcendental function $f$, given any "neighborhood of infinity" $U = [\{z; |z| > r\}, r \in R]$ $f(U)$ covers $C$ with exception of at most one point. This is certainly not true for polynomials because we find a neighborhood of $U$ so that $f(U) \subset U$.

The study of dynamical behavior of the transcendental functions were initiated by Fatou[12]. For transcendental function, points with unbounded orbits are not in Fatou sets but they must lie in Julia sets. Attractive points of a function have a basin of attraction, which may be disconnected. A point $z$ in Julia for cosine function has an orbit that satisfies $|\operatorname{Im} z| \geq 50$

A Julia set thus, satisfies the following properties:

(i) Closed
(ii) Nonempty
(iii) Forward invariant t(If $z \in J(F)$, then $F(z) \in J(F)$, where $F$ is the function).
(iv) Backward invariant

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(v) Equal to the closure of the set of repelling cycles of $F$ .

On the other hand, Fatou Set is the complement set of Julia set, also stated as stable set. Attracting cycles and their basins of attraction lie in the Fatou set, since iterates here tend to cycle and thus forms a normal family.
Thus, the iteration of complex analytic function $F$ decompose the complex plane into two disjoint sets

1. Stable Fatou sets on which the iterates are well behaved.
2. Julia sets on which the map is chaotic.

In trigonometric function, $S(z) = \sin z$ , 0 is defined as fixed point for $S$. If $x_0 \in R$, then either $S(x_0) = 0$ or $S^n(x_0) \to 0$. On the other hand, points on the imaginary axis have orbits that tend to infinity since $\sin(iy) = i\sin(hy)$. Sine and cosine functions are thus declared as "Topologically complete" [15 ].

The fixed point in topology, $z = z_0$ is declared as

(i) Attracting if $0 < |F'(z_0)| < 1$.

(ii) Superattracting if $F'(z_0) = 0$

(iii) Repelling if $|F'(z_0)| > 1$

(iv) Neutral if $F'(z_0) = e^{i2\pi\theta_0}$

If $\theta_0$ is rational, then $z_0$ is rationally indifferent or parabolic, otherwise $z_0$ is irrationally indifferent. For $S(z) = \sin z$, then 0 is rationally indifferent fixed point for $S$. There are two attracting petals along the real axis. Orbits on the imaginary axis leave a neighborhood of 0, so the orbits tend to infinity.

The fixed point 0 for $S(z) = \sin z$ also satisfies $S'(0) = 1$. Orbits on the real axis tend to 0, but on the imaginary axis tends to infinity. We cannot have uniform convergence in any neighborhood of 0.

The dynamics of sine function as revealed in the past literature studies that the points that converge to $\infty$ under iteration are organized in the form of rays. It is well known that the set of escaping points is an open neighborhood of $\infty$, which can be parameterized by dynamic rays. For the entire transcendental functions, the point $\infty$ is an essential singularity (rather than super attracting point). Ereneko[11] studied that for every entire transcendental functions, the set of escaping points is always non-empty. His query was answered in an affirmative way by R. L. Devaney[5,6 &7], for the special case of Exponential function, where every escaping point can be connected to $\infty$, along with unique curve running entirely through the escaping points.

A dynamic ray is connected component of escaping set, removing the landing points. It turns out to be union of all uncountable many dynamic rays, having Hausdroff dimension equal to one. However by a result of McMullen[15 ] the set of escaping points of a cosine family has an infinite planar Lesbegue measure. Therefore the entire measure of escaping points sits in the landing points of those rays which land at the escaping points.

In this past literature the sine function was considered of the following forms:

(i) $\sin(z^n + c)$ ,where $n \geq 2$

(ii) $f(z) = (e^{iz} - e^{-iz})/2$

We are using in our paper sine function of the type $\{\sin(z^n) + c\}$, where $n \geq 2$ and applied Relative Superior Ishikawa iterates to develop fractal images of this transcendental function. Escape criteria of polynomials are used to generate Relative Superior Mandelbrot Sets and Relative Superior Julia Sets. Our results are different from existing results in literature.

## 2. Preliminaries

The process of generating fractal images from $z \to \sin(z^n) + c$ is similar to the one employed for the self-squared function[17]. Briefly, this process consists of iterating this function up to N times. Starting from a value $z_0$ we obtain $z_1, z_2, z_3, z_4, \ldots$ by applying the transformation $z \to \sin(z^n) + c$ .

**Definition2.1: Ishikawa Iteration [13]:** Let X be a subset of real or complex numbers and $f : X \to X$ for $x_0 \in X$ , we have the sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_n = s'_n f(x_n) + (1 - s'_n)x_n$$

$$x_{n+1} = s_n f(y_n) + (1 - s_n)x_n$$

where $0 \leq s'_n \leq 1$, $0 \leq s_n \leq 1$ and $\{s'_n\}$ & $\{s_n\}$ are both convergent to non zero number.

**Definition 2.2[4, 18]:** The sequences $\{x_n\}$ and $\{y_n\}$ constructed above is called Ishikawa sequences of iterations or Relative Superior sequences of iterates.

We denote it by $RSO(x_0, s_n, s'_n, t)$. Notice that $RSO(x_0, s_n, s'_n, t)$ with $s'_n=1$ is $SO(x_0, s_n, t)$ i.e. Mann's orbit and if we place $s_n = s'_n = 1$ then $RSO(x_0, s_n, s'_n, t)$ reduces to $O(x_0, t)$.

We remark that Ishikawa orbit $RSO(x_0, s_n, s'_n, t)$ with $s'_n = 1/2$ is relative superior orbit.

Now we define Mandelbrot sets for function with respect to Ishikawa iterates. We call them as Relative Superior Mandelbrot sets.

**Definition 2.3[4, 18]:** Relative Superior Mandelbrot set RSM for the function of the form $Q_c(z) = z^n + c$, where $n$ = 1, 2, 3, 4… is defined as the collection of $c \in C$ for which the orbit of 0 is bounded *i.e.* $RSM = \{c \in C : Q_c^k(0) : k = 0, 1, 2...\}$ is bounded.

In functional dynamics, we have existence of two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number, such that the modulus of every point in the orbit is less than this number.

The collection of points that are bounded, *i.e.* there exists M, such that $|Q^n(z)| \le M$, for all *n*, is called as a prisoner set while the collection of points that are in the stable set of infinity is called the escape set. Hence, the boundary of the prisoner set is simultaneously the boundary of escape set and that is Julia set for Q.

**Definition 2.4[4, 18]]:** The set of points RSK whose orbits are bounded under relative superior iteration of the function Q (z) is called Relative Superior Julia sets. Relative Superior Julia set of Q is boundary of Julia set RSK

## 3. Generating the fractals:

We have used in this paper escape time criteria of Relative Superior Ishikawa iterates for function $z \to \sin(z^n) + c$.

**Escape Criterion for Quadratics:** Suppose that $|z| > \max\{|c|, 2/s, 2/s'\}$, then $|z_n| > (1+\lambda)^n |z|$ and $|z_n| \to \infty$ as $n \to \infty$. So, $|z| \ge |c|$ and $|z| > 2/s$ as well as $|z| > 2/s'$ shows the escape criteria for quadratics.

**Escape Criterion for Cubics:** Suppose $|z| > \max\{|b|, (|a| + 2/s)^{1/2}, (|a| + 2/s')^{1/2}\}$ then $|z_n| \to \infty$ as $n \to \infty$. This gives an escape criterion for cubic polynomials

**General Escape Criterion:** Consider $|z| > \max\{|c|, (2/s)^{1/n}, (2/s')^{1/n}\}$ then $|z_n| \to \infty$ s $n \to \infty$ is the escape criterion. (Escape Criterion derived in [4, 18]).

Note that the initial value $z_0$ should be infinity, since infinity is the critical point of $z \to \cos(z^n) + c$. However instead of starting with $z_0$ = infinity, it is simpler to start with $z_1 = c$, which yields the same result. (*A critical point of* $z \to F(z) + c$ *is a point where* $F'(z) = 0$ *).*

## 4. Geometry of Relative Superior Mandelbrot Sets and Relative Superior Julia Sets:

The fractals generated from the equation $z \to \sin(z^n) + c$ possesses symmetry along the real axis

**Relative Superior Mandelbrot Sets:**

- In case of quadratic polynomial, the central body is divided into two parts. The body is maintaining symmetry along the real axis. Secondary lobes are very small initially for $s$ = 1, $s'$ =1. As the value is changed to $s$ =0.3 $s'$ =0.7, the central body is divided into four parts but the middle part is quite larger in comparison to the head and tail. Secondary lobes seem to appear larger than initial stage. But as the value of the set changes to $s$ =0.1, $s'$ =0.5, the central body gets bifurcated into four parts and size of secondary lobes becomes larger. The fractal generated for $s$ =0.1, $s'$ =0.5 appears to be in the form of a fish.

- In case of Cubic polynomial, the central body is divided into two equal parts, each part containing one major secondary lobe and many minor secondary lobes. The symmetry of this body is maintained along both axes. For $s$ =0.3 $s'$ =0.7, the major secondary lobes starts enlarging in its size and also a tiny bulb seems to occur along the real axis.

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Satellites type structures are observed around the main body maintained in symmetry. As the value of relative Superior Mandelbrot set changes to $s$=0.1, $s'$=0.5, the secondary bulbs shows greater enlargement. Satellites type structures are observed more in number around the main body maintained in symmetry along the both axes.

- In case of Biquadratic polynomial, the central body is divided into three arts, each part having one major secondary bulb along with large number of minor secondary bulbs. The body is maintaining symmetry along the real axis. . For $s$=0.3, $s'$=0.7, the two of the major secondary lobes retains their size but one of them grows larger in size and undergoes bifurcation along the real axis. But as the value of relative Superior Mandelbrot set changes to $s$=0.1, $s'$=0.5, the secondary bulbs grows up in size while the central body shrinks. There occurs appearance of bifurcation along all three major secondary lobes. One major lobe that exists along the real axis shows maximum bifurcation *i.e.* it is divided into three parts.

**Relative Superior Julia Sets:**

- Relative Superior Julia Sets for the transcendental function $\sin(z)$ appears to follow law of having 2*n* wings. These sets maintained their symmetry along both the axes *i.e.* along real and imaginary axis.
- The Relative Superior Julia Sets for quadratic function is divided into four wings with central black body. Its symmetry exists along both axes.
- The Relative Superior Julia Sets for Cubic function is divided into six wings having reflectional and rotational symmetry, along with a larger black region.
- The Relative Superior Julia Sets for Biquadractic function is divided into eight wings possessing the reflectional and rotational symmetry, along with a large escape region.
- It is also observed from the graphical study of fixed points of Relative Superior Julia Sets that the convergence for $s$=0.3, $s'$= 0.7 is quite faster for all polynomials in comparison to the convergence for $s$=0.1, $s'$= 0.5.

## 5. Generation of Relative Superior Mandelbrot Sets:

### 5.1 Mandelbrot Sets of Quadratic function:

**Fig 1: Relative Superior Mandelbrot Set for s= s'=1**

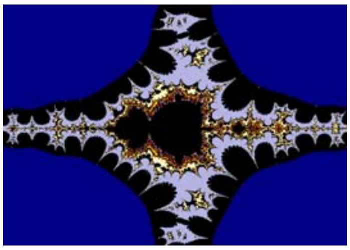


**Fig2: Relative Superior Mandelbrot Set for s=0.1, s'=0.5**

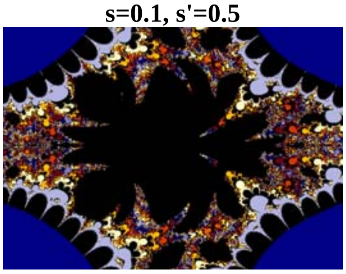
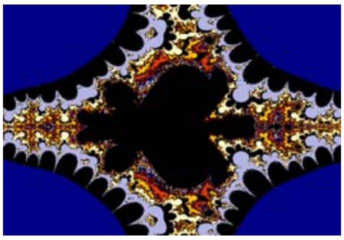


**Fig 3: Relative Superior Mandelbrot Set for s=0.3, s'=0.7**



### 5.2 Mandelbrot Sets of Cubic function:

**Fig 1: Relative Superior Mandelbrot Set for s= s'=1**

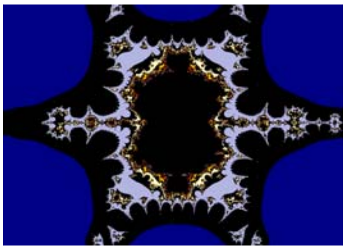


**Fig2: Relative Superior Mandelbrot Set for s=0.1, s'=0.5**

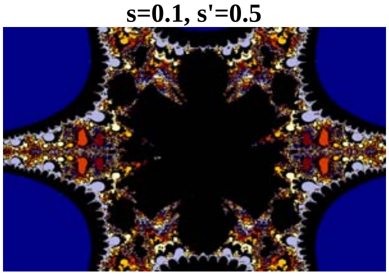


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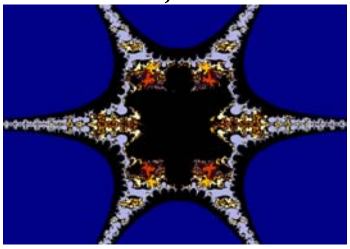
**Fig 3: Relative Superior Mandelbrot Set for s=0.3, s'=0.7**



### 5.3 Mandelbrot Sets of Biquadratic function:

**Fig 1: Relative Superior Mandelbrot Set for s= s'=1**

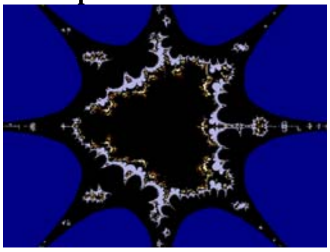


**Fig2: Relative Superior Mandelbrot Set for s=0.1, s'=0.5**

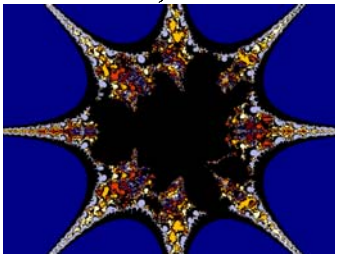
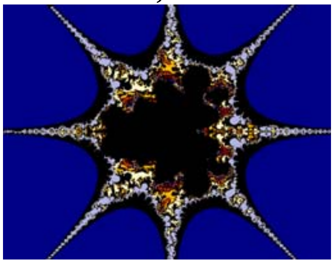
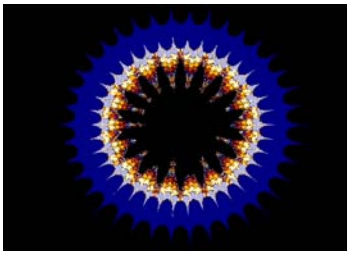


**Fig 3: Relative Superior Mandelbrot Set for s=0.3, s'=0.7**



### 5.4 Generalization of Relative Superior Mandelbrot Set

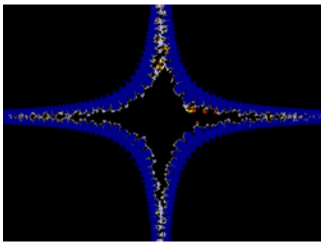
**Fig1: Relative Superior Mandelbrot Set for s=0.1, s'=0.5 n=19**



## 6. Generation of Relative Superior Julia Sets:

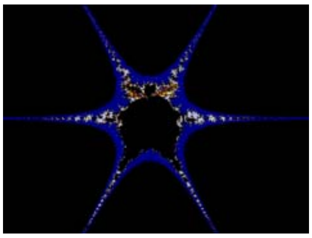
### 6.1 Julia sets of Quadratic function:

**Fig1: Relative Superior Julia Set for s=0.3, s'=0.7 c=-0.1848425651+0.2453514273i**



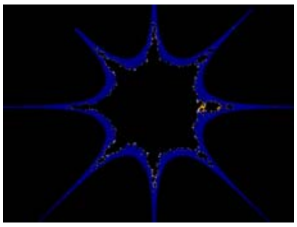
### 6.2 Julia Sets of Cubic function:

**Fig1: Relative Superior Julia Set for s=0.3, s'=0.7 c= 0.06553079165+1.052110021i**



### 6.3 Julia Sets of Biquadratic function:

**Fig1: Relative Superior Julia Set for s=0.3, s'=0.7 c= 0.02380189886-0.06066045312i**



## 7.Fixed points:

### 7.1 Fixed points of quadratic polynomial

**Table 1:** Orbit of F(z) at **s=0.1 and s'=0.5** for **($z_0$=3934870291+0i)**

| Number of iteration i | \|F(z)\| | Number of iteration i | \|F(z)\| |
|---|---|---|---|
| 57. | 0.4233 | 66. | 0.4237 |
| 58. | 0.4234 | 67. | 0.4237 |
| 59. | 0.4234 | 68. | 0.4237 |
| 60. | 0.4235 | 69. | 0.4237 |
| 61. | 0.4235 | 70. | 0.4237 |
| 62. | 0.4236 | 71. | 0.4237 |
| 63. | 0.4236 | 72. | 0.4237 |
| 64. | 0.4236 | 73. | 0.4237 |
| 65. | 0.4233 | 74. | 0.4238 |
| 66. | 0.4234 | 75. | 0.4238 |

***Here we skipped 56 iteration and observed that the value converges to a fixed point after 73 iterations***

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**Figure1**. Orbit of *F*(z) at **s=0.1 and s'=0.5** for **($z_0$= -0.3934870291+0i)**

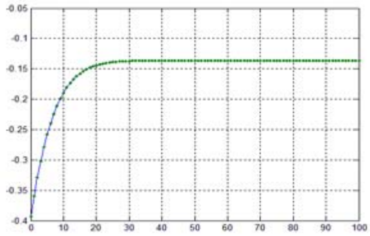


**Table 2:** Orbit of F(z) at **s=0.3 and s'=0.7** for **($z_0$=-0.1848425651+0.2453514273i)**

| Number of iteration i | \|F(z)\| | Number of iteration i | \|F(z)\| |
|---|---|---|---|
| 13. | 0.4384 | 23. | 0.4243 |
| 14. | 0.4342 | 24. | 0.4242 |
| 15. | 0.4313 | 25. | 0.4241 |
| 16. | 0.4291 | 26. | 0.424 |
| 17. | 0.4276 | 27. | 0.4239 |
| 18. | 0.4265 | 28. | 0.4239 |
| 19. | 0.4257 | 29. | 0.4239 |
| 20. | 0.4252 | 30. | 0.4239 |
| 21. | 0.4248 | 31. | 0.4238 |
| 22. | 0.4245 | 32. | 0.4238 |

***Here we skipped 12 iteration and observed that the value converges to a fixed point after 30 iterations***

**Figure2**. Orbit of F(z) at **s=0.3 and s'=0.7** for **($z_0$=-0.1848425651+0.2453514273i)**

**7.2 Fixed points of cubic polynomial**

**Table 1:** Orbit of F(z) at **s=0.1 and s'=0.5** for **($z_0$= 0.07944042258+0.03670696339i)**

| Number of iteration i | \|F(z)\| | Number of iteration i | \|F(z)\| |
|---|---|---|---|
| 44. | 0.4157 | 54. | 0.4162 |
| 45. | 0.4158 | 55. | 0.4163 |
| 46. | 0.4159 | 56. | 0.4163 |
| 47. | 0.4159 | 57. | 0.4163 |
| 48. | 0.416 | 58. | 0.4163 |
| 49. | 0.4161 | 59. | 0.4163 |
| 50. | 0.4161 | 60. | 0.4163 |
| 51. | 0.4161 | 61. | 0.4163 |
| 52. | 0.4162 | 62. | 0.4164 |
| 53. | 0.4162 | 63. | 0.4164 |

***Here we skipped 43 iteration and observed that the value converges to a fixed point after 61 iterations***

**Figure 1**. Orbit of F(z) at **s=0.1 and s'=0.5** for **($z_0$= 0.07944042258+0.03670696339i)**

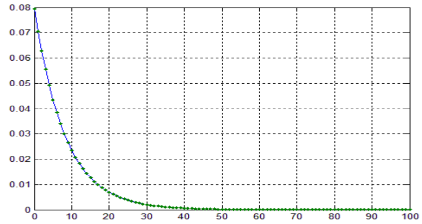
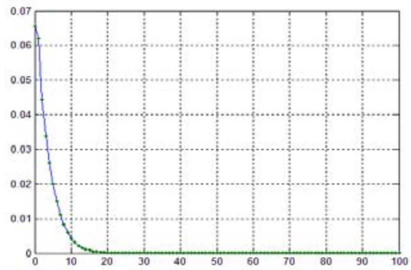


Table 2: Orbit of F(z) at **s=0.3 and s'=0.7** for **($z_0$ =0.06553079165+1.052110021i)**

| Number of iteration i | \|F(z)\| | Number of iteration i | \|F(z)\| |
|---|---|---|---|
| 1. | 0.3072 | 11. | 0.4161 |
| 2. | 0.3484 | 12. | 0.4162 |
| 3. | 0.3763 | 13. | 0.4163 |
| 4. | 0.3935 | 14. | 0.4163 |
| 5. | 0.4036 | 15. | 0.4164 |
| 6. | 0.4093 | 16. | 0.4164 |
| 7. | 0.4125 | 17. | 0.4164 |
| 8. | 0.4143 | 18. | 0.4164 |
| 9. | 0.4152 | 19. | 0.4164 |
| 10. | 0.4158 | 20. | 0.4164 |

***Here observe that the value converges to a fixed point after 14 iterations***

**Figure 2**. Orbit of F(z) at **s=0.3 and s'=0.7** for **($z_0$ =0.06553079165+1.052110021i)**



**7.3 Fixed points of Biquadratic polynomial**

**Table 1:** Orbit of F(z) at **s=0.1 and s'=0.5** for **($z_0$= 0.1350789463+0.1340743799i)**

| Number of iteration i | \|F(z)\| | Number of iteration i | \|F(z)\| |
|---|---|---|---|
| 54. | 0.4807 | 64. | 0.4811 |
| 55. | 0.4807 | 65. | 0.4811 |
| 56. | 0.4808 | 66. | 0.4811 |
| 57. | 0.4808 | 67. | 0.4811 |
| 58. | 0.4809 | 68. | 0.4811 |
| 59. | 0.4809 | 69. | 0.4811 |
| 60. | 0.481 | 70. | 0.4811 |
| 61. | 0.481 | 71. | 0.4811 |
| 62. | 0.481 | 72. | 0.4812 |
| 63. | 0.481 | 73. | 0.4812 |

***Here we skipped 53 iteration and observed that the value converges to a fixed point after 71 iterations***

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**Figure 1**. Orbit of F(z) at **s=0.1 and s'=0.5** for **($z_0$= 0.1350789463+0.1340743799i)**

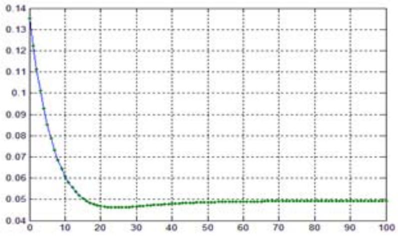
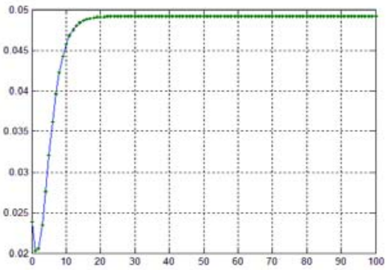



**Table 2:** Orbit of F(z) at **s=0.3 and s'=0.7** for **($z_{0=}$ 0.02380189886-0.06066045312i)**

| Number of iteration i | \|F(z)\| | Number of iteration i | \|F(z)\| |
|---|---|---|---|
| 6. | 0.4033 | 16. | 0.4801 |
| 7. | 0.4297 | 17. | 0.4805 |
| 8. | 0.4473 | 18. | 0.4807 |
| 9. | 0.459 | 19. | 0.4809 |
| 10. | 0.4667 | 20. | 0.481 |
| 11. | 0.4717 | 21. | 0.4811 |
| 12. | 0.475 | 22. | 0.4811 |
| 13. | 0.4772 | 23. | 0.4811 |
| 14. | 0.4786 | 24. | 0.4812 |
| 15. | 0.4795 | 25. | 0.4812 |

***Here we skipped 05 iteration and observed that the value converges to a fixed point after 23 iterations***

**Figure 2**. Orbit of F(z) at **s=0.3 and s'=0.7** for **($z_0$ =0.02380189886-0.06066045312i)**




## 8.Conclusion:

In this paper we studied the sine function which is one of the members of transcendental family. The fixed point 0 for $S(z) = \sin z$ also satisfies $S'(0) = 1$. Orbits on the real axis tend to 0, but on the imaginary axis tends to infinity. Relative Superior Julia sets possess $2n$ wings. The surrounding region of the Mandelbrot set appears to be an invariant Cantor set in the form of curve or "hair" that extends to $\infty$. The orbit of any point on hair tends to infinity under iteration. This geometry of hairs appears to be quite similar to that of exponential family and hence showed the property of transcendental function. On the other hand, Julia sets plane represented the region filled up of large number of escaping points.

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