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SPECTRAL RADIUS ALGEBRAS AND C_0 CONTRACTIONS. II

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Abstract

We consider spectral radius algebras associated with C_0 contractions. When the operator A is algebraic, we describe all invariant subspaces that are common for operators in its spectral radius algebra $\mathcal{B}_A$. When the operator A is not algebraic, $\mathcal{B}_A$ is weakly dense and we characterize a set of rank-one operators in $\mathcal{B}_A$ that is weakly dense in $\mathcal{L}(\mathcal{H})$.

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1. Introduction

Denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. Given an operator $A \in \mathcal{L}(\mathcal{H})$ with spectral radius r, we define a sequence of positive numbers d_m (or $d_m(A)$) by $d_m = m/(1 + rm)$, and we note that, for each $m \in \mathbb{N}$, the series $\sum_{n=0}^{\infty} d_m^{2n} A^{*n} A^n$ converges in the norm topology to a positive invertible operator. We denote by R_m (or $R_m(A)$), its positive square root $(\sum_{n=0}^{\infty} d_m^{2n} A^{*\,n} A^n)^{1/2}$. The *spectral radius algebra* $\mathcal{B}_A$ consists of all operators $T \in \mathcal{L}(\mathcal{H})$ such that $\sup_{m\in\mathbb{N}} \|R_m T R_m^{-1}\| < \infty$. The study of these algebras started in [6] where it was shown that, when A is compact, the algebra $\mathcal{B}_A$ has a nontrivial invariant subspace. A similar result followed for some normal operators [3]. A major role in these results was played by the ideal $\mathcal{Q}_A = \{T : \|R_m T R_m^{-1}\| \to 0\}$. We state the facts that are used in this paper and direct the reader to the articles [2–9] for more information.

PROPOSITION 1.1. *Let A be an operator in $\mathcal{L}(\mathcal{H})$. If $AT = \lambda TA$, where $\lambda \in \mathbb{C}$ and $|\lambda| \le 1$, then $T \in \mathcal{B}_A$. In particular, the commutant $\{A\}' \subseteq \mathcal{B}_A$. If there exists a nonzero compact operator in $\mathcal{Q}_A$, then $\mathcal{B}_A$ has a nontrivial invariant subspace. Finally, $B_A = \mathcal{L}(\mathcal{H})$ if and only if the operator A is similar to a constant multiple of an isometry.*

A contraction A is *completely nonunitary* if there is no invariant subspace $\mathcal{M}$ for A such that $A|_{\mathcal{M}}$ is a unitary operator. A completely nonunitary contraction A is said to be of *class* C_0 if there exists a nonzero function $h \in H^\infty$ such that $h(A) = 0$. The inner function v such that $vH^\infty = \{u \in H^\infty : u(A) = 0\}$ is the *minimal function* of A and is denoted by m_A. The operator A is *algebraic* if there is a polynomial p such that $p(A) = 0$.

One of the most studied concrete Hilbert spaces is the Hardy space H^2, and one of the best-understood operators is the unilateral shift. Throughout the paper we use S to denote the forward unilateral shift of multiplicity 1, and $\{e_n\}_{n=0}^\infty$ the orthonormal basis such that $Se_n = e_{n+1}$ when $n \geq 0$. It is known that S may be viewed as multiplication by z on H^2. A classical result of Beurling states that every invariant subspace of S is of the form θH^2 for some inner function θ. The compression of S to $H^2 \ominus \theta H^2$ is called a Jordan block. This subspace is denoted by $\mathcal{H}(\theta)$ and the compression in question by $S(\theta)$.

At this stage it is useful to point out that the term *Jordan block* has a different meaning in linear algebra. For example, if $\theta(z) = \mu_\alpha(z)^2\mu_\beta(z)^3$ for all $z \in \mathbb{C}$, where the Möbius transformation is given by $\mu_\lambda(z) = (z - \lambda)/(1 - \overline{\lambda}z)$, then $S(\theta)$ acts on a space of dimension five and is a direct sum of *two* Jordan blocks. To avoid confusion, we will say that, in this example, $S(\theta)$ is a direct sum of two *simple* Jordan blocks.

This paper may be viewed as a sequel to [9]. We continue the study of spectral radius algebras associated with C_0 contractions. However, in the previous paper, the emphasis was on establishing that the inclusion $\{A\}' \subset \mathcal{B}_A$ is proper. Here, our focus is on the structure of the algebra $\mathcal{B}_A$. In particular, we show that there are significant differences between the cases when m_A is a finite Blaschke product and when it is not. In the latter case, $\mathcal{B}_A$ is always weakly dense in $\mathcal{L}(\mathcal{H})$. (Throughout the paper, density will always mean *weak* density.) We establish this fact by characterizing the set of rank-one operators in $\mathcal{B}_A$ and by showing that its (finite) span is dense in $\mathcal{L}(\mathcal{H})$. This set is more easily understood in the case when $A = S(\theta)$ (Theorem 2.4) and less so for a general contraction of class C_0 (Theorem 4.3). The case where A is algebraic is studied using mostly finite-dimensional tools. For such an operator, the quasisimilarity model $S(\Theta)$ is a (possibly infinite) direct sum $\bigoplus_k S(\theta_k)$, but each operator $S(\theta_k)$ acts on a finite-dimensional space. Therefore, $S(\Theta)$ is similar to a direct sum of simple Jordan blocks and, moreover, $S(\Theta) = S(\Theta_1) \oplus S(\Theta_2)$ where $S(\Theta_2)$ contains all the blocks with maximal eigenvalues (that is, of absolute value equal to the spectral radius of A). Our main result for algebraic C_0 contractions (Theorem 4.6) is that if, relative to this decomposition,

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in \mathcal{B}_{S(\Theta)},$$

then $T_3 = 0$ and T_4 consists of upper triangular blocks, relative to the representation of $S(\Theta_2)$ as a direct sum of simple Jordan blocks.

The organization of this paper is as follows. In Sections 2 and 3 we investigate the basic C_0 contraction $S(\theta)$. In Section 2 we consider the case where θ is not a finite Blaschke product. We show that $\mathcal{B}_{S(\theta)}$ is weakly dense in $\mathcal{L}(\mathcal{H}(\theta))$ and characterize a set of rank-one operators with dense span that it contains (Theorem 2.4). In Section 3 we study the Jordan block $S(\theta)$, where θ is a finite Blaschke product, so that $S(\theta)$ acts on a finite-dimensional space. As a first step we show that if $S(\theta)$ is a simple Jordan block, then $\mathcal{B}_{S(\theta)}$ is the algebra of all upper triangular matrices (Theorem 3.4). We then consider a more general situation, where $S(\theta)$ is a direct sum of simple Jordan blocks but the corresponding eigenvalues are all of the same absolute value. In this case, $T \in \mathcal{B}_{S(\theta)}$ if and only if T is a block matrix (relative to the same decomposition), in which each block is upper triangular (Corollary 3.7). The main result of this section (Theorem 3.10) takes care of the most general $S(\theta)$, a direct sum of simple Jordan blocks with no restriction on their eigenvalues, and gives a complete characterization of operators in $\mathcal{B}_{S(\theta)}$. In Section 4, we consider general C_0 contractions and we describe the corresponding spectral radius algebras. We use a quasisimilarity model for $A \in C_0$ and we show that relevant properties are preserved under quasisimilarity. In particular, it turns out that the structure of $\mathcal{B}_A$ depends on whether the minimal function m_A is a finite Blaschke product or not. In the latter case, we get the analogue of Theorem 2.4, namely, $\mathcal{B}_A$ contains a set of rank-one operators with dense span (Theorem 4.3). When m_A is a finite Blaschke product, we obtain a complete characterization of $\mathcal{B}_A$ (Theorem 4.6), analogous to that in Section 3.

2. Jordan blocks on infinite-dimensional spaces

In this section, we consider the operators $S(\theta)$, where θ is an inner function that is not a finite Blaschke product. This implies that $\mathcal{H}(\theta)$ is an infinite-dimensional subspace of H^2. We demonstrate that, in this situation, the algebra $\mathcal{B}_{S(\theta)}$ is dense in $\mathcal{L}(\mathcal{H}(\theta))$ because it contains a set of rank-one operators with dense span. We make use of two operators acting on H^2. When $f = \sum_{k\geq 0} f_k e_k \in H^2$, we define

$$Df = \sum_{k\geq 1} \sqrt{k}\, f_k e_{k-1} \quad \text{and} \quad Jf = \sum_{k\geq 0} (f_k/\sqrt{k+1}) e_{k+1}.$$

Although D is an unbounded operator on H^2, it is not hard to see that the operator DJ^* is bounded. We start by introducing an important dense subset of $\mathcal{H}(\theta)$.

PROPOSITION 2.1. *Let θ be an inner function that is not a finite Blaschke product, and let $\mathcal{N} = \{u \in \mathcal{H}(\theta) : Du \in H^2\}$. Then the set $\mathcal{N}$ is dense in $\mathcal{H}(\theta)$.*

PROOF. Suppose, to the contrary, that there exists $h \in \mathcal{H}(\theta)$ such that $h \perp \mathcal{N}$. Note that, if g is any function satisfying $g \perp J(\theta H^2)$, then $J^* g \in \mathcal{N}$. Therefore $h \perp J^* g$ and $Jh \perp g$, which implies that Jh belongs to the closure of $J(\theta H^2)$. In other words, there exists a sequence of polynomials $\{p_n\}$ such that $J(\theta p_n) \to Jh$ in the norm of H^2. Moreover, $J(\theta p_n - h) \to 0$ weakly. Let $f \in H^2$. Then

$$\langle \theta p_n - h, J^* f\rangle = \langle J(\theta p_n - h), f\rangle \to 0.$$

Since the range of J^* is dense, it follows that $\theta p_n - h \to 0$ weakly. In particular, $\langle \theta p_n - h, \theta v\rangle \to 0$ for all $v \in H^2$. But $\langle h, \theta v\rangle = 0$, so $\langle \theta p_n, \theta v\rangle \to 0$. Taking into account that multiplication by θ is an isometry, we see that $\langle p_n, v\rangle \to 0$, that is,

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the sequence p_n converges weakly to 0. Consequently, the same is true of $J(\theta p_n)$. However, $J(\theta p_n) \to Jh$, and it follows that $Jh = 0$, and hence that $h = 0$. We conclude that $\mathcal{N}$ is dense in $\mathcal{H}(\theta)$. $\square$

Next we demonstrate the relevance of $\mathcal{N}$.

THEOREM 2.2. *Let θ be an inner function that is not a finite Blaschke product. A rank-one operator $u \otimes v$ is in $\mathcal{B}_{S(\theta)^*}$ if and only if $u \in \mathcal{N}$.*

PROOF. Suppose first that $u \in \mathcal{N}$. Since $\|R_m(u \otimes v)R_m^{-1}\| = \|R_m u\| \|R_m^{-1} v\|$ and $\|R_m^{-1}\| \le 1$ hold universally, it suffices to show that $\sup_m \|R_m(S(\theta)^*)u\| < \infty$. The assumption on θ guarantees that the spectral radius $r(S(\theta))$ is equal to 1, so

$$d_m(S(\theta)) = d_m(S^*) = m/(m+1).$$

Relative to the decomposition $H^2 = \theta H^2 \oplus \mathcal{H}(\theta)$,

$$R_m^2(S^*) = \begin{pmatrix} \star & \star \\ \star & R_m^2(S(\theta)^*) \end{pmatrix},$$

while u may be identified with $w = 0 \oplus u$. Clearly,

$$\langle R_m^2(S^*)w, w\rangle = \langle R_m^2(S(\theta)^*)u, u\rangle,$$

so $\|R_m(S^*)w\| = \|R_m(S(\theta)^*)u\|$. In order to prove that $\sup_m \|R_m(S^*)w\| < \infty$, we note that $R_m(S^*)$ may be represented in the basis $\{e_k\}$ as a diagonal matrix $\operatorname{diag}(\alpha_{m,0}, \alpha_{m,1}, \ldots)$ where $\alpha_{m,k} = 1 + d_m^2 + d_m^4 + \cdots + d_m^{2k}$. Now

$$R_m(S^*)w = R_m(S^*) \sum w_k e_k = \sum w_k R_m(S^*)e_k = \sum w_k \alpha_{m,k} e_k$$

and

$$\|R_m(S^*)w\|^2 = \sum |w_k|^2 |\alpha_{m,k}|^2 \le \sum |w_k|^2 (k+1).$$

Since $Dw = Du \in H^2$, the last series converges, which shows that the condition $u \in \mathcal{N}$ is sufficient and in addition that the algebra $\mathcal{B}_{S(\theta)^*}$ contains the set $\mathcal{N} \otimes \mathcal{H}(\theta)$, which is dense in the set of all rank-one operators on $\mathcal{H}(\theta)$. Consequently, $\mathcal{B}_{S(\theta)^*}$ is dense in $\mathcal{L}(\mathcal{H}(\theta))$.

Suppose now that $u \otimes v \in \mathcal{B}_{S(\theta)^*}$. Then $\|R_m(S(\theta)^*)u\| \|R_m^{-1}(S(\theta)^*)v\|$ is a bounded sequence, so $\sup_m \|R_m(S(\theta)^*)u\| < \infty$ or $\lim_m \|R_m^{-1}(S(\theta)^*)v\| = 0$. However, the latter is impossible. Indeed, if there exists such a nonzero vector v, then $\|R_m(S(\theta)^*)u_0\| \|R_m^{-1}(S(\theta)^*)v\| \to 0$ for all $u_0 \in \mathcal{N}$. In other words, $u_0 \otimes v \in \mathcal{Q}_{S(\theta)^*}$ and it would follow from Proposition 1.1 that the algebra $\mathcal{B}_{S(\theta)^*}$ has a nontrivial invariant subspace, contradicting the fact that it is dense. Thus, $\|R_m(S(\theta)^*)u\|$ must be bounded and, as above, if $w = 0 \oplus u$, then $\sup_m \|R_m(S^*)w\| < \infty$. Consequently, there exists $M > 0$ such that $\sum |w_k|^2 |\alpha_{m,k}|^2 \le M$ for all $m \in \mathbb{N}$. Since the last series converges uniformly in m, we may pass to the limit as $m \to \infty$. We obtain that $\sum |w_k|^2 (k+1) \le M$, which implies that $Dw \in H^2$ and $u \in \mathcal{N}$. $\square$

As a consequence of Proposition 2.1 and Theorem 2.2 we obtain the following characterization.

THEOREM 2.3. *Let θ be an inner function that is not a finite Blaschke product. Then the algebra $\mathcal{B}_{S(\theta)^*}$ is dense in $\mathcal{L}(\mathcal{H}(\theta))$. Moreover, it contains a set of rank-one operators with dense span, and $u \otimes v \in \mathcal{B}_{S(\theta)^*}$ if and only if $u \in \mathcal{N}$, with $\mathcal{N}$ as in Proposition 2.1.*

In order to describe $\mathcal{B}_{S(\theta)}$, we employ a connection between the Jordan block $S(\theta)$ and the operator $S(\tilde{\theta})^*$, where $\tilde{\theta}(z) = \overline{\theta(\overline{z})}$. We recall (see [1, Corollary 3.1.7]) that there exists a unitary operator $U : \mathcal{H}(\theta) \to \mathcal{H}(\tilde{\theta})$ such that $S(\tilde{\theta})^* U = U S(\theta)$. Further, [3, Theorem 2.4] implies that there exists an isomorphism $\mathcal{U} : \mathcal{B}_{S(\theta)} \to \mathcal{B}_{S(\theta)^*}$, defined by $\mathcal{U}(X) = U X U^*$. Using Theorem 2.3, we obtain that $\mathcal{B}_{S(\theta)}$ is dense. We omit the proof since it is straightforward.

THEOREM 2.4. *Let θ be an inner function that is not a finite Blaschke product and let $\mathcal{N}' = \{u \in \mathcal{H}(\theta) : DUu \in H^2\}$. Then the algebra $\mathcal{B}_{S(\theta)}$ is weakly dense in $\mathcal{L}(\mathcal{H}(\theta))$. Moreover, it contains a dense set of rank-one operators and $u \otimes v \in \mathcal{B}_{S(\theta)}$ if and only if $u \in \mathcal{N}'$, where U is the unitary operator such that $S(\tilde{\theta})^* U = U S(\theta)$.*

REMARK 2.5. In [1, Exercise 5, p. 42] the operator U is given explicitly. Using this formula, a short calculation shows that the condition $DUu \in H^2$ may be written as

$$\sum_{m \geq 1} m \left| \sum_{j \geq 0} \overline{\theta}_{m+j+1} u_j \right|^2 < \infty,$$

where θ_k and u_k are Taylor coefficients of θ and u, respectively.

3. Jordan blocks on finite-dimensional spaces

We now turn our attention to the case where θ is a finite Blaschke product, and $S(\theta)$ acts on a finite-dimensional space. In this situation, $S(\theta)$ may be represented as a direct sum of simple Jordan blocks. More precisely, $S(\theta) = \bigoplus_{i=1}^{n} J_{\alpha_i}$, where

$$J_{\alpha_i} = \begin{pmatrix} \alpha_i & 1 & & & \\ & \alpha_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \alpha_i & 1 \\ & & & & \alpha_i \end{pmatrix}.$$

We start with the case where $n = 1$. The following analysis is based on results and techniques from [4]. We review them here in order to make the article self-contained.

LEMMA 3.1 (See [4, Lemma 4.5]). *Let $|x| < 1$ and let $s_k(x) = \sum_{n=0}^{\infty} n^k x^n$. Then $s_k(x)$ is a polynomial of degree $k+1$ in $(1-x)^{-1}$, whose leading coefficient is $k!$.*

PROPOSITION 3.2 (See [4, Proposition 4.6]). *Let B be the $n \times n$ matrix whose (i, j)th entry is $\binom{i+j}{i}$, when $0 \leq i, j \leq n-1$. Then* $\det(B) = 1$.

Next we present a result that is a combination of [4, Theorem 4.7] and a fact that may be found in its proof. Following [4], we denote $1/(1 - |\alpha|^2 d_m^2)$ by λ_m.

THEOREM 3.3. *Let α be a nonzero complex number and let J_α be the simple $N \times N$ Jordan block with eigenvalue α. If $R_m = R_m(J_\alpha)$, then the (i, j)th entry of R_m^2 is a polynomial in λ_m of degree $i + j + 1$. Also, $\det(R_m)$ is a polynomial (in λ_m) of degree N^2. Finally, the (j, j)th entry of R_m^{-2} is a rational function $P(\lambda_m)/Q(\lambda_m)$, where P and Q are polynomials of degrees $N^2 - 2j - 1$ and N^2, respectively.*

PROOF. Note that A^n is an upper triangular Toeplitz matrix whose (k, j)th entry is $\binom{n}{j-k} \alpha^{n+k-j}$ when $0 \le j - k \le n$ and 0 if $j < k$ or $j - k > n$. Consequently, the (i, j)th entry of $A^{*n} A^n$ is

$$\sum_{k=0}^{\min\{i,j\}} \binom{n}{i-k} \overline{\alpha}^{n+k-i} \binom{n}{j-k} \alpha^{n+k-j}.$$

It is not hard to see that this expression may be written as a sum of terms $c_l n^l |\alpha|^{2n+2k}/\overline{\alpha}^i \alpha^j$, where $0 \le l \le i + j$, and $c_{i+j} = |\alpha|^{2n}/(i!j!\overline{\alpha}^i \alpha^j)$ (consider what happens when $k = 0$). It follows that the (i, j)th entry of R_m^2 satisfies

$$(R_m^2)_{i,j} = \sum_{n \ge 0} d_m^{2n} (n^{i+j} |\alpha|^{2n}/(i!j!\overline{\alpha}^i \alpha^j) + p_{i+j-1}(n)),$$

and, using Lemma 3.1, we deduce that

$$(R_m^2)_{i,j} = (i + j)!/(i!j!\overline{\alpha}^i \alpha^j)\lambda_m^{i+j+1} + q_{i,j}(\lambda_m),$$

where $q_{i,j}$ is a polynomial of degree at most $i + j$.

To prove the second assertion, note that the determinant of R_m is a polynomial in λ_m. When polynomial is calculated, its leading term is obtained without using the nonleading terms in any of the entries of R_m. Thus, we concentrate on the matrix F_m, whose (i, j)th entry is $\binom{i+j}{i} \lambda_m^{i+j+1}/(\overline{\alpha}^i \alpha^j)$. This matrix may be written as a product $G_m B L_m$, where G_m stands for the diagonal matrix $\mathrm{diag}(\lambda_m^i/\overline{\alpha}^i)_{i \ge 0}$, while $L_m = \mathrm{diag}(\lambda_m^{i+1}/\alpha^i)_{i \ge 0}$ and B is the matrix with (i, j)th entry $\binom{i+j}{i}$. A calculation shows that

$$\det(G_m B L_m) = \lambda_m^{N^2} \det(B) |\alpha|^{N-N^2}.$$

The result now follows from Proposition 3.2.

Finally, we turn our attention to the (j, j)th entry of R_m^{-2}. It is known that this entry may be calculated by dividing the appropriate cofactor A_m by the determinant of R_m^2. Let F'_m, G'_m, B', and L'_m denote the matrices F_m, G_m, B, and L_m with the jth rows and columns deleted. The determinant A_m (obtained by deleting the jth row and column from $\det(R_m^2)$) is a polynomial, and in order to calculate its leading term, we need to consider only the matrix F'_m. It is not hard to see that $F'_m = G'_m B' L'_m$, so $\det(F'_m) = \det(G'_m) \det(B') \det(L'_m)$. Now,

$$\det(G'_m) = (\lambda_m/\overline{\alpha})^{1+2+\cdots+(N-1)-j} = (\lambda_m/\overline{\alpha})^{(N-1)N/2-j},$$
$$\det(L'_m) = \lambda_m^{N(N+1)/2-(j+1)}/\alpha^{(N-1)N/2-j}.$$

Of course, $\det(B')$ is independent of m, and so A_m is a polynomial of degree $N^2 - (2j+1)$. Consequently $\|R_m^{-1}e_j\|^2$ is a rational function P/Q, where $\deg(P) = N^2 - 2j - 1$ and $\deg(Q) = N^2$. $\square$

We can now describe the algebra $\mathcal{B}_{J_\alpha}$.

THEOREM 3.4. *The spectral radius algebra associated with a simple Jordan block J_α is the algebra of all upper triangular matrices.*

PROOF. We consider separately the cases where $\alpha \neq 0$ and $\alpha = 0$. Take $\alpha \neq 0$ and J_α of size $N \times N$, and let $e_0, e_1, e_2, \ldots, e_{N-1}$ be the corresponding basis for $\mathbb{C}^N$. It suffices to show that the rank-one operator $e_i \otimes e_j$ belongs to $\mathcal{B}_{J_\alpha}$ if and only if $i \leq j$. Indeed, any upper triangular matrix is a finite linear combination of these rank-one operators. On the other hand, let $A = (a_{ij})$; if $A \in \mathcal{B}_{J_\alpha}$, then $e_i \otimes e_i A e_j \otimes e_j = a_{ij} e_i \otimes e_j \in \mathcal{B}_{J_\alpha}$ too, and $a_{ij} \neq 0$ only when $i \leq j$.

By definition, $e_i \otimes e_j \in \mathcal{B}_{J_\alpha}$ if and only if $\sup_m \|R_m(e_i \otimes e_j)R_m^{-1}\| < \infty$. Since

$$\|R_m(e_i \otimes e_j)R_m^{-1}\| = \|R_m e_i\| \|R_m^{-1} e_j\|,$$

we can determine when $\|R_m e_i\| \|R_m^{-1} e_j\|$ is bounded. Note that $\|R_m e_i\|^2 = \langle R_m^2 e_i, e_i\rangle$, the (i, i)th entry of R_m^2, and similarly, $\|R_m^{-1} e_j\|^2$ is equal to the (j, j)th entry of R_m^{-2}. By Theorem 3.3, $\|R_m e_i\|^2 \|R_m^{-1} e_j\|^2$ is a rational function $\hat{P}/Q$, where

$$\deg(\hat{P}) = (2i+1) + (N^2 - 2j - 1) = N^2 + 2i - 2j \quad \text{and} \quad \deg(Q) = N^2.$$

If $m \to \infty$, then $d_m \to 1/r(J_\alpha) = 1/|\alpha|$, so $\lambda_m \to \infty$. Therefore $\|R_m e_i\|^2 \|R_m^{-1} e_j\|^2$ is bounded if and only if $i \leq j$.

The case where $\alpha = 0$ leads to a different form for R_m. Let J_0 be the simple Jordan block of size $N \times N$ corresponding to the eigenvalue $\alpha = 0$. A calculation shows that

$$R_m = \operatorname{diag}(1, \alpha_{m,1}, \alpha_{m,2}, \ldots, \alpha_{m,N-1}),$$

where $\alpha_{m,k} = (1 + d_m^2 + \cdots + d_m^{2k})^{1/2}$. If $T = (t_{ij})$, then $R_m T R_m^{-1} = (\alpha_{m,i} t_{ij} \alpha_{m,j}^{-1})$. Since the spectral radius of J_0 is 0, it follows that $d_m = m$ and $\alpha_{m,k} \to \infty$ as $m \to \infty$. Consequently, $T \in \mathcal{B}_{J_0}$ if and only if $t_{ij} = 0$ for $i > j$, that is, if and only if T is an upper triangular matrix. $\square$

Next we consider a slightly more complicated scenario: we allow θ to have more than one zero, but require that they all be of the same modulus. The corresponding operator $S(\theta)$ is then a direct sum of simple Jordan blocks, which need not be of the same size. Thus, in the block representation of the matrix for this operator, the off-diagonal blocks may be rectangular. We extend the meaning of an upper triangular matrix to apply to such blocks. Namely, if $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$, then we say that A is upper triangular if $a_{ij} = 0$ whenever $i > j$. Similarly, we say that A is diagonal if $a_{ij} = 0$ for $i \neq j$. Now we can prove an extension of Theorem 3.4.

THEOREM 3.5. *Let N and K be positive integers, and let J_α and J_β be simple Jordan blocks of sizes $N \times N$ and $K \times K$ with eigenvalues α and β, and suppose* *that $|\alpha| = |\beta|$. If $J = J_\alpha \oplus J_\beta$ and $\{e_k\}_{k=0}^{N+K-1}$ is the corresponding basis for $\mathbb{C}^{N+K}$, then $e_i \otimes e_j \in \mathcal{B}_J$ if and only if i and j satisfy: $i \le j$ when $0 \le i, j \le N-1$ or $N \le i, j \le N+K-1$; $i \le j-N$ when $0 \le i \le N-1$ and $N \le j \le N+K-1$; $i \le j+N$ when $N \le i \le N+K-1$ and $0 \le j \le N-1$. In other words, a block matrix $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ belongs to $\mathcal{B}_J$ if and only if each of the four blocks is upper triangular.*

PROOF. We note that $R_m(J) = R_m(J_\alpha) \oplus R_m(J_\beta)$ so the estimates for $\|R_m e_i\|^2$ depend on whether $i \le N-1$ or $i \ge N$. Using the same computations as in the proof of Theorem 3.4, together with the fact that the quantity λ_m depends only on the modulus of the eigenvalue, we see, when $\alpha \neq 0$, that $\|R_m e_i\|^2$ is a polynomial of degree $2i+1$ if $0 \le i \le N-1$ or $2(i-N)+1$ if $N \le i \le N+K-1$. Similarly, $\|R_m^{-1} e_j\|^2$ is a rational function P/Q, where

$$\deg(P) = \begin{cases} N^2 - 2j - 1 & \text{if } 0 \le j \le N-1 \\ N^2 - 2(j-N) - 1 & \text{if } N \le j \le N+K-1, \end{cases}$$

and the degree of Q is N^2. The rest of the proof, including the case where $\alpha = 0$, is straightforward. □

REMARK 3.6. If the ordered basis $\{e_k\}$ is replaced by its permutation

$$e_0, e_N, e_1, e_{N+1}, \ldots, e_{K-1}, e_{N+K-1}, e_K, e_{K+1}, \ldots, e_{N-1},$$

then the matrix for T becomes an $N \times K$ block upper triangular matrix. (We have assumed that $N \ge K$. A similar permutation may be written if $N < K$.)

It is easy to see that Theorem 3.5 and the previous remark may be extended to the case where θ has any finite number of zeros of the same absolute value.

COROLLARY 3.7. *Let $J = J_{\alpha_1} \oplus J_{\alpha_2} \oplus \cdots \oplus J_{\alpha_n}$, where $|\alpha_k| = \alpha$ and the simple Jordan block J_{α_k} is of dimension $N_k \times N_k$ when $1 \le k \le n$. If we set $N = N_1 + N_2 + \cdots + N_n$, and the operator T is of the form $(T_{ij})_{i,j=1}^n$, then $T \in \mathcal{B}_J$ if and only if each block T_{ij} is an upper triangular $N_i \times N_j$ matrix. Furthermore, if the ordered basis $\{e_k\}$ is replaced by its permutation*

$$e_0, e_{N_1}, e_{N_2}, \ldots, e_{N_{n-1}}, e_1, e_{N_1+1}, e_{N_2+1}, \ldots, e_{N_{n-1}+1}, \ldots,$$

then an operator $T \in \mathcal{B}_J$ if and only if it is block upper triangular relative to the new basis.

It remains to consider the situation in which the zeros of θ may be of different absolute values. Here we prove a more general result, which is true regardless of the dimension of the Hilbert space.

PROPOSITION 3.8. *Let $A_k \in \mathcal{L}(\mathcal{H}_k)$ where $k = 1, 2$ and $r(A_1) < r(A_2)$, and let $A = A_1 \oplus A_2$. Suppose that there exists an orthonormal basis $\{e_n\}$ of $\mathcal{H}_2$ such that $e_i \otimes e_i \in \mathcal{B}_{A_2}$ and $\lim_m \|R_m(A_2)e_i\| = \infty$ when $i \ge 0$. If T is an operator on $\mathcal{H}_1 \oplus \mathcal{H}_2$*

with matrix

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

relative to this decomposition, then $T \in \mathcal{B}_A$ *if and only if* $T_3 = 0$ *and* $T_4 \in \mathcal{B}_{A_2}$.

PROOF. Let $C_m = \sum_{n\geq 0} d_m(A)^{2n} A_1^{*n} A_1^n$. Note that $R_m^2(A) = C_m \oplus R_m^2(A_2)$ since $d_m(A) = d_m(A_2)$. The inequality $r(A_1) < r(A_2)$ implies that the sequence C_m is norm bounded. Since $C_m \geq 1$ and $R_m^2(A_2) \geq 1$, we see that C_m^{-1} and $R_m^{-1}(A_2)$ are contractions. Consequently, $\sup_m \|C_m T_1 C_m^{-1}\| < \infty$ and $\sup_m \|C_m T_2 R_m(A_2)^{-1}\| < \infty$ for all T_1 and T_2. Further, $\|R_m(A_2) T_4 R_m(A_2)^{-1}\|$ is bounded if and only if $T_4 \in \mathcal{B}_A$. Finally, let $\{e_n\}$ be the basis as stipulated, and let (t_{ij}) be the matrix for T_3, relative to the same basis $\{e_n\}$ for both $\mathcal{H}_1$ and $\mathcal{H}_2$. (When $N = \dim(\mathcal{H}_1) < \dim(\mathcal{H}_2)$ the basis of $\mathcal{H}_1$ is $\{e_n\}_{n\leq N}$; when $\dim(\mathcal{H}_1) > \dim(\mathcal{H}_2)$ the basis of $\mathcal{H}_1$ is obtained by extending $\{e_n\}$ to an arbitrary orthonormal basis.) We note that

$$\begin{pmatrix} 0 & 0 \\ 0 & e_i \otimes e_i \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} e_j \otimes e_j & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ t_{ij} e_i \otimes e_j & 0 \end{pmatrix}. \tag{3.1}$$

If $T \in \mathcal{B}_A$, then in (3.1), all $T_{ij} \in \mathcal{B}_A$, and it follows that $\|R_m^2(A_2) t_{ij} e_i \otimes e_j C_m^{-1}\|$ is a bounded sequence. However, $C_m^{-1} \geq 1$ and $\lim_m \|R_m(A_2) e_i\| = \infty$ so $t_{ij} = 0$ for all i and j, and hence $T_3 = 0$. Since the other direction is trivial, the proof is complete. $\square$

REMARK 3.9. The existence of an orthonormal basis satisfying the conditions listed in Proposition 3.8 is essential for the conclusion that $T_3 = 0$. Indeed, if $A_2 = 0 \oplus 1$ and $T_3 = 1 \oplus 0$, then $R_m(A_2) T_3 = T_3$, whence the boundedness of C_m^{-1} implies that $\sup_m \|R_m(A_2) T_3 C_m^{-1}\| < \infty$.

We now establish the most general result for the case where $S(\theta)$ acts on a finite-dimensional space.

THEOREM 3.10. *Let* $N_1, N_2, \ldots, N_n$ *and* $K_1, K_2, \ldots, K_m$ *be positive integers, let* $N = N_1 + \cdots + N_n$ *and* $K = K_1 + \cdots + K_m$, *and let* $\{\alpha_i\}_{i=1}^n$ *and* $\{\beta\}_{j=1}^m$ *be sequences of complex numbers such that*

$$|\alpha_1| < |\alpha_2| < \cdots < |\alpha_n| < |\beta_1| = |\beta_2| = \cdots = |\beta_m|.$$

Suppose that simple Jordan blocks J_{α_i} *and* J_{β_j} *are of dimensions* $N_i \times N_i$ *and* $K_j \times K_j$ *respectively, and let* J *denote* $J_\alpha \oplus J_\beta$, *where* $J_\alpha = J_{\alpha_1} \oplus \cdots \oplus J_{\alpha_n}$ *and* $J_\beta = J_{\beta_1} \oplus \cdots \oplus J_{\beta_m}$. *Relative to the decomposition* $\mathbb{C}^{N+K} = \mathbb{C}^N \oplus \mathbb{C}^K$, *let*

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}.$$

Then $T \in \mathcal{B}_J$ *if and only if* $T_3 = 0$ *and* $T_4 \in \mathcal{B}_{J_\beta}$.

PROOF. Clearly $r(J_\alpha) < r(J_\beta)$ and $e_i \otimes e_i \in \mathcal{B}_{J_\beta}$ by Corollary 3.7. Furthermore, $\lim_m \|R_m e_i\|^2 = \lim_m \langle R_m^2 e_i, e_i\rangle = \infty$, by Theorem 3.3. The result now follows from Proposition 3.8. $\square$

4. Operators of class C_0

In the remainder of the paper we apply the results about Jordan blocks to describe $\mathcal{B}_A$ for all $A \in C_0$. From [1, Theorem 3.5.1], there exist inner functions $\{\theta_k\}$ and Hilbert spaces $\mathcal{H}_k$ such that $\theta_{k+1}|\theta_k$ and A is quasisimilar to a direct sum of Jordan blocks $S(\Theta) \equiv \bigoplus_k S(\theta_k)$, acting on $\bigoplus_k \mathcal{H}_k$. Therefore, we need to establish some ties between spectral radius algebras associated with quasisimilar operators. We start with a result from [9]. Recall that an operator $Z \in \mathcal{L}(\mathcal{H})$ is a quasiaffinity if it has trivial kernel and dense range.

LEMMA 4.1. *Suppose that A and B are quasisimilar C_0 contractions and let Y, Z be quasiaffinities such that $AY = YB$ and $ZA = BZ$. If $T \in \mathcal{B}_B$, then $YTZ \in \mathcal{B}_A$.*

We now establish a much stronger result.

THEOREM 4.2. *Suppose that A and B are quasisimilar C_0 contractions and let Y, Z be quasiaffinities such that $AY = YB$ and $ZA = BZ$. Then $\mathcal{B}_A$ is weakly dense in $\mathcal{L}(\mathcal{H})$ if and only if the same is true of $\mathcal{B}_B$. Moreover, if one of the algebras possesses a set of rank-one operators with a dense span, then so does the other. In fact, if there is a dense set $\mathcal{N}$ such that $\sup_m \|R_m(A)u\| < \infty$ for all $u \in \mathcal{N}$, then $\sup_m \|R_m(B)w\| < \infty$ for all w in the dense set $Z\mathcal{N}$. On the other hand, if one of the algebras has a nontrivial invariant subspace, then the same is true of the other algebra.*

PROOF. Suppose that $\mathcal{B}_B$ is dense. It suffices to show that the weak closure of $\mathcal{B}_A$ contains all rank-one operators in $\mathcal{L}(\mathcal{H})$, because the closure of $\mathcal{B}_A$ is an algebra that contains all finite rank operators, and hence is dense. So let $\epsilon > 0$, and let W be a rank-one operator in $\mathcal{L}(\mathcal{H})$. Since Y and Z^* have dense ranges, there are $u, v \in \mathcal{H}$ such that $W_1 \equiv Yu \otimes Z^*v$ satisfies $|\langle (W_1 - W)x, y\rangle| < \epsilon\|x\|\|y\|$ for all $x, y \in \mathcal{H}$. Also, $\mathcal{B}_B$ is dense, hence there exists an operator $W_2 \in \mathcal{B}_B$ such that $|\langle (W_2 - u \otimes v)x, y\rangle| < \epsilon\|x\|\|y\|$ for all $x, y \in \mathcal{H}$. By Lemma 4.1, $YW_2Z \in \mathcal{B}_A$ and it is easy to see that

$$|\langle (YW_2Z - W)x, y\rangle| < \epsilon(\|Z\|\|Y\| + 1)\|x\|\|y\|,$$

whence $\mathcal{B}_A$ is dense in $\mathcal{L}(\mathcal{H})$. Also, if W_2 is a finite-rank operator, then so is YW_2Z. This shows that if $\{u_\alpha \otimes v_\alpha\}$ is a collection of rank-one operators with dense span, then the same is true of $\{Y(u_\alpha \otimes v_\alpha)Z\}$. Finally, since A and B share the same quasisimilarity model, they have the same spectral radius and thus $d_m(A) = d_m(B)$. Since $\|B^nZ\| = \|ZA^n\| \le \|Z\|\|A^n\|$, we obtain that $\|R_m(B)Zu\| \le \|Z\|\|R_m(A)u\|$.

We now turn our attention to the existence of an invariant subspace. This part of the proof is based on the proof of [10, Theorem 6.19]. Let $\mathcal{M}_A$ be an invariant subspace for $\mathcal{B}_A$. We define the subspace $\mathcal{M}_B$ to be the closure of $\{TZx : x \in \mathcal{M}_A, T \in \mathcal{B}_B\}$. Since $\mathcal{B}_B$ is an algebra, it is easy to see that $\mathcal{M}_B$ is invariant for $\mathcal{B}_B$. Clearly $\mathcal{M}_B \neq \{0\}$, so it remains to prove that $\mathcal{M}_B$ is not the whole space. To that end, we show that $Y\{TZx : x \in \mathcal{M}_A, T \in \mathcal{B}_B\} \subseteq \mathcal{M}_A$, whence the result follows from the fact that Y has dense range, except that the last inclusion follows from the facts that $YTZ \in \mathcal{B}_A$ and $\mathcal{M}_A$ is invariant for $\mathcal{B}_A$. □

With Theorem 4.2 in hand, we proceed to analyze the operator $S(\Theta)$. It turns out that, as before, there are two very different cases, depending on the type of the minimal function of A. We present these results separately, in Theorems 4.3 and 4.6.

THEOREM 4.3. *Let A be a C_0 contraction and let m_A be its minimal function. If m_A is not a finite Blaschke product, then the algebra $\mathcal{B}_A$ contains a set of rank-one operators with a dense span, so it is dense.*

PROOF. Suppose first that none of the functions θ_k in the quasisimilarity model $S(\Theta)$, defined to be $\bigoplus_{k\in\mathbb{N}} S(\theta_k)$, is a finite Blaschke product. By Theorem 2.4, for each k there is a dense set of vectors $\mathcal{N}_k \subseteq \mathcal{H}_k$ such that $\sup_m \|R_m(S(\theta_k)u\| < \infty$ for all $u \in \mathcal{N}_k$. Define the subset $\mathcal{N}$ of $\bigoplus_{k\in\mathbb{N}} \mathcal{N}_k$ as follows: if $x = \bigoplus_{k\in\mathbb{N}} x_k \in \bigoplus_{k\in\mathbb{N}} \mathcal{N}_k$, then $x \in \mathcal{N}$ if there are at most a finite number of k such that $x_k \neq 0$. Then $\mathcal{N}$ is dense in $\bigoplus_k \mathcal{H}_k$ and $\sup_m \|R_m(S(\Theta))u\| < \infty$ for all $u \in \mathcal{N}$. Further, if Y, Z are quasiaffinities such that $S(\Theta)Y = YA$ and $ZS(\Theta) = AZ$, then $Z\mathcal{N}$ is dense and, by Theorem 4.2, it follows that $\sup_m \|R_m(A)w\| < \infty$ for all $w \in Z\mathcal{N}$.

Thus, we turn our attention to the case where there exists $k_0 > 0$ such that θ_k is a finite Blaschke product for $k \geq k_0$ but not for $k < k_0$. In this situation, we use the notation $S(\Theta_1) = \bigoplus_{k\geq k_0} S(\theta_k)$ and $S(\Theta_2) = \bigoplus_{k<k_0} S(\theta_k)$, so $S(\Theta) = S(\Theta_1) \oplus S(\Theta_2)$. Note that $r(S(\Theta_1)) < r(S(\Theta_2)) = 1$. If

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in \mathcal{B}_{S(\Theta)}$$

(relative to the same decomposition), then it follows as in the proof of Proposition 3.8 that T_3 and T_4 must satisfy the conditions

$$\sup_m \|R_m(S(\Theta_2))T_3R_m(S(\Theta_1))^{-1}\| < \infty$$

and

$$\sup_m \|R_m(S(\Theta_2))T_3R_m(S(\Theta_2))^{-1}\| < \infty.$$

Since R_m^{-1} is always a contraction, each of these conditions is met when the relevant operator (T_3 or T_4) is the rank-one operator $u \otimes v$ and $\sup_m \|R_m(S(\Theta_2))u\| < \infty$. The first part of the proof shows that this is true when $u \in \bigoplus_{k<k_0} \mathcal{N}_k$, which is dense in $\bigoplus_{k<k_0} \mathcal{H}_k$. Consequently, $\mathcal{B}_{S(\Theta)}$ contains a set of rank-one operators with a dense span, and by Theorem 4.2, the same is true of $\mathcal{B}_A$. □

It remains to consider the case where m_A is a finite Blaschke product. We note that, due to the relation $\theta_{k+1}|\theta_k$ between the inner functions in the quasisimilarity model $S(\Theta)$, the function θ_0 is a finite Blaschke product, and each zero of each of the functions θ_k must be a zero of θ_0. Let $\{\alpha_i\}_{i=1}^n$ and $\{\beta\}_{j=1}^m$ be the zeros of θ_0, labelled so that

$$|\alpha_1| < |\alpha_2| < \cdots < |\alpha_n| < |\beta_1| = |\beta_2| = \cdots = |\beta_m|.$$

We denote by J' and J'' direct sums of copies of simple Jordan blocks with eigenvalues α_i (where $1 \leq i \leq n$) and β_j (where $1 \leq j \leq m$) respectively, so that $S(\Theta)$ is quasisimilar to the direct sum $J' \oplus J''$. In order to apply Proposition 3.8, we need to understand the algebra $\mathcal{B}_{J''}$. Note that J'' is a (possibly infinite) direct sum of a finite number of distinct simple Jordan blocks. We split these blocks into two sets—those that are repeated infinitely many times and those that are repeated finitely many times. Of course, if the former set is empty, the characterization of $\mathcal{B}_{J''}$ was obtained in Corollary 3.7. Our first step is to consider the case where the latter set is empty.

THEOREM 4.4. *Let $J = J_{\alpha_1} \oplus J_{\alpha_2} \oplus \cdots \oplus J_{\alpha_n}$, where $|\alpha_k| = \alpha$ and the simple Jordan block J_{α_k} is of dimension $N_k \times N_k$ when $1 \le k \le n$. Let $N = N_1 + N_2 + \cdots + N_n$ and let A be a direct sum of infinitely many copies of J. If $T = (T_{ij})_{i,j=1}^{\infty}$ relative to the decomposition*

$$\mathcal{H} = \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2} \oplus \cdots \oplus \mathbb{C}^{N_n} \oplus \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2} \oplus \cdots \oplus \mathbb{C}^{N_n} \oplus \cdots,$$

then $T \in \mathcal{B}_A$ if and only if each block T_{ij} is upper triangular.

PROOF. Relative to the given decomposition, let $\mathcal{H}_1 = \mathbb{C}^{N_1}$, $\mathcal{H}_2 = \mathbb{C}^{N_2}$, and so on. Suppose now that $T \in \mathcal{B}_A$, and that one of its blocks, say T_{pq}, is not upper triangular. Let A' denote the restriction of A to $\mathcal{H}_p \oplus \mathcal{H}_q$. Since both $R_m(A)$ and $R_m^{-1}(A)$ are block diagonal matrices with blocks of the same spectral radius α, the (p, q)th block of $R_m(A)TR_m(A)^{-1}$ is equal to $R_m(A')T_{pq}R_m(A')^{-1}$. Therefore

$$\sup_m \|R_m(A')T_{pq}R_m(A')^{-1}\| \le \sup_m \|R_m(A)TR_m(A)^{-1}\| < \infty$$

and $T_{pq} \in \mathcal{B}_{A'}$, contradicting Theorem 3.5. That shows that the upper triangularity condition is necessary.

To prove that it is sufficient, let $T = (T_{ij})$ be a matrix relative to the given decomposition of $\mathcal{H}$, and suppose that each block is upper triangular. We now replace the basis $\{e_n\}$ by its permutation $\{\tilde{e}_n\}$, so that in the new basis the matrix of T becomes block upper triangular. In the first of two steps, we write $\mathcal{H} = \bigoplus_{k \ge 0} \mathcal{G}_k$, where each $\mathcal{G}_k$ is a copy of $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n$, and we permute the basis vectors within each $\mathcal{G}_k$ as described in the remark following Theorem 3.5. Relative to the new decomposition and basis of $\mathcal{H}$, T is now a block matrix and each block is an $N \times N$ matrix that is itself block upper triangular. We denote this new basis of $\mathcal{G}_k$ by $\{f_i^{(k)}\}_{i=1}^N$, or just $\{f_i\}_{i=1}^N$. Next we perform what is sometimes called 'the canonical shuffle': we write $\mathcal{H}$ as a direct sum $\mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \cdots \oplus \mathcal{K}_N$, where $\mathcal{K}_i$ has as an ordered basis $f_i^{(1)}, f_i^{(2)}, \ldots$. If we denote this new basis of $\mathcal{H}$ by $\{\tilde{e}_j\}$, we see that the corresponding matrix for T is block upper triangular; more precisely, $T = (C_{ij})_{i,j=1}^N$, and $C_{ij} = 0$ when $i > j$. The transition from $\{f_n\}$ to $\{\tilde{e}_n\}$ also affects the matrices of $R_m(A)$ and $R_m(A)^{-1}$. Since A is a direct sum of the same operator J, the operators $R_m(A)$ and $R_m(A)^{-1}$ exhibit the same pattern: $R_m(A)$ is a direct sum of infinitely many copies of $R_m(J)$, and $R_m(A)^{-1}$ is a direct sum of infinitely many copies of $R_m(J)^{-1}$. Therefore, if in the basis $\{f_n\}$ the matrices for $R_m(J)$ and $R_m^{-1}(J)$ are $(r_{ij}^{(m)})_{i,j=1}^N$ and $(s_{ij}^{(m)})_{i,j=1}^N$ respectively, then in $\{\tilde{e}_n\}$ the matrices for $R_m(A)$ and $R_m^{-1}(A)$ are $(r_{ij}^{(m)}I)_{i,j=1}^N$ and $(s_{ij}^{(m)}I)_{i,j=1}^N$.

We can now prove that $T \in \mathcal{B}_A$. Clearly, $R_m(A)TR_m(A)^{-1}$ is an $N \times N$ matrix with operator entries, so we need to show that each of its N^2 blocks remains bounded as $m \to \infty$. To that end, fix i and j. Then the (i, j)th block of $R_m(A)TR_m(A)^{-1}$ is $\sum_{k,l=1}^{N} r_{ik}^{(m)} C_{kl} s_{lj}^{(m)}$, so it suffices to prove that $\sup_m \|r_{ik}^{(m)} C_{kl} s_{lj}^{(m)}\| < \infty$ for each pair (k, l) where $k \le l$. We fix such a pair (k, l). Since C_{kl} is a bounded operator, it remains to prove that $\sup_m |r_{ik}^{(m)} s_{lj}^{(m)}| < \infty$. Note that

$$|r_{ik}^{(m)}| \le \|(r_{1k}^{(m)}, r_{2k}^{(m)}, \ldots, r_{Nk}^{(m)})\| = \|R_m(J) f_k\|.$$

Also, $R_m(J)^{-1}$ is a Hermitian matrix, and it follows that

$$|s_{lj}^{(m)}| = |s_{jl}^{(m)}| \le \|(s_{1l}^{(m)}, s_{2l}^{(m)}, \ldots, s_{Nl}^{(m)})\| = \|(R_m(J)^{-1}) f_l\|.$$

Thus

$$\sup_m |r_{ik}^{(m)} s_{lj}^{(m)}| \le \sup_m \|R_m(J) f_k\| \|(R_m(J)^{-1}) f_l\| = \sup_m \|R_m(J) f_k \otimes f_l (R_m(J)^{-1})\|.$$

It is not hard to see that the second assertion of Corollary 3.7 applies to the operator J and the basis $\{f_i\}_{i=1}^{N}$. Since $k \le l$, the theorem is proved. □

Next we address the situation when J'' is a direct sum of simple Jordan blocks, in which some blocks are repeated finitely many times, and others infinitely many times.

THEOREM 4.5. *Let $J_1 = J_{\alpha_1} \oplus J_{\alpha_2} \oplus \cdots \oplus J_{\alpha_n}$ and $J_2 = J_{\alpha_{n+1}} \oplus J_{\alpha_{n+2}} \oplus \cdots \oplus J_{\alpha_{n+m}}$, where $|\alpha_k| = \alpha$ and the simple Jordan block J_{α_k} is of dimension $N_k \times N_k$ whenever $1 \le k \le n+m$. Let A be a direct sum of infinitely many copies of J_1 followed by J_2. If $T = (T_{ij})_{i,j=1}^{\infty}$ relative to the decomposition*

$$\mathcal{H} = \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2} \oplus \cdots \oplus \mathbb{C}^{N_n} \oplus \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2} \oplus \cdots \oplus \mathbb{C}^{N_n} \oplus \cdots \oplus \mathbb{C}^{N_{n+1}} \oplus \mathbb{C}^{N_{n+2}} \oplus \cdots \oplus \mathbb{C}^{N_{m+n}},$$

then $T \in \mathcal{B}_A$ if and only if each block T_{ij} is upper triangular.

PROOF. We cannot apply Theorem 4.4 directly, because when $k \ge n+1$, the blocks J_{α_k} are not repeated infinitely many times. We correct this 'error' by defining the operator $\hat{J}$ as a direct sum of A with infinitely many copies of J_2. This operator acts on the direct sum $\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}'$, where

$$\mathcal{H}' = \mathbb{C}^{N_{n+1}} \oplus \mathbb{C}^{N_{n+2}} \oplus \cdots \oplus \mathbb{C}^{N_{m+n}} \oplus \mathbb{C}^{N_{n+1}} \oplus \mathbb{C}^{N_{n+2}} \oplus \cdots \oplus \mathbb{C}^{N_{m+n}} \oplus \cdots.$$

Further, we identify the operator T acting on $\mathcal{H}$ with the operator $\hat{T} = T \oplus 0$ acting on $\hat{\mathcal{H}}$. Then

$$R_m(\hat{J})\hat{T}R_m^{-1}(\hat{J}) = R_m(J)TR_m^{-1}(J),$$

so $T \in \mathcal{B}_A$ if and only if $\hat{T} \in \mathcal{B}_{\hat{J}}$. Now the result follows from Theorem 4.4 since each block of $\hat{T}$ is upper triangular if and only if the same is true of T. □

Combining Corollary 3.7, Theorems 4.4 and 4.5 we obtain the general case.

THEOREM 4.6. *Let A be a C_0 contraction on $\mathcal{H}$ and let m_A be its minimal function. If m_A is a finite Blaschke product, then A is quasisimilar to $S(\Theta)$, which is a finite or* *infinite direct sum of simple Jordan blocks. Further,* $S(\Theta) = S(\Theta_1) \oplus S(\Theta_2)$*, where all blocks in* $S(\Theta_1)$ *have eigenvalues of absolute value less than the spectral radius of* A *and all blocks in* $S(\Theta_2)$ *have eigenvalues of absolute value equal to the spectral radius of* A*. If*

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

relative to this decomposition, then $T \in \mathcal{B}_{S(\Theta)}$ *if and only if* $T_3 = 0$ *and* $T_4 \in \mathcal{B}_{S(\Theta_2)}$*. Moreover,* $S(\Theta_2) = \bigoplus J_{\alpha_k}$ *and, relative to this decomposition, an operator* $T = (T_{ij}) \in \mathcal{B}_{S(\Theta_2)}$ *if and only if each* T_{ij} *is upper triangular.*

Theorem 4.6 shows that the algebra $\mathcal{B}_{S(\Theta)}$ has a nontrivial invariant subspace. Using Theorem 4.2, we obtain our final result.

THEOREM 4.7. *Let* A *be a* C_0 *contraction on* $\mathcal{H}$ *and let* m_A *be its minimal function. If* m_A *is a finite Blaschke product, then the algebra* $\mathcal{B}_{S(\Theta)}$ *possesses a nontrivial invariant subspace.*

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