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## On Some Properties of the Sandpile Model of Self-Organized Critical Systems *Sobre Algunas Propiedades del Modelo de la Pila de los Sistemas Críticamente Auto Organizados*

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### Abstract

In this paper we analyze the sandpile model proposed by Bak, Tang and Wiesenfeld as the canonical example of self-organized critical systems. We find that the sandpile-model can reproduce staircase graphics and also that the distribution of large avalanches recurrence times in this model is log-normal. We also find that the slope of cumulative activity characterize a “province” of generation of avalanches in the same way as the seismic or evolutionary provinces do.

**Keywords:** Sandpile, criticality, self-organization.

### Resumen

En este artículo se analiza el modelo de la pila de arena propuesto por Bak, Tang y Weinsfeld como ejemplo canónico de los sistemas críticamente auto-organizados. Encontramos que la pila de arena puede reproducir gráficas tipo escalera, así como que la distribución de tiempos de recurrencia en este modelo es log-normal. Hemos también encontrado que existe una pendiente característica de la actividad acumulada que caracteriza a una “provincia” de generación de avalanchas de la misma manera que se hace para provincias sísmicas o evolutivas.

**Palabras Clave:** Pila de arena, criticalidad, auto-organización.

**PACS:** 89.75.Da; 05.65.+b

## 1 Introduction

For many years the standard statistical mechanics was a powerful theory for the study of equilibrium and uniform systems; however, when we try to study a nonequilibrium system possessing spatio-temporal structure there is no general theory to apply [1]. Indeed, it is not even clear what quantities might characterize the physics of such systems.

Recently, a great advance had been made in order to understand these systems, for example, with the aid of very simple algorithms, it had been possible to study low-dimensional discrete systems [2], and it's well known that nonlinear maps can have time series with very complex structures. Besides, between all the symmetries governing the universe, there are one that has been cherished in the last years by all the scientific community, this is the invariance against changes of size. It had been observed that there are many objects in nature who possess identical scaling properties. So, the hope has been that by studying this simple ideas the scientists can answer more general questions about the complex systems [3].

With this in mind a group of physicists led by P. Bak [4,8,12] studied the behavior of complex systems based in a new idea: What would happen if there exist a simple mechanism in nature which behaves typically and this behavior was shared by systems with many interacting parts?.

The first hypothesis for answer this question was given in 1987 by Bak, Tang and Wiesenfeld (BTW) [4]. They suggested that under certain general conditions the complex systems self-organize into a state with a complex structure. Bak et al. also proposed, that this behavior is established without any external “tuning” (In fact the term self-organization have been used to describe the ability of certain non-equilibrium systems to develop structures and patterns in absence of an external agent

[5]), and all the states into which the system organize, behave like the equilibrium systems at the critical point. So they described the behavior of these systems as "Self-Organized Criticality" (SOC) behavior.

The original ambition of the BTW article was to explain two phenomena that occur very frequently in nature: fractal structures [6] and fractal times series known as " $1/f$  fluctuations" [7]. To explain this, BTW suggest that when we have a complex system, a signal who wants to evolve through the system needs to find regions that are able to transmit it. One could think that these regions form some sort of random network which would be modified by the action of internal dynamics induced by external driving. The dynamics will stop every time that the internal dynamics have relaxed the system, so that all local regions are below a threshold and one more time the external perturbation will bring some other region above the threshold and the relaxation process will begin once again.

So it's natural to imagine that this dynamical paths have certain geometry (the fractal geometry for example) and the duration of the induced relaxation process can led to  $1/f$  noise. All this lack of typical scales even in time and in space generate correlated algebraic functions known as power laws, which has been characteristic of many phenomena in nature like earthquakes for example (Gutenberg-Richter law), river flows (Horton's law), galaxies, mountains, clouds, turbulent fluids etc. [3,8], and this is precesly the idea of what happens in SOC systems.

In order to illustrate the last ideas BTW introduced a cellular automaton model called the sandpile model that quickly became a paradigm of SOC models. In this model a pile is gradually built adding individual grains into an open system, after a transient period the sandpile reaches his critical state characterized by the slope of the pile. In this state, the minor perturbation (the adition of a single grain) can produce avalanches of all sizes, giving a power law distribution [7].

Following the publication of the theoretical sandpile model there was a spurt of worldwilde experimental activity including experiments on sand and other granular materials. It was realized that the sandpile model was an oversimplification of what really happens. Firstly, real grains have different sizes and shapes, the inestabilities in real sandpiles occur not only at the surface but also through the formation of cracks in the bulk, secondly, the sandpile model was conservative, that is, all the sand that topples ends up at the neighboring sites, there was no sand lost in the process [8].

All of this was supported by experimental facts, the first experiment was performed by Sidney Nagel and Henz Jaeger working with Leo Kadanoff [9], however the inertial effects were the responsible for an oscillatory behavior which were not included in the theorical simple sandpile model. Glen Held and co-workers at IBM's research center set up a different type of experiment more in line with the initial suggestion but with the same inertial problems [10], these expermiental results showed that, the real physical sandpiles didn't exhibit scale-invariant behavior in space and time.

Frette and co-workers [11] choose to study grains of rice and not sand. Frette et al. perfomed an experiment where rice grains were added slowly in a narrow gap between two plates. They found that the avalanche size distribution for grains with large aspect ratio presents a power law behavior while a stretched exponential behavior was observed for rounder grains. The new results were that small avalanches in real sandpiles exhibit a behavior that was consistent with SOC theory. However, the SOC-like behavior was cutt-off and overwhelmed by the large avalanches in the system.

On the other hand, once the SOC's idea was developed out, new physical applications appeared. In 1989, Bak and Tang [12] asserted that earthquakes could be the most direct example of SOC behavior in nature. Ito and Matzuzaki [13] proposed a cellular automaton model similar to the sandpile, but adapted to the study of earthquakes. In the same year, Sornette and Sornette [14] suggested that the earth crust is organized in a self-consistent way. Carlson and Langer [15] elaborated simulations with other conservative models. However, Feder and Feder [16] showed that special non-conservative models with a global perturbation exhibit SOC behavior. In 1992, Olami, Fedder and Christensen [17] proposed a model in two dimensions realated to the "Spring-Block" model of Burridge and Knopoff [18] for earthquakes and they reported properties apparently related to real seismicity. Other applications of the SOC's idea were developed in geophysics [8] (river flows, volcanic activity), in astrophysics [8] (pulsar glitches, starquakes, solar flares), economy, traffic jams [8], etc., every phenomenon where exists a power law, was a candidate for using the SOC's theory.

However, the most important application was in biology, Bak and Sneppen (BS) [19] proposed a model for biological evolution at a level of entire species. The BS model is a cellular automaton which attempts to mimic the effects of landscapes similar to the NK model introduced by Kauffman [20,21] in terms of "fitness barriers". In this model they found a power law distribution of coevolutionary avalanches and extintion events, however when the exponent who represents the distribution of extintion events  $\tau=1.5$  was compared with the real exponent obtained with the fossil records

$\tau=2.0$ , they didn't match. So, some authors [22,23,24] considered that the BS model as the sandpile model, in the sense that it was a oversimplified model because it doesn't take into account other important biological interactions like the biological speciation for example.

Although there were these problems with the SOC's theory, in recent years it has been shown that models like the spring-block earthquake model and the BS evolution model have many numerical properties reminiscent of real seismic and biological properties [25,26,27]. These models reproduced stair-shaped graphs, power laws distribution, punctuated equilibrium [25,26,27], and the idea that the slope of the cumulative seismicity or the slope of the evolutionary activity characterize a seismic or a evolutionary province [27] respectively, between others features related to real properties in geophysics and biological evolution.

In the present work we show that the sandpile model (the paradigm of SOC's theory), in essence, has the same numerical properties that both the spring-block earthquake model and the Bak-Sneppen model for evolution. We find that the sandpile model qualitatively reproduces the staircase graphics. We analyze such graphics and also we obtain some other results. This paper is organized as follows: In Section 1 we present a brief introduction to SOC, in Section 2 we present the sandpile model; in Section 3 we present the spring-block model, in section 4 we present the numerical results and some further properties of the sandpile model, finally in section 5 we present the concluding remarks.

## 2 The Sandpile Model

The sandpile model is a celular automaton in which we are trying to represent the next physical situation: Consider a pile of sand on a table, where sand is added slowly, starting from a flat configuration. This is a dynamical system with many interacting degrees of freedom, represented by the grains of sand. The flat state represents the general equilibrium state, this state has the lowest energy.

Initially, the grains of sand will stay more or less where they land. Eventually, the pile becomes steeper, and small avalanches occur. The addition of a single grain of sand can cause a local disturbance, but nothing dramatic happens. As the time passes the system reaches a statistically stationary state, where the amount of sand added is balanced, on average, by the amount of sand leaving the system along the edges of the table. In this stationary state, there are avalanches of all sizes, up to the entire system. The collection of grains of sand has been transformed from one where the individual grains follow their own independent dynamics to one where the dynamics is global [26,27]. This last ideas can be resumed in the next model: Imagine that each site  $i$ , ( $i=1,2,...,N$ ) is characterized by an integer variable  $h_i$  which gives the height of the pile at a given point (see Figure 1).

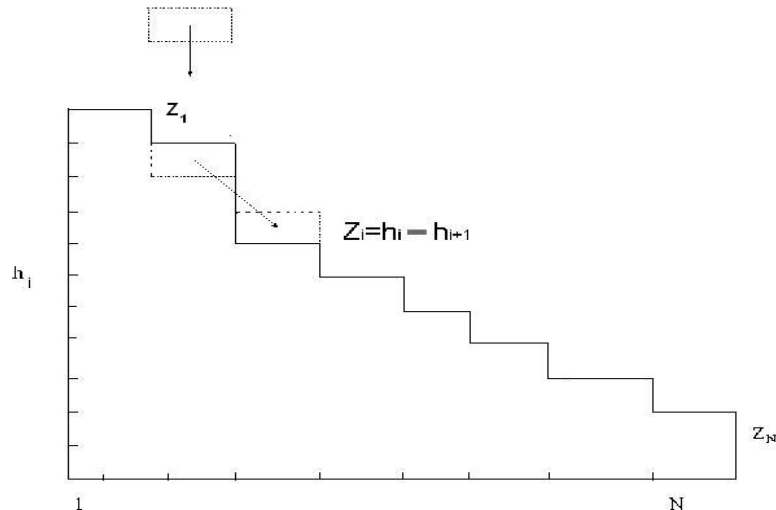


Fig. 1. A sketch of the sandpile model

We define the slope of a site as:

$$z_i = h_i - h_{i+1} \quad (1)$$

The addition of a sand grain on a randomly chosen site  $i$  results in the following changes in the slopes:

$$z_i \rightarrow z_{i+1} \quad (2)$$

$$z_{i-1} \rightarrow z_{i-1} - 1 \quad (3)$$

We proceed by dropping grains at random until one site reaches a slope which is larger than some critical value  $z_i > z_c$ , ( $z_c = 2$  usually in one-dimensional models) and the site topples by transferring one grain to its neighboring site on the right. In this case the slope changes accordingly to (see Fig. 1a):

$$z_i \rightarrow z_i - 2 \quad (4)$$

$$z_{i\pm 1} \rightarrow z_{i\pm 1} + 1 \quad (5)$$

unless at the rightmost site where the sand grains fall off the pile. The neighbors that are affected by the toppling can topple in turn generating a chain reaction or avalanche. During the avalanche no more grains are added to the pile thus separating the two scales of time involved in the dynamic evolution of the pile, one for the addition of grains and one for the relaxation process.

When the avalanche is over, we add more grains following (Eq. 2) until a new avalanche starts and so on. After some transient, whose duration depends on the initial conditions, the system reaches a critical state in which  $z_i = z_c$ , for all  $i$ . This is the stable state, since after any perturbation the system relaxes returning to the stable state, the addition of a single grain to the pile results in an avalanche that makes this grain simply fall off the pile. If we define the size of the avalanche as the number of sites that topple (which can easily see that in this case it is equal to the distance from the site where sand is added to the open boundary), this is the critical state in the sense that we can observe that the avalanches do not have a characteristic scale. We also define the duration of an avalanche as the number of iteration needed to reach one more time the stable state.

Now we analyze the two-dimensional model, in which we describe the state of the system by the slope in each site and this slope is assumed to be a scalar variable. The rules are a simple generalization of the one dimensional ones; thus the addition of a sand grain makes the slope changes as follows,

$$\begin{aligned} z_{i,j} &= z_{i,j} + 2 \\ z_{i-1,j} &= z_{i-1,j} - 1 \\ z_{i,j-1} &= z_{i,j-1} - 1 \end{aligned} \quad (6)$$

Usually this rule is modified such that the perturbation on the system is made directly in terms of the slope; thus we just add a unit of slope,

$$z_{i,j} = z_{i,j} + 1 \quad (7)$$

This makes the two-dimensional sandpile just a little unclear, in the sense that it is difficult to imagine how to add a unit of slope into a the pile. We then, proceed with this rule until we have reached the critical value  $z_{i,j} > z_c = 4$ , then

$$\begin{aligned}
 z_{i,j} &= z_{i,j} - 4 \\
 z_{i\pm 1,j} &= z_{i\pm 1,j} + 1 \\
 z_{i,j\pm 1} &= z_{i,j\pm 1} + 1
 \end{aligned} \tag{8}$$

For the two dimensional sandpile model we can also think of a critical state with all sites having the same critical slope but now this state is unstable, with respect small perturbations since any avalanche would propagate in two directions on the lattice, and in this case we also would have more than one grain falling off the pile.

So, one more time, after some transient time, the system reaches, through a self-organized process, a stable critical state, with a non-homogeneous distributionn of slopes. This state respondes to the addition of a sand grain with an avalanche of unpredictable size and duration. Statistically, we measure the distribution of avalanches sizes and lifetimes averaged over a large number of perturbations and we can observe power laws with no characteristic time or length scales.

### 3 The Spring Block Model

Bak et al. proposed the notion of self-organized criticality as a general organizing principle governing the behavior of spatially extended dynamical systems with both temporal and spatial degrees of freedom. According to this principle, composite open systems having many interacting elements organize themselves into stationary critical states, with no length or time scales, others than those imposed by the size of the system.

These critical states are characterized by the appearance of power law distributions. According to Bak et al. [4], the temporal "fingerprint" of the SOC state is the presence of  $1/f$  noise. Their spatial signature is the emergence of scale invariant (fractal structure). Equilibrium systems require fine-tuning of some parameters to be held at a critical point, while no fine-tuning is required by SOC systems.

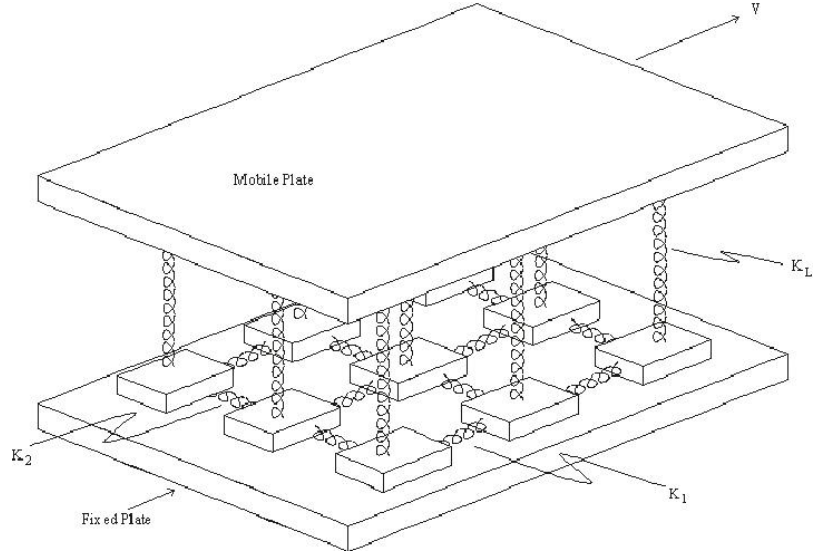
Self-organized criticality represents an attempt towards the elucidation of a general mechanism responsible for the ubiquitous appearance of natural phenomena lacking a spatial or temporal scale. Between the great variety of phenomena showing scale and time invariance there are several geophysical processes. Since the Bak et al., pioneering papers, numerous SOC physical realizations have been proposed. For instance, Sornette and Sornette [14], suggested that SOC is relevant to understand earthquakes, as a relaxation mechanism which organizes the crust.

In 1989, Bak and Tang asserted that earthquakes could be the most direct example of a self-organized critical system in nature. Ito and Matsuzaki [13], proposed a cellular automaton, similar to the sand pile model (introduced by Bak et al in order to explain the SOC property [4]), adapted to the study of earthquakes occurrence. Sornette and Sornette [14] suggested that if earthquakes are natural consequence of stationary dynamical states of the crust subject to growing tensions, these also organize the crust in a self-consistent way.

Using SOC concepts, other models were elaborated for earthquake simulation, as those of Carlson and Langer [15] and Nakanishi [33]. Until 1991, most of these simulations were limited to conservative models; but in real earthquakes there are losses of energy by friction and all the relieved energy contributes to the slipping. For a time it was believed that a necessary condition for systems to be SOC was that they were completely conservative. However, Feder and Feder [16] showed that special non-conservative models, with global perturbations, also exhibit SOC properties. Later Olami, Feder and Christensen (OFC) [17] introduced a cellular automaton model where the conservation level could be controlled. The proposed model was a version in two dimensions of the spring-block model of earthquakes by Burridge and Knopoff (BK) [18]. The OFC model is represented by a two dimensional array of blocks interconnected by Hooke's springs (with elastic constants  $K_1, K_2$  in each of the two perpendicular dimensions).

Each block is connected to its four nearest neighbors and to a driving rigid plate by an another array of Hookes springs (with elastic constant  $K_L$ ), there are other forces, due to the friction between the blocks and the rigid plate. The blocks are

forced to move by the relative movement of the two rigid plates (see Fig.2). When the bet force acting on a block is greater than some threshold value  $F_s$  (the maximal static friction) that block slips. Olami et al assumed that when a block moves it slips to the zero force state. The Block slipping, will redefine the forces in its nearest neighbors. This can result in more slippings and a chain reaction can evolve.



**Fig. 2.** A sketch of the spring block model

If we define an  $L \times L$  arrangement of blocks by  $(i,j)$ , where  $i$  and  $j$  are integers whose values are between 1 and  $L$  and if the displacement of each block from its relaxed position on the lattice is  $X_{i,j}$  then the total fore exerted by the springs on a given block is expressed by [34],

$$F_{i,j} = K_1(2X_{i,j} - X_{i-1,j} + X_{i+1,j}) + K_2(2X_{i,j} - X_{i,j-1} + X_{i,j+1}) + K_L X_{i,j} \quad (9)$$

where  $K_1, K_2$  and  $K_L$  are the elastic constants. The force redistribution in the position  $(i,j)$  is given by the following relationship,

$$\begin{aligned} F_{i\pm 1,j} &\rightarrow F_{i\pm 1,j} + \delta F_{i\pm 1,j} \\ F_{i,j\pm 1} &\rightarrow F_{i,j\pm 1} + \delta F_{i,j\pm 1} \\ F_{i,j} &\rightarrow 0 \end{aligned} \quad (10)$$

where the force increase in the nearest neighbors is given by,

$$\begin{aligned} \delta F_{i\pm 1,j} &= \frac{K_1}{2K_1 + 2K_2 + K_L} F_{i,j} = \alpha_1 F_{i,j} \\ \delta F_{i,j\pm 1} &= \frac{K_2}{2K_1 + 2K_2 + K_L} F_{i,j} = \alpha_2 F_{i,j} \end{aligned} \quad (11)$$

where  $\alpha_1$  and  $\alpha_2$  are the elastic ratios. Observe that the force distribution is not conservative.

Olami et al. first restricted the simulation to the isotropic case,  $K_1 = K_2$  ( $\alpha_1 = \alpha_2 = \alpha$ ) with a rigid frontier condition, implying that in it,  $F=0$ . They made the mapping of the spring-block model into a continuous, non conservative cellular automaton modeling earthquakes which is described by the following algorithm: a) Initialize all the sites of a matrix to a random value between 0 and  $F_s$ ; b) locate the block with the largest force  $F_{\max}$ ; c) add  $F_s - F_{\max}$  to all sites (global perturbation); d) for all, redistribute the force in

the neighbors of  $F_{i,j}$  according to the rule:

$$\begin{aligned} F_m &\rightarrow F_m + \alpha F_{i,j} \\ F_{i,j} &\rightarrow 0 \end{aligned} \quad (12)$$

where  $F_m$  are the forces for the four nearest neighbors, and  $\alpha$  is an elastic ratio (see Eq. (11) of reference [25]). A synthetic earthquake is in process; d) Repeat step c) until the earthquake has totally evolved; e) Once the earthquake has thoroughly ended, return to step b). For these conditions Olami et al obtained a robust SOC behavior for the probability distribution of the earthquakes size, however some authors [23,25] believe that the most important result obtained by Olami et al is the possibility to calculate the exponents of the Gutenberg-Richter law.

The spring block model studied by Olami et al some years later was also investigated by some other authors [25], who found new properties related to real seismicity. Following the same procedure that Brown, Scholz and Rundle [30], Diosdado and Brown [25] found that the results of the OFC model can produce real seismicity staircase graphics, and that also produce seismicity patterns similar to the real ones. When they plotted the cumulative seismicity against time, where cumulative seismicity is obtained by adding the number of blocks which relax at each synthetic event (the magnitude of a synthetic earthquake is supposed to be proportional to the number of blocks which are relaxed in each event), the results were stair-shaped plots, similar to those reported by Macnally [28] for real seismicity in many places of the world.

They also found that the distribution of large earthquakes recurrence times in the OFC model is log-normal for certain elastic parameters. The recurrence time is the time that passes until another earthquake of similar or greater magnitude is produced. It is well known that if an earthquake of great magnitude has been preceded by a recurrence time larger than the mean recurrence time, then the following earthquake will be preceded by a smaller recurrence time. The same result was obtained when Diosdado and Angulo Brown studied the OFC model.

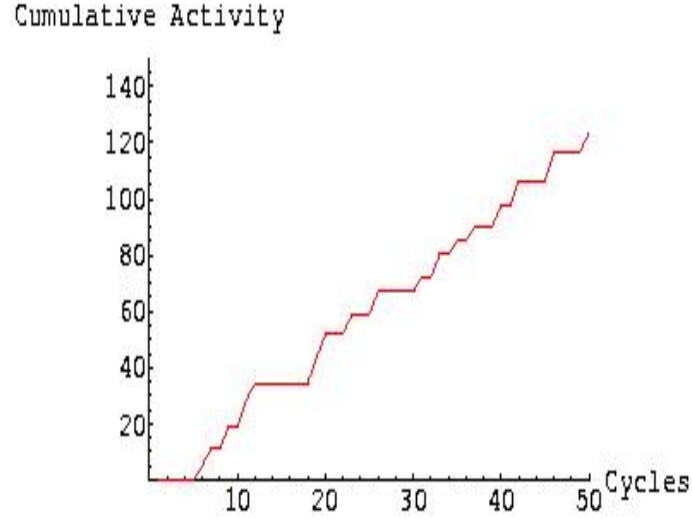
In summary, the spring block model proposed by OFC for the simulation of seismicity has many features that are reminiscent of real seismicity, such as stair shaped graphics for cumulative seismicity, the Gutenberg-Richter law, the log-normal distribution of recurrence times and others reported by Diosdado and Angulo Brown [25].

## 4 Numerical Results

In order to observe if the sandpile model reproduce some features that the spring block model had as we saw before, we implemented the sandpile model presented in the last section on a personal computer. If we define the size of the avalanche as the number of sites that topple (which can easily see that in this case it is equal to the distance from the site where sand is added to the open boundary), this is the critical state in the sense that we can observe that the avalanches do not have a characteristic scale (we can also define the duration of an avalanche as the number of iteration needed to reach one more time the stable state), so we can define the cumulative activity of the sand pile model as the sum of the sizes for each avalanche every time step, the definition is similar to that given by Chimal-Eguía et al. [27] for the Bak-Sneppen model for biological evolution and by Diosdado and Angulo Brown [25] for the spring block model.

We can observe that the mean cumulative activity behaves like a stair-shaped function as that of Fig.3 which was obtained from 100,000 cycles after thermalization (1,000,000 cycles) with  $N=50$ . No matter what the size  $N$  and the number of cycles were, the behavior always was the same.

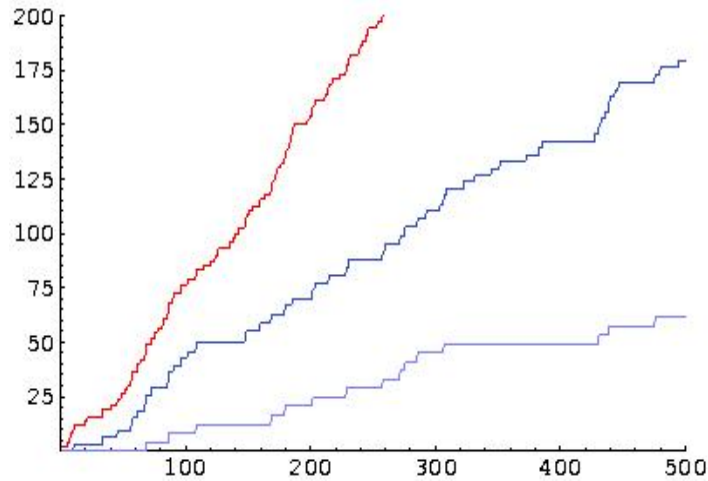




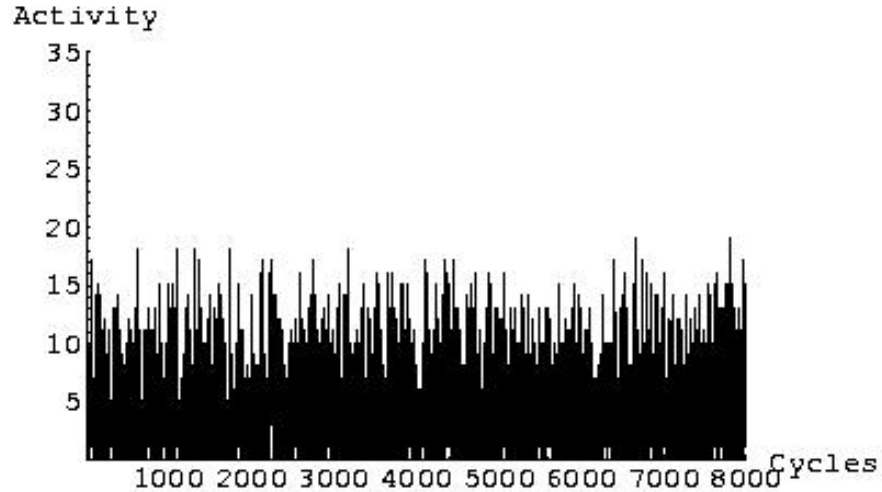
**Fig. 3.** A stair-shape cumulative activity exhibiting punctuated equilibrium for the first 50 of 100000 cycles, stasis periods are present before activity episodes. Here we used a threshold  $u=0.6$

It can be seen that the behavior of the mean cumulative activity (see Fig. 3) follows a pattern which is expected from a system with punctuated equilibrium [19]. No matter what the size of the system is, there are periods of activity (bursts) which are followed by periods of no activity (stasis). Not all of the events have been considered in making the plot of mean cumulative activity.

Following the ideas of Chimal-Eguía et al. And Diosdado et al. [25,26,27], we considered only those events whose activity is larger than some prescribed threshold in the activity  $u_j$ , with  $j = 1, 2, 3, \dots$  labeling the corresponding threshold for a particular stair-step graph. Fig. 4 shows plots of mean cumulative activity for different values of this threshold. In Fig. 5 we can see a typical time series of cumulative activity with a certain threshold. This threshold defines the counting of stasis and burst periods.

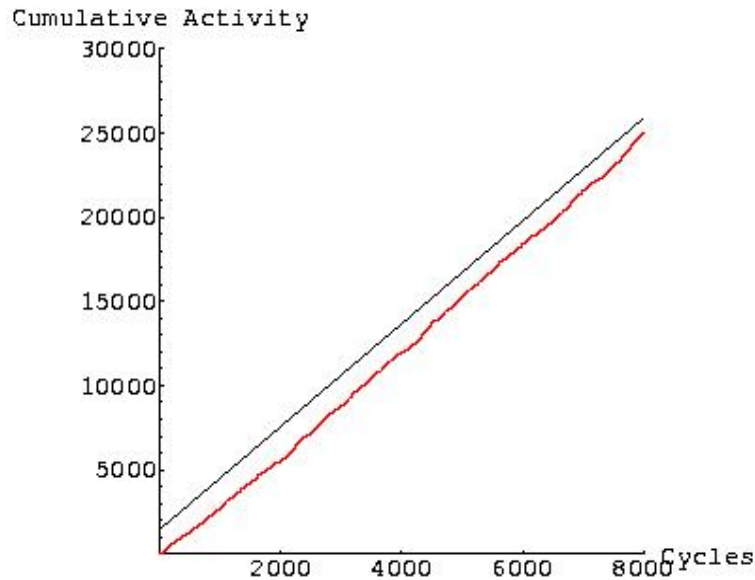


**Fig. 4.** Three cases of cumulative activity along the time (iterative updates of the program). The slopes decrease with increasing threshold of activity. In the decreasing order of the slopes of the threshold values are  $u_1 = 3, u_2 = 4, u_3 = 5$  respectively



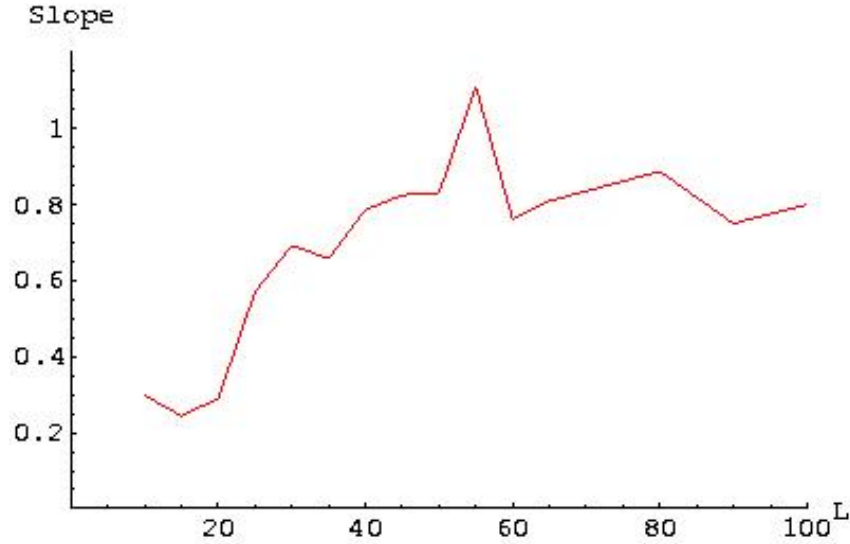
**Fig. 5.** Plot of a typical time-series with a threshold of  $u=5$ , here we show the first 8000 cycles of 100000

Figure 6 shows cumulative activity during 100000 cycles. A straight line has been added to show that in the long term the staircase plots (corresponding to the steady state of the model) have a unique very well defined slope. This property has been found for different values of the parameters. This behavior is also present in the "Spring-block" model for earthquakes [25,26,27] and the Bak-Sneppen model for biological evolution [27] and qualitatively reproduces the cumulative graphs of real seismicity [30].



**Fig. 6.** After thermalization (1,000,000 cycles), the slope of the stair-shaped curve stabilizes around a straight line in the long term. Here the staircase graph is accompanied for a straight line with slope  $m=0.3926$

It seems that there is a maximum of the characteristic slope (see Fig. 7). This, we think, models a property of the real sandpiles. If the slope is too large, in order to get back to the historical trend, after a quiescence period, the system has to develop a huge activity. However, the sandpile has some limitations (gravitational and others), the sandpile can not grow and get a steeper slope forever, and therefore a very large slope would imply that the sandpile could be destroyed entirely and this kind of phenomenon could be a frequent fact, which according with our experience and the conservation of energy principle, is not the case.

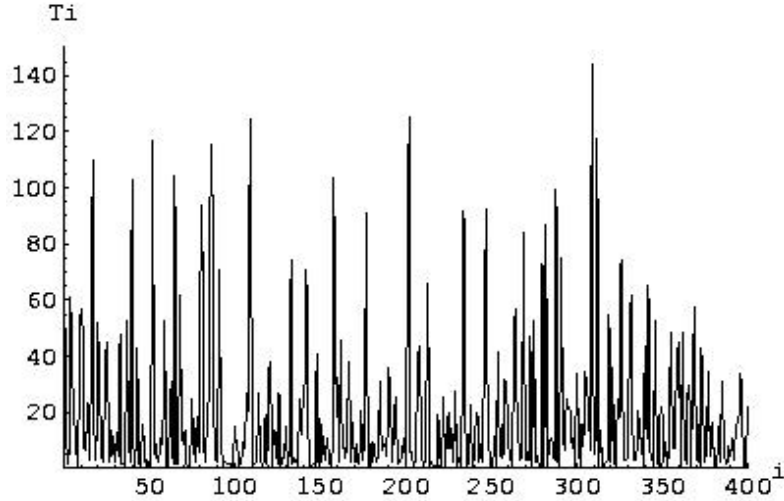


**Fig. 7.** Apparently, the slopes of graphs as the Figure 3 depend on the size  $N$  of the system and they have a maximum value, which bounds the size of a burst following a stasis. Here we depict 15 slopes values

Reasoning by analogy, we propose that once attained the characteristic slope of some sandpile, it remains constant as it happens with the spring-block model and the Bak-Snnepen model. From this emerges the notion of pile provinces, in the sense that different piles would have distinct dynamical properties, for example, the sandpile has different dynamical properties than the ricepile and other piles [33], which could be discriminated, at least by a characteristic slope.

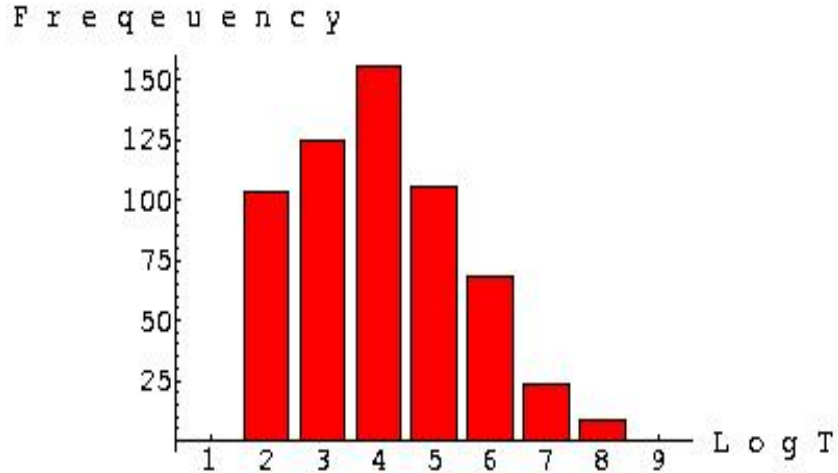
In real seismicity, the time between events whose magnitude is greater or equal than some prescribed value, is known as the recurrence time for events of that magnitude. Nishenko and Buland [32] analyzed the recurrence times  $T_i$ , normalized by their arithmetic mean value  $\langle T \rangle$ , of fifty well characterized real great earthquakes; they found that these recurrence times had a log-normal distribution, that is, 68 per cent of the events were contained within  $\pm \sigma$  of the mean, where  $\sigma$  is the standard deviation. For some spring-block models, it has been found that the recurrence times of large events of synthetic seismicity have also a log-normal distribution [25,26].

Therefore, in the BS-model following a similar procedure analyzing the recurrence times, it was founded that between 65 and 70 per cent of the data where within the  $\pm \sigma$  interval around the mean value [29]. If we follow a similar procedure to analyze recurrence times of the sandpile model of the previous section, we obtained plots as the one of Fig.8 for the normalized recurrence times  $T_i / \langle T \rangle$ , against the index event  $i$ .



**Fig. 8.** Recurrence times normalized by the mean  $T_i / \langle T \rangle$  against the number of cycle  $i$ ,  $n=8000$  (iterative updates of the program). The recurrence time used is the number of cycles that separate activity values that exceed a certain threshold. Here threshold of  $T_i / \langle T \rangle$  is 4

After several runs of the numerical model, recording the mean value and the standard deviation  $\sigma$  of  $\log T$ , we found that between 60 and 70 per cent of the data were within the  $\pm \sigma$  interval around the mean value (see Fig. 9).



**Fig. 9.** Approximated log-normal distribution for recurrence times for  $n=8000$

In summary, we can assert that many properties are shared by the sandpile, the BS and the Spring-block models in a self-organized critical state.

## 5 Conclusions

Bak, Tang and Wiesenfeld [1] proposed the concept of self-organized criticality as a general organizing principle governing the behavior of spatially extended dynamical systems with both temporal and spatial degrees of freedom. According to this principle, composite open systems having many interacting parts organize themselves into a stationary critical state with no length or time scales other than those imposed by the finite size of the system. The critical state is characterized by spatial and temporal power laws.

Since the Bak et al. pioneering papers [4] a number of numerical models and natural phenomena have been discussed that

may or may not exhibit self-organized critical behavior. The essential question is whether a broad range of real complex phenomena exhibits similar behavior under very broad conditions. This seems to be true for the earthquakes, landslides and evolution. It may also be true for a variety of other examples in the physical, biological and social sciences (a few examples are species extinctions, epidemics, stock market crashes and wars).

It seems that a universal feature of these phenomena is that they are driven systems that involve "avalanches" with a fractal (power law) frequency-size distribution. However, we think there are other important features of the behavior of models (the sandpile, the spring block model and the BS model) and the natural phenomena that are associated with self-organized criticality. In this paper we have studied some of these features that the sandpile model shares with the spring-block model, and the BS-model.

One of the common properties that these models share is punctuated equilibrium, i.e., no matter what the size of the system is, there are periods of activity (bursts) which are followed by periods of no activity (stasis). This implies that, when a straight line has been added to showing the long term behavior of these staircase plots (corresponding to the steady state of the model) we found that the model exhibits a unique very well defined slope (this property was found for different values of the parameters) we observed the same behavior in the "Spring-block" model for earthquakes [25,26,27] and the Bak-Sneppen model for biological evolution [27].

We also show that there is a maximum of the characteristic slope. This, we think, models a property of the real sandpiles, and permits us to suggest, in the same way as we did for seismic or evolutionary provinces [25,26,27] (that these provinces can be characterized by its own stable slope of cumulative activity). So the sandpile also can be characterized by its own slope of cumulative activity, creating the concept of pile province as synonymous of different kinds of piles. This result is supported by some other results that show the different kind of dynamics developed for distinct piles [29].

The last feature that these models share was found when we analyzed the recurrence times for the sandpile model, and it was found that between 65 and 70 per cent of the data were within the  $\pm \sigma$  interval around the mean value. In real seismicity, Nishenko and Buland [32] analyzed the recurrence times  $T_i$ , normalized by their arithmetic mean value  $\langle T \rangle$ , of fifty well characterized real great earthquakes and found that these recurrence times had a log-normal distribution, that is, 68 per cent of the events were contained within  $\pm \sigma$  of the mean. Therefore, in the BS-model following a similar procedure, it was found that between 65 and 70 per cent of the data were within the  $\pm \sigma$  interval around the mean value [32].

This empirical evidence must force us to recognize that power laws, punctuated equilibrium, provinces, log-normal distribution, thresholds, metastability, fluctuations, and other important features (like for example, the size and shape of the grains in the sandpile or the well-defined correlation length for seismic activation prior to a major earthquake), play a key role in the spatio-temporal behavior of a large class of many-body systems and this insight is sufficiently important to inspire more theoretical, observational and experimental research from the point of the self-organized criticality.

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