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THE LINK BETWEEN REGULARITY AND STRONG-PI-REGULARITY

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Abstract

It is shown that if all powers of a ring element a are regular, then a is strongly pi-regular exactly when a suitable word in the powers of a and their inner inverses is a unit.

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1. Introduction

An element m in a ring R is said to be *regular* if there exists m^-, referred to as an inner inverse, such that $mm^-m = m$. The set of all inner inverses of m is denoted by $m\{1\}$. We say that m is *strongly pi-regular* if it has a *Drazin inverse* m^d that satisfies $xmx = x$ and $mx = xm$, as well as $m^k xm = m^k$ for some k [2]. The smallest such k is called the *index* of m and is denoted by $i(m)$. When $i(m) \le 1$, we say that m has a group inverse, and this is denoted by $m^\#$. In particular, m is a unit if and only if $i(m) = 0$. The index $i(m)$ can also be characterized as the smallest k for which there exist x and y such that $a^{k+1}x = a^k = ya^{k+1}$. Given ring elements x and y, we say they are *orthogonal*, and we write $x \perp y$, if $xy = yx = 0$.

It is known that if m is strongly pi-regular, then $m^{i(m)}$ is regular, and in fact belongs to a multiplicative group, which ensures that $(m^{i(m)})^\#$ exists. We propose to solve the converse problem, namely, that of characterizing strong-pi-regularity in terms of the regularity of suitable powers of m together with the existence of a word, in powers of m and their inner inverses, that is a unit.

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2. The regular stack

Suppose m is an element in R, and assume that m and all its powers are regular. For each power, we pick a fixed inner inverse. That is, we fix a list

$$\{m^-, (m^2)^-, \ldots, (m^k)^-, \ldots\}.$$

We define the fixed idempotents $E_k = m^k(m^k)^-$, when $k = 1, 2, \ldots$, and we also set $e = E_1 = mm^-$. It is easily seen that

$$em = m, \quad eE_k = E_k = E_k^2, \quad E_k m E_k = mE_k, \quad E_k E_{k+1} = E_{k+1}.$$

We now consider the map $\phi : R \to R$ defined by $\phi(x) = mxe + 1 - exe$, and construct the sequence of elements $m_k = \phi(E_k) = x_k + y_k$, where $x_k = mE_k e$ and $y_k = 1 - eE_k e$, when $k = 1, 2, \ldots$. Observe that $\phi(1) = \phi(e)$. We recall that $\phi(e)$ is a unit precisely when m has a group inverse [7], and that $\phi(a)$ is a unit exactly when am has a group inverse [3]. In addition, we see that

$$x_k y_k = mE_k e - mE_k eE_k e = 0,$$
$$y_k x_k = mE_k e - eE_k emE_k e = mE_k e - E_k mE_k e = 0,$$

and therefore we have an orthogonal splitting $m_k = x_k + y_k$.

We now claim that the elements m_k are in fact regular and may be generated recursively.

Lemma 2.1. *If $m_k = \phi(m^k(m^k)^-)$, then there exists an inner inverse m_{k-1}^- such that*

$$m_k = m_{k-1}^2 m_{k-1}^- + 1 - m_{k-1} m_{k-1}^-,$$

with $m_0 = m$.

Proof. If $i \geq 1$, then we have $m_i = x_i + y_i$, in which *both* components are regular. Indeed, $y_i = 1 - m^i(m^i)^- mm^-$ and so y_i is idempotent, and x_i has an inner inverse, namely, $m^i(m^{i+1})^- mm^-$; calling this x_i^-, we deduce that $x_i x_i^- = m^{i+1}(m^{i+1})^- mm^-$ and $y_i x_i^- = 0$ since

$$eE_i em^i = mm^- m^i(m^i)^- mm^- m^i = m^i.$$

We can, therefore, take $m_{k-1}^- = x_{k-1}^- + y_{k-1}$, and this in turn gives

$$\begin{aligned} m_k &= m^{k+1}(m^k)^- mm^- + 1 - m^k(m^k)^- mm^- \\ &= x_{k-1}x_{k-1}x_{k-1}^- + y_{k-1} + 1 - x_{k-1}x_{k-1}^- - y_{k-1} \\ &= (x_{k-1} + y_{k-1})(x_{k-1}x_{k-1}^- + y_{k-1}) + 1 - (x_{k-1}x_{k-1}^- + y_{k-1}) \\ &= m_{k-1}^2 m_{k-1}^- + 1 - m_{k-1}m_{k-1}^-, \end{aligned}$$

as desired. □

Using this lemma, we can now express m_k alternatively:

$$m_k = m^{k+1}(m^k)^- mm^- + 1 - m^k(m^k)^- mm^-.$$

3. Index results

Let us now use the above regular stack to obtain suitable index results. Suppose that m is strongly pi-regular, and consider the associated sequences

$$u_k = m^{k+1}(m^k)^- + 1 - m^k(m^k)^-,$$
$$w_k = m^- m^{k+1}(m^k)^- m + 1 - m^- m^k(m^k)^- m,$$
$$v_k = (m^k)^- m^{k+1} + 1 - (m^k)^- m^k.$$

We shall need the following fact.

LEMMA 3.1 [1]. *If $1 + ab$ has a Drazin inverse, then $1 + ba$ has a Drazin inverse and*

$$i(1 + ab) = i(1 + ba).$$

PROOF. Suppose $1 + ab$ has a Drazin inverse and its index $i(1 + ab)$ is equal to k. Then

$$(1 + ab)^{k+1}x = (1 + ab)^k = y(1 + ab)^{k+1},$$

for some x and y in R. This means that

$$(1 + ba)^{k+1}(1 - bxa) = (1 + ba)^k = (1 - bya)(1 + ba)^{k+1},$$

and thus $i(1 + ba) \le i(1 + ab)$. By interchanging a and b, we obtain the equality. □

By applying this lemma we may conclude that $i(m_k) = i(u_k) = i(w_k) = i(v_k)$.

We now recall the following lemma.

LEMMA 3.2 [5]. *If m is strongly pi-regular, then*

$$i(m^2 m^- + 1 - mm^-) = i(m) - 1.$$

As a consequence, we may deduce that $i(m_k) = t$ if and only if $i(m_{k+1}) = t - 1$.

We shall also need the following result, which can be deduced from the proof of [2, Theorem 4].

LEMMA 3.3. *If $a^{k+1}x = a^k = ya^{k+1}$, then we have $a^d = a^k x^{k+1} = y^{k+1}a^k$ and $aa^d = a^k x^k = y^k a^k$.*

PROOF. Repeatedly premultiplying the first equality by a and postmultiplying it by x shows that $a^{k+r}x^r = a^k$ when $r = 1, 2, \ldots,$ and in particular, if $r = k$, then $a^{2k}x^k = a^k$. By symmetry, $a^k = y^k a^{2k}$. The latter two equalities ensure that a^k has a group inverse of the form

$$(a^k)^\# = y^k a^k x^k = y^k a^{2k} x^{2k} = a^k x^{2k} = y^{2k} a^k.$$

This implies that

$$a^d = a^{k-1}(a^k)^\# = a^{k-1}a^k x^{2k} = (a^{k+(k-1)}x^{k-1})x^{k+1} = a^k x^{k+1},$$

and by symmetry $a^d = y^{k+1}a^k$.

Finally, we also see that $aa^d = a^{k+1}x^{k+1} = (a^{k+1}x)x^k = a^k x^k$, and so $aa^d = y^k a^k$ by symmetry. □

Combining these results, we now may state the following theorem.

THEOREM 3.4. *The following conditions are equivalent.*

(a) $i(m) = s$.
(b) *s is the smallest integer such that $m^s + 1 - m^s(m^s)^-$ is a unit.*
(c) *s is the smallest integer such that $m^{2s}(m^s)^- + 1 - m^s(m^s)^-$ is a unit.*
(d) *s is the smallest integer such that m_s is a unit.*
(e) *s is the smallest integer such that u_s is a unit.*
(f) *m_ℓ is strongly pi-regular and $i(m_\ell) = s - \ell$, for one and hence all ℓ such that $0 \leq \ell \leq s$.*
(g) *u_ℓ is strongly pi-regular and $i(u_\ell) = s - \ell$, for one and hence all ℓ such that $0 \leq \ell \leq s$.*

If the conditions are satisfied, then

$$\begin{aligned} m^d &= u_s^{-1} m^s v_s^{-s} = m^s v_s^{-s-1} = u_s^{-s} m^s v_s^{-1} = u_s^{-s-1} m^s \\ &= m^{s-1} u_s^{-(s+1)} m^{s+2} v_s^{-(s+1)}. \end{aligned}$$

PROOF. The equivalences between (a), (b) and (c) are known (see [6]). Since $i(m_\ell) = t$ if and only if $i(m_{\ell+1}) = t - 1$, we may, by using this argument recursively, conclude that $i(m) = s$ if and only if $i(m_\ell) = s - \ell$.

The equivalence of (f) and (g), and that of (d) and (e), may be seen by applying Lemma 3.1, setting $b = mm^-$ and first $a = m^{\ell+1}(m^\ell)^- - m^\ell(m^\ell)^-$ and then $a = m^{s+1}(m^s)^- - m^s(m^s)^-$. It is obvious that (f) implies (d) and that (g) implies (e).

Finally, we now prove that (e) implies (a). As u_s is a unit and $u_s m^s = m^{s+1}$, we have $m^s = u_s^{-1} m^{s+1}$. Likewise, $v_s = (m^s)^- m^{s+1} + 1 - (m^s)^- m^s$, so u_s being a unit implies that v_s is a unit, and this in turns yields $m^s = m^{s+1} v_s^{-1}$. Therefore, $m^s \in m^{s+1} R \cap R m^{s+1}$ and $m^d = m^{s-1} u_s^{-(s+1)} m^{s+2} v_s^{-(s+1)}$. □

We may in fact compute the Drazin inverses of the three associated sequences $\{u_k\}$, $\{v_k\}$ and $\{w_k\}$. It suffices to compute the former.

THEOREM 3.5. *If $i(m) = s$ and $0 \leq \ell \leq s$, then*

$$u_\ell^d = m^d m^\ell (m^\ell)^- + 1 - m^\ell (m^\ell)^-.$$

PROOF. Set $X = m^\ell$ and $A = m(m^\ell)^-$, so that $u_\ell = XA + (1 - E_\ell)$. From the last theorem, we recall that $i(u_\ell) = i(m) - \ell$. Now observe that u_ℓ is a sum of two orthogonal elements, and since u_ℓ is strongly pi-regular, so are each of the two orthogonal summands. In particular, $m^{\ell+1}(m^\ell)^-$ is strongly pi-regular and we obtain the expression

$$(u_\ell)^d = (mE_\ell)^d + 1 - E_\ell = (XA)^d + 1 - E_\ell, \tag{3.1}$$

where $E_\ell = m^\ell (m^\ell)^-$.

Next, we turn our attention to the computation of $(XA)^d = (mE_\ell)^d$. We claim that $(XA)^{k+1}y = (XA)^k$, where $y = m^d m^\ell (m^\ell)^-$. Indeed, it follows by induction that $(XA)^i = m^{i+\ell}(m^\ell)^-$, and hence

$$\begin{aligned}(XA)^{k+1}y &= m^{k+\ell+1}(m^\ell)^- m^\ell m^d (m^\ell)^- = m^\ell m^{k+1} m^d (m^\ell)^- \\ &= m^{k+\ell}(m^\ell)^- = (XA)^k.\end{aligned}$$

We now apply Lemma 3.3 to obtain $(XA)^d = (XA)^k y^{k+1}$.

Again, by induction, $y^i = (m^d)^i m^\ell (m^\ell)^-$, whence $y^{k+1} = (m^d)^{k+1} m^\ell (m^\ell)^-$, and this gives

$$\begin{aligned}(XA)^d &= (XA)^k y^{k+1} = m^{\ell+k}(m^\ell)^-(m^d)^{k+1} m^\ell (m^\ell)^- \\ &= m^{\ell+k}(m^\ell)^- m^\ell (m^d)^{k+1}(m^\ell)^- = m^\ell m^k (m^d)^{k+1}(m^\ell)^- \\ &= m^d m^\ell (m^\ell)^-,\end{aligned}$$

and

$$(XA)^d XA = (mE_\ell)^d m E_\ell = m^d m^\ell (m^\ell)^- m^{\ell+1}(m^\ell)^- = m^d m^{\ell+1}(m^\ell)^-.$$

Finally, substituting the expression for $(XA)^d$ in (3.1), we arrive at

$$(u_\ell)^d = m^d E_\ell + 1 - E_\ell = m^d m^\ell (m^\ell)^- + 1 - m^\ell (m^\ell)^-,$$

which is the desired expression. □

We close with some pertinent remarks.

Remarks

(a) If m_k is a unit for one choice of $(m^k)^-$, then it is a unit for all such choices. Indeed, the fact that m_k is a unit implies that $i(m) = s$, which implies, from the proof of Theorem 3.4, that $m_s = m^{s+1}(m^s)^{=} mm^- + 1 - m^s (m^s)^{=} mm^-$ is also a unit.

(b) If u_s is a unit for one choice of $(m^s)^-$, then it is a unit for all such choices.

(c) In a ring, a^2 may be regular without a being regular. For example, take $a = 4$ in $\mathbb{Z}_8$.

(d) In a ring, a may be regular without a^2 being regular. Indeed, in $\mathbb{Z}_4$, consider

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then A has an inner inverse, namely B, but A^2 has no inner inverse, since $A^2 = 2A$, and so $(2A)X(2A) = 0 \neq 2A$.

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