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A Computational Method for Solving Two Point Boundary Value Problems of Order Four

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Abstract

Present paper portrays a computational method using cubic B-spline to solve fourth order boundary value problems. The proposed scheme is first applied for solution of special case fourth order boundary value problem. The method is, then, extended to solution of non-linear and singular problems. Selected Examples from the literature are solved numerically using computer programs in MATLAB.

Key words: *Fourth order boundary value problem, Singular boundary value problem, Cubic B-spline, Nodal points, Maximum absolute errors.*

1. Introduction

Engineers are researching solutions to resolve many of today's technical challenges. Numerical techniques are used to solve the mathematical models in engineering problems. Many of the mathematical models of engineering problems are expressed in terms of Boundary Value Problems, which are ordinary differential equations with boundary conditions.

Fourth-order Boundary Value Problems arise in the mathematical modeling of two-dimensional channel with porous walls, viscoelastic and inelastic flows, deformation of beams, plate deflection theory, beam element theory and a number of other engineering and applied mathematics applications.

Solving such type of boundary value problems analytically is possible only in very rare cases. Many researchers worked for the numerical solutions of fourth order boundary value problems. Some numerical methods such as finite difference method, differential transformation method, Adomian's decomposition method, homotopy perturbation method, variational iteration method, spline methods have been developed for solving such boundary value problems.

Two-point and multi-point boundary value problems for fourth order ordinary differential equations have attracted a lot of attention recently. Many authors have studied the beam equation under various boundary conditions and by different approaches. Consider smooth approximation to the problem of bending a rectangular clamped beam of length l resting on elastic foundation. The vertical deflection w of the beam satisfies the system

$$\frac{d^4w}{dx^4} + (k/D)\,w = D^{-1}\,q(x), \tag{1}$$

$$w(0) = w(l) = w'(0) = w'(l) = 0.$$

where D is the flexural rigidity of the beam, and k is the spring constant of the elastic foundation, and the load $q(x)$ acts vertically downwards per unit length of the beam. The details of the mechanical interpretation are given in [1]. Mathematically, the system (1) belongs to a general class of boundary problems of the form

$$\frac{d^4y}{dx^4} + f(x)\,y(x) = g(x), \qquad a < x < b \tag{2}$$

$$y(a) = A_1,\ y(b) = A_2,\ y'(a) = B_1,\ y'(b) = B_2 \tag{3}$$

where $f(x)$ and $g(x)$ are continuous on $[a,b]$ and $A_i^s, B^s{}_i$ are finite real arbitrary constants. The analytical solution of (2) for arbitrary choices of $f(x)$ and $g(x)$ cannot be determined. So, numerical methods are developed to overcome this limitation. Usmani [2] has formulated a simple condition that guarantees the uniqueness of the solution of the problem (2) and (3).

Among many numerical methods, as enumerated above, Spline methods have been widely applied for the approximation solutions of boundary value problems including fourth order boundary value problems. (See [3, 4] and references therein). Also, Cubic B-spline has been used to solve boundary value problems and system of boundary value problems [5, 6, 7], singular boundary value problems [8] and also, second order perturbation problems by some authors. In the present paper, we will use cubic B-spline to solve fourth order boundary value problems.

This paper is organized as follows. In Section 2, preliminary results and derivations of cubic B-splines are presented; Section 3 contains cubic B-spline solutions of the special linear fourth order boundary value problems based on the results in section 2; general case of the boundary value problem is discussed in section 4; section 5 deals with non-linear problem; while section 6 contains treatment of singular problem. Section 7 concludes the paper.

2. Derivations for Cubic B-spline

The given range of independent variable is $[a,b]$. For this range we choose equidistant points given by $\pi = \{a = x_0, x_1, x_2, ...x_n = b\}$ i.e. $x_i = a + ih \ \ (i = 0,1,2.....n)$ where $h = \frac{b-a}{n}$. Let us define $S_3(\pi) = \left\{ p(t) \in C^2[a,b] \right\}$ such that $p(t)$ reduces to cubic polynomial on each sub-interval (x_i, x_{i+1}). The basis function is defined as

$$B_i(x) = \frac{1}{6h^3} \begin{cases} (x - x_{i-2})^3, & if\ x \in [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 & \\ -3(x - x_{i-2})^3, & if\ x \in [x_{i-1}, x_i] \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 & \\ -3(x_{i+1} - x)^3, & if\ x \in [x_i, x_{i+1}] \\ (x_{i+2} - x)^3, & if\ x \in [x_{i+1}, x_{i+2}] \\ 0, & otherwise. \end{cases}$$

Let us introduce four additional knots $x_{-2} < x_{-1} < x_0$ and $x_{n+2} > x_{n+1} > x_n$. From the above expression, it is obvious that each $B_i(x) \in C^2(\square)$. Also, the values of $B_i(x), B_i'(x)$ and $B_i''(x)$ at nodal points are given by Table I.

Table I: Values of $B_i(x), B_i'(x)$ and $B_i''(x)$ at nodes

	$B_i(x)$	$B_i'(x)$	$B_i''(x)$
x_{i-2}	0	0	0
x_{i-1}	$1/6$	$1/2h$	$1/h^2$
x_i	$4/6$	0	$-2/h^2$
x_{i+1}	$1/6$	$-1/2h$	$1/h^2$
x_{i+2}	0	0	0

Since each $B_i(x)$ is also a piecewise cubic polynomial with knots at π, each $B_i(x) \in S_3(\pi)$. Let $\Omega = \{B_{-1}, B_0, B_1, ..., B_{n+1}\}$ and let $B_3(\pi) = span\,\Omega$. The functions in Ω are linearly independent on $[a,b]$, thus $B_3(\pi)$ is $(n+3)$- dimensional. Also $B_3(\pi) = S_3(\pi)$ [9]. Let $s(x)$ be the B-spline interpolating function at the nodal points and $s(x) \in B_3(\pi)$. Then $s(x)$ can be written as $s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$. Therefore, for a given function $y(x)$, there exists a unique cubic spline $s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$ satisfying the interpolating conditions:

$$s(x_i) = y(x_i) \text{ and } s'(a) = y'(a),\ s'(b) = y'(b) \qquad (4)$$

Let $m_i = s'(x_i)$ and $M_i = s''(x_i)$, we have [10]

$$m_i = s'(x_i) \cong y'(x_i) - \frac{1}{180} h^4 y^{(v)}(x_i) \quad (5)$$

$$M_i = s''(x_i) \cong y''(x_i) - \frac{1}{12} h^2 y^{(iv)}(x_i) + \frac{1}{360} h^4 y^{(vi)}(x_i) \quad (6)$$

M_i can be applied to construct numerical difference formulae for $y^{(iii)}(x_i)$, $y^{(iv)}(x_i)$ $(i = 1,2,...,n-1)$ and $y^{(v)}(x_i)$ $(i = 2,3...,n-2)$ as follows;

$$\frac{M_{i+1} - M_{i-1}}{2h} = \frac{s^{(iii)}(x_{i-}) + s^{(iii)}(x_{i+})}{2} \cong y^{(iii)}(x_i) + \frac{1}{12} h^2 y^{(v)}(x_i) \quad (7)$$

$$\frac{M_{i+1} - 2M_i + M_{i-1}}{h^2} = \frac{s^{(iii)}(x_{i+}) - s^{(iii)}(x_{i-})}{h} \cong y^{(iv)}(x_i) - \frac{1}{720} h^4 y^{(viii)}(x_i) \quad (8)$$

$$\frac{M_{i+2} - 2M_{i+1} + 2M_{i-1} - M_{i-2}}{2h^3} = \frac{\frac{M_{i+2} - M_i}{2h} - 2\frac{M_{i+1} - M_{i-1}}{2h} + \frac{M_i - M_{i-2}}{2h}}{h^2} \cong y^{(v)}(x_i) \quad (9)$$

Now, since $s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$, using Table I and above equations, we get approximate values of $y(x_i)$, $y'(x_i)$, $y''(x_i)$, $y^{(iii)}(x_i)$ and $y^{(iv)}(x_i)$ as

$$y(x_i) = s(x_i) \cong \frac{c_{i-1} + 4c_i + c_{i+1}}{6} \quad (10)$$

$$y'(x_i) = s'(x_i) \cong \frac{c_{i+1} - c_{i-1}}{2h} \quad (11)$$

$$y''(x_i) = s''(x_i) \cong \frac{c_{i-1} - 2c_i + c_{i+1}}{h^2} \quad (12)$$

$$y^{(iii)}(x_i) = s^{(iii)}(x_i) \cong \frac{c_{i+2} - 2c_{i+1} + 2c_{i-1} - 2c_{i-2}}{2h^3} \quad (13)$$

$$y^{(iv)}(x_i) = s^{(iv)}(x_i) \cong \frac{c_{i+2} - 4c_{i+1} + 6c_i - 4c_{i-1} + c_{i-2}}{h^4} \quad (14)$$

3. Solution of special case fourth order boundary value problem

Let $y(x) = s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$ be the approximate solution of BVP

$$y^{(iv)}(x) + f(x)\, y(x) = g(x) \quad (15)$$

Discretizing BVP at the knots, we get

$$y^{(iv)}(x_i) + f(x_i)\, y(x_i) = g(x_i) \quad (i = 1,2.....n-1) \quad (16)$$

Putting values in terms of c_i^s using equations (10, 14), we get

$$\frac{c_{i+2} - 4c_{i+1} + 6c_i - 4c_{i-1} + c_{i-2}}{h^4} + f_i \frac{c_{i-1} + 4c_i + c_{i+1}}{6} = g_i \quad (17)$$

Where $f_i = f(x_i)$ and $g_i = g(x_i)$ are the values of $f(x)$ and $g(x)$ at the knots x_i.

Simplifying (17) becomes

$$6(c_{i+2} - 4c_{i+1} + 6c_i - 4c_{i-1} + c_{i-2}) + f_i h^4 (c_{i-1} + 4c_i + c_{i+1}) = 6h^4 g_i \quad (18)$$

This gives a system of $(n-1)$ linear equations for $(i = 1,2.....n-1)$ in $(n+3)$ unknowns *viz.* c_i^s $(i = -1,0,.....n+1)$. Remaining four equations will be obtained using the boundary conditions as follows;

$$y(a) = A_1 \quad \Rightarrow \quad c_{-1} + 4c_0 + c_1 = 6A_1 \quad (19)$$

$$y(b) = A_2 \quad \Rightarrow \quad c_{n-1} + 4c_n + c_{n+1} = 6A_2 \quad (20)$$

$$y'(a) = B_1 \quad \Rightarrow \quad -c_{-1} + c_1 = 2hB_1 \quad (21)$$

$$y'(b) = B_2 \quad \Rightarrow \quad -c_{n-1} + c_{n+1} = 2hB_2 \quad (22)$$

The approximate solution $y(x) = s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$ is obtained by solving the above system of $(n+3)$ linear equations in $(n+3)$ unknowns using equations (18) and (19) to (22).

Numerical Examples

In this section we illustrate the numerical techniques discussed in the previous sections by the following two boundary-value problems:

Problem 1.

$$y^{(iv)} + xy = -(8+7x+x^3)e^x \quad with$$
$$y(0) = y(1) = 0,\ y'(0) = 1,\ y'(1) = -1 \tag{23}$$

The analytical solution is $y(x) = x(1-x)e^x$. Table II compares the numerical results for problem 1 of present method and numerical method in [11].

Problem 2.

$$y^{(iv)} + 4y = 1 \quad with\ y(-1) = y(1) = 0,$$
$$y'(-1) = -y'(1) = \frac{\sinh 2 - \sin 2}{4(\cosh 2 + \cos 2)} \tag{24}$$

Given fourth order boundary value problem has analytical solution as

$$y(x) = 0.25\left[1 - 2\frac{\sin 1 \sinh 1 \sin x \sinh x + \cos 1 \cosh 1 \cos x \cosh x}{\cos 2 + \cosh 2}\right]$$

Comparison of numerical results by present method and method of [11] is demonstrated in Table III.

Table II: Max absolute errors $\|e\|$ for problem 1

h	Present method	Method in[11]
1/8	$2.37E-8$	$1.51E-5$
1/16	$5.75E-9$	$3.96E-6$
1/32	$1.47E-9$	$3.54E-8$

Table III: Max absolute errors $\|e\|$ for problem 1

h	Present method	Method in[11]
1/8	$1.29E-7$	$1.83E-5$
1/16	$3.08E-8$	$4.67E-6$
1/32	$7.54E-9$	$1.01E-6$

4. General case linear 4th order boundary value problem

Consider the boundary value problem

$$y^{(iv)}(x) + p(x)y^{(iii)}(x) + q(x)y''(x) + r(x)y'(x) + t(x)y(x) = u(x) \tag{25}$$

Subject to boundary conditions given by (3).

Let $y(x) = s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$ be the approximate solution of BVP. Discretizig at knots

$$y^{(iv)}(x_i) + p_i y^{(iii)}(x_i) + q_i y''(x_i) + r_i y'(x_i) + t_i y(x_i) = u_i \tag{26}$$

Where

$p_i = p(x_i),\ q_i = q(x_i),\ r_i = r(x_i),\ t_i = t(x_i), u_i = u(x_i)$.

Putting the values of derivatives using (10-14), we get

$$\frac{c_{i+2} - 4c_{i+1} + 6c_i - 4c_{i-1} + c_{i-2}}{h^4} + p_i \frac{c_{i+2} - 2c_{i+1} + 2c_{i-1} - 2c_{i-2}}{2h^3}$$
$$+ q_i \frac{c_{i-1} - 2c_i + c_{i+1}}{h^2} + r_i \frac{c_{i+1} - c_{i-1}}{2h} + t_i \frac{c_{i-1} + 4c_i + c_{i+1}}{6} = u_i \tag{27}$$

On simplification, it becomes

$$6\,c_{i+2} - (4c_{i+1} + 6c_i - 4c_{i-1} + c_{i-2}) + 3h\,{}_i(c_{i+2}p - 2c_{i+1} + 2c_{i-1}$$
$$-2c_{i-2}) + 6h^2 q_i (c_{i-1} - 2c_i + c_{i+1}) + 3h^3 r_i (c_{i+1} - c_{i-1}) + h^4 t_i$$
$$(c_{i-1} + 4c_i + c_{i+1}) = 6h^4 u_i \tag{28}$$

Now, the approximate solution is obtained by solving the system given by (28) and (19-22).

5. Non-linear 4th order boundary value problem

Consider non-linear fourth order BVP of the form

$$y^{(iv)}(x) = f(x, y(x), y'(x), y''(x), y^{(iii)}(x)) \tag{29}$$

Subject to boundary conditions given in (3).

Let $y(x) = s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$ be the approximate solution of BVP. It must satisfy the BVP at knots. So, we have

$$y^{(iv)}(x_i) = f(x_i, y(x_i), y'(x_i), y''(x_i), y^{(iii)}(x_i)) \tag{30}$$

Using (10-14) ,we get

$$\frac{c_{i+2}-4c_{i+1}+6c_i-4c_{i-1}+c_{i-2}}{h^4}=$$

$$f\left(\begin{array}{l} x_i, \frac{c_{i-1}+4c_i+c_{i+1}}{6}, \frac{c_{i+1}-c_{i-1}}{2h}, \\ \frac{c_{i-1}-2c_i+c_{i+1}}{h^2}, \frac{c_{i+2}-2c_{i+1}+2c_{i-1}-2c_{i-2}}{2h^3} \end{array}\right) \quad (31)$$

This eqn (31) together with eqns (19-22) gives a non-linear system of equations, which is solved to get the required solution of BVP.

6. Singular 4th order boundary value problem

Consider singular fourth order BVP of the form

$$y^{(iv)}(x)+\frac{\gamma}{x}y^{(iii)}(x)=f(x,y(x)); \quad 0\le x\le 1 \quad (32)$$

Under the boundary conditions

$$y(0)=A_1,\ y'(1)=A_2,\ y''(1)=B_1,\ y'''(0)=0. \quad (33)$$

Since $x=0$ is singular point of eqn (32), we first modify it at $x=0$ to get transformed problem as,

$$y^{(iv)}(x)+p(x)y^{(iii)}(x)=r(x,y) \quad (34)$$

where

$$p(x)=\begin{cases}0 & x=0\\ \frac{\gamma}{x} & x\neq 0\end{cases} \quad (35)$$

And

$$r(x,y)=\begin{cases}\frac{f(0,y)}{\gamma+1} & x=0\\ f(x,y) & x\neq 0\end{cases} \quad (36)$$

Now, as in previous sections, let $y(x)=s(x)=\sum_{i=-1}^{n+1}c_iB_i(x)$ be the approximate solution of BVP. Discretizig at knots, we get

$$y^{(iv)}(x_i)+p(x_i)y^{(iii)}(x_i)=r(x_i,y(x_i)) \quad (37)$$

Putting the values of derivatives using (10-14),

$$\frac{c_{i+2}-4c_{i+1}+6c_i-4c_{i-1}+c_{i-2}}{h^4}+p_i\frac{c_{i+2}-2c_{i+1}+2c_{i-1}-2c_{i-2}}{2h^3}$$

$$=r(x_i,\frac{c_{i-1}+4c_i+c_{i+1}}{6}) \quad (38)$$

And boundary conditions provide,

$$y(0)=A_1 \ \Rightarrow\ c_{-1}+4c_0+c_1=6A_1 \quad (39)$$

$$y'(1)=A_2 \ \Rightarrow\ -c_{n-1}+c_{n+1}=2hA_2 \quad (40)$$

$$y''(1)=B_1 \ \Rightarrow\ c_{n-1}-2c_n+c_{n+1}=h^2B_1 \quad (41)$$

$$y'''(0)=0 \ \Rightarrow\ c_2-2c_1+2c_{-1}-2c_{-2}=0 \quad (42)$$

This eqn (38) together with eqns (39-42) gives a non-linear system of equations, which is solved to get the required solution of BVP (32).

Numerical examples

Problem 3.

$$\frac{d^4y}{dx^4}=6e^{-4y}-12(1+x)^{-4},\ 0<x<1, \quad (43)$$

$$y(0)=0, y(1)=\ln 2,\ y'(0)=1,\ y'(1)=1/2.$$

The maximum absolute errors by our method and by finite difference method (Twizell [12]) for problem 3 are presented in following Table IV.

Problem 4.

$$\frac{d^4y}{dx^4}+\frac{4}{x}\frac{d^3y}{dx^3}=15y^5(1-x^2y^2)(1-7x^2y^2),\ x\in(0,1), \quad (44)$$

$$y(0)=\frac{1}{2}, y'(1)=-\frac{1}{\sqrt{5}}, y''(1)=\frac{1}{\sqrt{5}},\ y'''(0)=0$$

Comparison of numerical results by present method and that of [13] is demonstrated in Table V.

Table IV: Max absolute errors $\|e\|$ for problem 3

h	Present method	Method in[12]
1/8	1.22E-5	0.22E-4
1/16	7.97E-7	0.42E-5
1/32	5.38E-8	0.67E-6

Table IV: Max absolute errors $\|e\|$ for problem 4

h	Present method	Method in[13]
1/8	3.67E-6	1.10E-04
1/16	2.41E-7	2.75E-05
1/32	1.49E-8	2.86E-06

7. Conclusion

A numerical algorithm for solution of fourth order boundary value problems has been envisaged. The proposed method has been extended to solve non-linear and singular problems as well. The numerical results demonstrate that the present method approximates solution better than previously applied methods with same number of intervals.

References

[1] E.L. Reiss, A.J. Callegari, D.S. Ahluwalia, Ordinary Differential Equation with Applications, Holt, Rinehart and Winston, New Cork, 1976.

[2] R. A. Usmani, Discrete methods for boundary-value problems with Engineering application, Mathematics of Computation, 32 (1978) 1087–1096.

[3] M. Kumar, P. K. Srivastava, Computational techniques for solving differential equations by cubic, quintic and sextic spline, International Journal for Computational Methods in Engineering Science & Mechanics , 10(1) (2009) 108 – 115.

[4] M. Kumar, P. K. Srivastava, Computational techniques for solving differential equations by quadratic, quartic and octic Spline, Advances in Engineering Software 39 (2008) 646-653.

[5] N. Caglar, H. Caglar, B-spline method for solving linear system of second order boundary value problems, Computers and Mathematics with Applications 57 (2009) 757-762.

[6] H. Caglar, N. Caglar, K. Elfaituri, B-spline interpolation compared with finite difference, finite element and finite volume methods which applied to two point boundary value problems, Applied Mathematics and Computation 175 (2006) 72–79.

[7] M. Dehghan, M. Lakestani, Numerical solution of nonlinear system of second-order boundary value problems using cubic B-spline scaling functions, International Journal of Computer Mathematics, 85(9) 2008 1455–1461.

[8] M. Kumar, Y. Gupta, Methods for solving singular boundary value problems using splines: a review, Journal of Applied Mathematics and Computing 32(2010) 265–278.

[9] P. M. Prenter, Splines and variation methods, John Wiley & sons, New York, 1989

[10] F. Lang, Xiao-ping Xu, A new cubic B-spline method for linear fifth order boundary value problems, Journal of Applied Mathematics and Computing 36 (2011) 101-116.

[11] Siraj-ul-Islam, Ikram A. Tirmizi , Saadat Ashraf, A class of methods based on non-polynomial spline functions for the solution of a special fourth-order boundary value problems with engineering applications, Applied Mathematics and Computation 174 (2006) 1169-1180

[12] E. H. Twizell, A two-grid, fourth order method for nonlinear fourth order boundary value problems, Brunel University department of mathematics and Statistics Technical report TR/12/85 (1985).

[13] R. K. Sharma, C.P. Gupta, Iterative solutions to nonlinear fourth-order differential equations through multi integral methods, International Journal of Computer Mathematics, 28(1989) 219–226.