J. Aust. Math. Soc. **88** (2010), 205–230
doi:10.1017/S1446788709000433

TENSOR EXTENSION PROPERTIES OF $C(K)$-REPRESENTATIONS AND APPLICATIONS TO UNCONDITIONALITY

CHRISTOPH KRIEGLER and CHRISTIAN LE MERDY ✉

(Received 8 January 2009; accepted 10 July 2009)

Communicated by A. M. Hassell

Abstract

Let K be any compact set. The C^*-algebra $C(K)$ is nuclear and any bounded homomorphism from $C(K)$ into $B(H)$, the algebra of all bounded operators on some Hilbert space H, is automatically completely bounded. We prove extensions of these results to the Banach space setting, using the key concept of R-boundedness. Then we apply these results to operators with a uniformly bounded H^∞-calculus, as well as to unconditionality on L^p. We show that any unconditional basis on L^p 'is' an unconditional basis on L^2 after an appropriate change of density.

2000 *Mathematics subject classification*: primary 47A60; secondary 46B28.

Keywords and phrases: functional calculus, R-boundedness, unconditionality, tensor products.

1. Introduction

Throughout this paper, we let K be a nonempty compact set and we let $C(K)$ be the algebra of all continuous functions $f: K \to \mathbb{C}$, equipped with the supremum norm. A representation of $C(K)$ on some Banach space X is a bounded unital homomorphism $u: C(K) \to B(X)$ into the algebra $B(X)$ of all bounded operators on X. Such representations appear naturally and play a major role in several fields of operator theory, including functional calculi, spectral theory and spectral measures, and the classification of C^*-algebras. Several recent papers, in particular [8, 12, 21, 23], have emphasized the rich and fruitful interplays between the notion of R-boundedness, unconditionality and various functional calculi. The aim of this paper is to establish new properties of the $C(K)$-representations involving R-boundedness, and to give applications to H^∞-calculus (in the sense of [6, 21]) and to unconditionality in L^p-spaces.

We recall the definition of R-boundedness (see [2, 4]). Let $(\epsilon_k)_{k\geq 1}$ be a sequence of independent Rademacher variables on some probability space Ω_0. That is, the ϵ_k

The first author is supported by the Karlsruhe House of Young Scientists and the Franco-German University DFH-UFA, the second author is supported by the research program ANR-06-BLAN-0015.

take values in the set $\{-1, 1\}$ and $\mathrm{Prob}(\{\epsilon_k = 1\}) = \mathrm{Prob}(\{\epsilon_k = -1\}) = 1/2$. For any Banach space X, we let $\mathrm{Rad}(X) \subset L^2(\Omega_0; X)$ be the closure of $\mathrm{Span}\{\epsilon_k \otimes x : k \geq 1, x \in X\}$ in $L^2(\Omega_0; X)$. Thus, for all $x_1, \ldots, x_n$ in X,

$$\Big\| \sum_k \epsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(X)} = \bigg(\int_{\Omega_0} \Big\| \sum_k \epsilon_k(\lambda) x_k \Big\|_X^2 \, d\lambda \bigg)^{1/2}.$$

By definition, a set $\tau \subseteq B(X)$ is R-bounded if there is a constant $C \geq 0$ such that, for all finite families $T_1, \ldots, T_n$ in τ, and $x_1, \ldots, x_n$ in X,

$$\Big\| \sum_k \epsilon_k \otimes T_k x_k \Big\|_{\mathrm{Rad}(X)} \leq C \Big\| \sum_k \epsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(X)}.$$

In this case, we let $R(\tau)$ denote the smallest possible C. It is called the R-bound of τ. By convention, we write $R(\tau) = \infty$ if τ is not R-bounded.

It will be convenient to let $\mathrm{Rad}_n(X)$ denote the subspace of $\mathrm{Rad}(X)$ of all finite sums $\sum_{k=1}^n \epsilon_k \otimes x_k$. If $X = H$ is a Hilbert space, then $\mathrm{Rad}_n(H) = \ell_n^2(H)$ isometrically and all bounded subsets of $B(H)$ are automatically R-bounded. Conversely, if X is not isomorphic to a Hilbert space, then $B(X)$ contains bounded subsets which are not R-bounded [1, Proposition 1.13].

In order to provide motivation for the results in this paper, we recall two well-known properties of $C(K)$-representations on the Hilbert space H. First, any bounded homomorphism $u \colon C(K) \to B(H)$ is completely bounded, and $\|u\|_{\mathrm{cb}} \leq \|u\|^2$, that is for all integers $n \geq 1$, the tensor extension $I_{M_n} \otimes u \colon M_n(C(K)) \to M_n(B(H))$ satisfies $\|I_{M_n} \otimes u\| \leq \|u\|^2$ when $M_n(C(K))$ and $M_n(B(H))$ are both equipped with their natural C^*-algebra norms. This in turn implies that any bounded homomorphism $u \colon C(K) \to B(H)$ is similar to a $*$-representation, a result going back at least to [3]. We refer to [28, 30] and the references therein for some information on completely bounded maps and similarity properties.

Second, let $u \colon C(K) \to B(H)$ be a bounded homomorphism. Then for all $b_1, \ldots, b_n$ lying in the commutant of the range of u and for all $f_1, \ldots, f_n$ in $C(K)$,

$$\Big\| \sum_k u(f_k) b_k \Big\| \leq \|u\|^2 \sup_{t \in K} \Big\| \sum_k f_k(t) b_k \Big\|. \tag{1.1}$$

This property is essentially a rephrasing of the fact that $C(K)$ is a nuclear C^*-algebra. More precisely, nuclearity means that the above property holds true for $*$-representations (see, for example, [19, Ch. 11] or [28, Ch. 12]), and its extension to arbitrary bounded homomorphisms easily follows from the similarity property mentioned above (see [25] for more explanations and developments).

Now let X be a Banach space and let $u \colon C(K) \to B(X)$ be a bounded homomorphism. In Section 2, we will show the following analog of (1.1):

$$\Big\| \sum_k u(f_k) b_k \Big\| \leq \|u\|^2 R\bigg(\Big\{ \sum_k f_k(t) b_k : t \in K \Big\} \bigg), \tag{1.2}$$

provided that the b_k commute with the range of u.

 https://doi.org/10.1017/S1446788709000433

Section 3 is devoted to the sectorial operators A which have a uniformly bounded H^∞-calculus, in the sense that they satisfy an estimate

$$\|f(A)\| \le C \sup_{t>0} |f(t)| \tag{1.3}$$

for all bounded analytic functions f on a sector Σ_θ surrounding $(0, \infty)$. Such operators turn out to have a natural $C(K)$-functional calculus. Applying (1.2) to the resulting representation $u\colon C(K) \to B(X)$, we show that (1.3) can be automatically extended to operator-valued analytic functions f taking their values in the commutant of A. This is an analog of a remarkable result of Kalton and Weis [21, Theorem 4.4] which says that if an operator A has a bounded H^∞-calculus and f is an operator-valued analytic function taking its values in an R-bounded subset of the commutant of A, then the operator $f(A)$ arising from 'generalized H^∞-calculus' is bounded.

In Section 4, we introduce matricially R-bounded maps $C(K) \to B(X)$, a natural analog of completely bounded maps in the Banach space setting. We show that if X has property (α), then any bounded homomorphism $C(K) \to B(X)$ is automatically matricially R-bounded. This extends both the Hilbert space result mentioned above, and a result of de Pagter and Ricker [8, Corollary 2.19] which says that any bounded homomorphism $C(K) \to B(X)$ maps the unit ball of $C(K)$ into an R-bounded set, provided that X has property (α).

In Section 5 , we give an application of matricial R-boundedness to the case when $X = L^p$. A classical result of Johnson and Jones [18] asserts that any bounded operator $T\colon L^p \to L^p$ acts, after an appropriate change of density, as a bounded operator on L^2. We show versions of this theorem for bases (more generally, for finite-dimensional decompositions). Indeed, we show that any unconditional basis (any R-basis) on L^p becomes an unconditional basis (respectively a Schauder basis) on L^2 after an appropriate change of density. These results rely on Simard's extensions of the Johnson–Jones theorem established in [32].

We end this introduction with a few preliminaries and some notation. For any Banach space Z, we denote by $C(K; Z)$ the space of all continuous functions $f\colon K \to Z$, equipped with the supremum norm

$$\|f\|_\infty = \sup\{\|f(t)\|_Z : t \in K\}.$$

We may regard $C(K) \otimes Z$ as a subspace of $C(K; Z)$ by identifying $\sum_k f_k \otimes z_k$ with the function $t \mapsto \sum_k f_k(t) z_k$, for all finite families $(f_k)_k$ in $C(K)$ and $(z_k)_k$ in Z. Moreover, $C(K) \otimes Z$ is dense in $C(K; Z)$. Note that, for all integers $n \ge 1$, $C(K; M_n)$ coincides with the C^*-algebra $M_n(C(K))$ mentioned above.

We will need the so-called 'contraction principle' which says that, for all $x_1, \ldots, x_n$ in a Banach space X and all $\alpha_1, \ldots, \alpha_n$ in $\mathbb{C}$,

$$\Big\|\sum_k \epsilon_k \otimes \alpha_k x_k\Big\|_{\mathrm{Rad}(X)} \le 2 \sup_k |\alpha_k| \Big\|\sum_k \epsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)}. \tag{1.4}$$

We also recall that any unital commutative C^*-algebra is a $C(K)$-space (see, for example, [19, Ch. 4]). Thus our results concerning $C(K)$-representations apply as well to all these algebras. For example, we will apply them to ℓ^∞ in Section 5.

We let I_X denote the identity mapping on a Banach space X, and we let χ_B denote the indicator function of a set B. If X is a dual Banach space, we let $w^*B(X) \subseteq B(X)$ be the subspace of all w^*-continuous operators on X.

2. The extension theorem

Let X be an arbitrary Banach space. For any compact set K and any bounded homomorphism $u\colon C(K) \to B(X)$, we denote by

$$E_u = \{b \in B(X) : bu(f) = u(f)b \;\forall f \in C(K)\}$$

the commutant of the range of u.

Our main purpose in this section is to prove (1.1). We start with the case when $C(K)$ is finite-dimensional.

PROPOSITION 2.1. *Let $N \geq 1$ and let $u\colon \ell^\infty_N \to B(X)$ be a bounded homomorphism. Let $(e_1, \ldots, e_N)$ be the canonical basis of ℓ^∞_N and set $p_i = u(e_i)$, $i = 1, \ldots, N$. Then, for all $b_1, \ldots, b_N \in E_u$,*

$$\Big\| \sum_{i=1}^N p_i b_i \Big\| \leq \|u\|^2 R(\{b_1, \ldots, b_N\}).$$

PROOF. Since u is multiplicative, each p_i is a projection and $p_i p_j = 0$ when $i \neq j$. Hence for all choices of signs $(\alpha_1, \ldots, \alpha_N) \in \{-1, 1\}^N$,

$$\sum_{i=1}^N p_i b_i = \sum_{i,j=1}^N \alpha_i \alpha_j p_i p_j b_j.$$

Furthermore,

$$\Big\| \sum_i \alpha_i p_i \Big\| = \|u(\alpha_1, \ldots, \alpha_N)\| \leq \|u\| \|(\alpha_1, \ldots, \alpha_N)\|_{\ell^\infty_N} = \|u\|.$$

Therefore, for all $x \in X$, we have the following chain of inequalities which prove the desired estimate:

$$\begin{aligned}
\Big\| \sum_i p_i b_i x \Big\|^2 &= \int_{\Omega_0} \Big\| \sum_i \epsilon_i(\lambda) p_i \sum_j \epsilon_j(\lambda) p_j b_j x \Big\|^2 d\lambda \\
&\leq \int_{\Omega_0} \Big\| \sum_i \epsilon_i(\lambda) p_i \Big\|^2 \Big\| \sum_j \epsilon_j(\lambda) p_j b_j x \Big\|^2 d\lambda \\
&\leq \|u\|^2 \int_{\Omega_0} \Big\| \sum_j \epsilon_j(\lambda) b_j p_j x \Big\|^2 d\lambda
\end{aligned}$$

$$
\begin{aligned}
&\leq \|u\|^2 R(\{b_1, \dots, b_N\})^2 \int_{\Omega_0} \Big\| \sum_j \epsilon_j(\lambda) p_j x \Big\|^2 \, d\lambda \\
&\leq \|u\|^4 R(\{b_1, \dots, b_N\})^2 \|x\|^2.
\end{aligned}
$$

This concludes the proof. □

The study of infinite-dimensional $C(K)$-spaces requires the use of second duals and w^*-topologies. We recall a few well-known facts that will be used later on in this paper. According to the Riesz representation theorem, the dual space $C(K)^*$ can be naturally identified with the space $M(K)$ of Radon measures on K. Next, the second dual space $C(K)^{**}$ is a commutative C^*-algebra for the so-called Arens product. This product extends the product on $C(K)$ and is separately w^*-continuous, which means that, for all $\xi \in C(K)^{**}$, the two linear maps

$$
\nu \in C(K)^{**} \longmapsto \nu\xi \in C(K)^{**} \quad \text{and} \quad \nu \in C(K)^{**} \longmapsto \xi\nu \in C(K)^{**}
$$

are w^*-continuous.

Equip the space $\mathcal{B}^\infty(K)$ of all bounded, Borel measurable functions from K to $\mathbb{C}$ with the supremum norm. According to the duality pairing

$$
\langle f, \mu \rangle = \int_K f(t)\, d\mu(t) \quad \forall \mu \in M(K),\ f \in \mathcal{B}^\infty(K),
$$

one can regard $\mathcal{B}^\infty(K)$ as a closed subspace of $C(K)^{**}$. Moreover, the restriction of the Arens product to $\mathcal{B}^\infty(K)$ coincides with the pointwise product. Thus we have the natural C^*-algebra inclusions

$$
C(K) \subseteq \mathcal{B}^\infty(K) \subseteq C(K)^{**}. \tag{2.1}
$$

See, for example, [7, pp. 366–367] and [5, Section 9] for further details.

Let $\widehat{\otimes}$ denote the projective tensor product on Banach spaces. We recall that, for any two Banach spaces Y_1, Y_2, we have a natural identification

$$
(Y_1 \widehat{\otimes} Y_2)^* \simeq B(Y_2, Y_1^*),
$$

see, for example, [10, Section VIII.2]. This implies that when X is a dual Banach space, $X = (X_*)^*$ say, then $B(X) = (X_* \widehat{\otimes} X)^*$ is a dual space. The next two lemmas are elementary.

LEMMA 2.2. *Let $X = (X_*)^*$ be a dual space, $S \in B(X)$, and define the right and left multiplication operators $R_S, L_S \colon B(X) \to B(X)$ by $R_S(T) = TS$ and $L_S(T) = ST$, respectively. Then R_S is w^*-continuous whereas L_S is w^*-continuous if (and only if) S is w^*-continuous.*

PROOF. The tensor product mapping $I_{X_*} \otimes S$ on $X_* \otimes X$ uniquely extends to a bounded map $r_S \colon X_* \widehat{\otimes} X \to X_* \widehat{\otimes} X$, and we have $R_S = r_S^*$. Thus R_S is w^*-continuous.

Likewise, if S is w^*-continuous and if we let $S_*\colon X_* \to X_*$ be its pre-adjoint map, the tensor product mapping $S_* \otimes I_X$ on $X_* \otimes X$ extends to a bounded map $l_S\colon X_*\widehat{\otimes}X \to X_*\widehat{\otimes}X$, and $L_S = l_S^*$. Thus L_S is w^*-continuous. The converse (which we will not use) is left to the reader. $\square$

LEMMA 2.3. *Let $u\colon C(K) \to B(X)$ be a bounded map. Suppose that X is a dual space. Then there exists a (necessarily unique) w^*-continuous linear mapping $\widetilde{u}\colon C(K)^{**} \to B(X)$ whose restriction to $C(K)$ coincides with u. Moreover, $\|\widetilde{u}\| = \|u\|$.*

Furthermore, if u is a homomorphism and u takes values in $w^*B(X)$, then $\widetilde{u}$ is also a homomorphism.

PROOF. Let $j\colon (X_*\widehat{\otimes}X) \hookrightarrow (X_*\widehat{\otimes}X)^{**}$ be the canonical injection and consider its adjoint $p = j^*\colon B(X)^{**} \to B(X)$. Then set

$$\widetilde{u} = p \circ u^{**}\colon C(K)^{**} \longrightarrow B(X).$$

By construction, $\widetilde{u}$ is w^*-continuous and extends u. The equality $\|\widetilde{u}\| = \|u\|$ is clear.

Assume now that u is a homomorphism and that u takes values in $w^*B(X)$. Let $\nu, \xi \in C(K)^{**}$ and let $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ be bounded nets in $C(K)$ w^*-converging to ν and ξ, respectively. By Lemma 2.2, we have the following equalities, where limits are taken in the w^*-topology of either $C(K)^{**}$ or $B(X)$:

$$\begin{aligned}\widetilde{u}(\nu\xi) &= \widetilde{u}(\lim_\alpha \lim_\beta f_\alpha g_\beta) = \lim_\alpha \lim_\beta u(f_\alpha g_\beta) = \lim_\alpha \lim_\beta u(f_\alpha)u(g_\beta)\\ &= \lim_\alpha u(f_\alpha)\widetilde{u}(\xi) = \widetilde{u}(\nu)\widetilde{u}(\xi).\end{aligned}$$ $\square$

We refer, for example, to [17, Lemma 2.4] for the following fact.

LEMMA 2.4. *Consider $\tau \subseteq B(X)$ and set $\tau^{**} = \{T^{**} : T \in \tau\} \subseteq B(X^{**})$. Then τ is R-bounded if and only if τ^{**} is R-bounded, and in this case*

$$R(\tau) = R(\tau^{**}).$$

For any $F \in C(K; B(X))$, we set

$$R(F) = R(\{F(t) : t \in K\}).$$

Note that $R(F)$ may be infinite. If $F = \sum_k f_k \otimes b_k$ belongs to the algebraic tensor product $C(K) \otimes B(X)$, we set

$$\Big\|\sum_k f_k \otimes b_k\Big\|_R = R(F) = R\bigg(\bigg\{\sum_k f_k(t)b_k : t \in K\bigg\}\bigg).$$

Note that, by (1.4),

$$\|f \otimes b\|_R \leq 2\|f\|_\infty \|b\| \quad \forall f \in C(K),\ b \in B(X). \tag{2.2}$$

From this it is easy to check that $\|\cdot\|_R$ is finite and is a norm on $C(K) \otimes B(X)$.

Whenever $E \subseteq B(X)$ is a closed subspace, we let

$$C(K) \overset{R}{\otimes} E$$

denote the completion of $C(K) \otimes E$ for the norm $\|\cdot\|_R$.

REMARK 2.5. Clearly $\|\cdot\|_\infty \leq \|\cdot\|_R$ on $C(K) \otimes B(X)$, since the R-bound of a set is greater than its uniform bound. Hence the canonical embedding of $C(K) \otimes B(X)$ into $C(K; B(X))$ extends uniquely to a contraction

$$J : C(K) \overset{R}{\otimes} B(X) \longrightarrow C(K; B(X)).$$

Moreover, J is one-to-one and, for all $\varphi \in C(K) \overset{R}{\otimes} B(X)$, we have $R(J(\varphi)) = \|\varphi\|_R$. To see this, let $(F_n)_{n\geq 1}$ be a sequence in $C(K) \otimes B(X)$ such that $\|F_n - \varphi\|_R \to 0$ and let $F = J(\varphi)$. Then $\|F_n\|_R \to \|\varphi\|_R$ and $\|F_n - F\|_\infty \to 0$. According to the definition of the R-bound, the latter property implies that $\|F_n\|_R \to \|F\|_R$, which yields the result.

THEOREM 2.6. *Let $u : C(K) \to B(X)$ be a bounded homomorphism.*

(1) *For all finite families $(f_k)_k$ in $C(K)$ and $(b_k)_k$ in E_u,*

$$\Big\| \sum_k u(f_k) b_k \Big\| \leq \|u\|^2 \Big\| \sum_k f_k \otimes b_k \Big\|_R .$$

(2) *There is a (necessarily unique) bounded linear map*

$$\widehat{u} : C(K) \overset{R}{\otimes} E_u \longrightarrow B(X)$$

such that $\widehat{u}(f \otimes b) = u(f)b$ for all $f \in C(K)$ and all $b \in E_u$. Furthermore, $\|\widehat{u}\| \leq \|u\|^2$.

PROOF. Part (2) clearly follows from part (1). To prove part (1) we introduce

$$w : C(K) \longrightarrow B(X^{**}), \quad w(f) = u(f)^{**} .$$

Then w is a bounded homomorphism and $\|w\| = \|u\|$. We let $\widetilde{w} : C(K)^{**} \to B(X^{**})$ be its w^*-continuous extension given by Lemma 2.3. Note that w takes values in $w^* B(X^{**})$, so $\widetilde{w}$ is a homomorphism. We claim that

$$\{b^{**} : b \in E_u\} \subseteq E_{\widetilde{w}}.$$

Indeed, let $b \in E_u$. Then, for all $f \in C(K)$,

$$b^{**}\, w(f) = (bu(f))^{**} = (u(f)b)^{**} = w(f)b^{**} .$$

Next, for all $\nu \in C(K)^{**}$, let $(f_\alpha)_\alpha$ be a bounded net in $C(K)$ which converges to ν in the w^*-topology. Then, by Lemma 2.2,

$$b^{**}\widetilde{w}(\nu) = \lim_\alpha b^{**}w(f_\alpha) = \lim_\alpha w(f_\alpha)b^{**} = \widetilde{w}(\nu)b^{**},$$

and the claim follows.

Now fix $f_1, \ldots, f_n \in C(K)$ and $b_1, \ldots, b_n \in E_u$. For each $m \in \mathbb{N}$, there is a finite family $(t_1, \ldots, t_N)$ of K and a measurable partition $(B_1, \ldots, B_N)$ of K such that

$$\Big\| f_k - \sum_{l=1}^{N} f_k(t_l)\chi_{B_l} \Big\|_\infty \leq \frac{1}{m} \quad \forall k \in \{1, \ldots, n\}.$$

We set $f_k^{(m)} = \sum_{l=1}^N f_k(t_l)\chi_{B_l}$. Let $\psi : \ell_N^\infty \to \mathcal{B}^\infty(K)$ be defined by

$$\psi(\alpha_1, \ldots, \alpha_N) = \sum_{l=1}^{N} \alpha_l \chi_{B_l}.$$

Then ψ is a norm 1 homomorphism. According to (2.1), we can consider the bounded homomorphism

$$\widetilde{w} \circ \psi : \ell_N^\infty \longrightarrow B(X^{**}).$$

Applying Proposition 2.1 to that homomorphism, together with the above claim and Lemma 2.4, we find that

$$\begin{aligned}
\Big\| \sum_k \widetilde{w}(f_k^{(m)}) b_k{}^{**} \Big\| &= \Big\| \sum_{k,l} f_k(t_l) \widetilde{w} \circ \psi(e_l) b_k{}^{**} \Big\| \\
&\leq \|\widetilde{w} \circ \psi\|^2 R\bigg(\Big\{ \sum_k f_k(t_l) b_k{}^{**} : 1 \leq l \leq N \Big\} \bigg) \\
&\leq \|u\|^2 R\bigg(\Big\{ \sum_k f_k(t) b_k{}^{**} : t \in K \Big\} \bigg) \\
&\leq \|u\|^2 \Big\| \sum_k f_k \otimes b_k \Big\|_R.
\end{aligned}$$

Since $\|f_k^{(m)} - f_k\|_\infty \to 0$ for all k,

$$\Big\| \sum_k \widetilde{w}(f_k^{(m)}) b_k{}^{**} \Big\| \longrightarrow \Big\| \sum_k w(f_k) b_k{}^{**} \Big\| = \Big\| \sum_k u(f_k) b_k \Big\|,$$

and the result follows at once. □

The following notion is implicit in several recent papers on functional calculi (see, in particular, [8, 21]).

DEFINITION 2.7. Let Z be a Banach space and let $v\colon Z \to B(X)$ be a bounded map. We set

$$R(v) = R(\{v(z) : z \in Z,\ \|z\| \leq 1\}),$$

and we say that v is R-bounded if $R(v) < \infty$.

COROLLARY 2.8. *Suppose that $u\colon C(K) \to B(X)$ is a bounded homomorphism and that $v\colon Z \to B(X)$ is an R-bounded map. Assume further that $u(f)v(z) = v(z)u(f)$ for all $f \in C(K)$ and all $z \in Z$. Then there exists a (necessarily unique) bounded linear map*

$$u\cdot v\colon C(K; Z) \longrightarrow B(X)$$

such that $u\cdot v(f \otimes z) = u(f)v(z)$ for all $f \in C(K)$ and all $z \in Z$. Moreover, we have

$$\|u\cdot v\| \leq \|u\|^2 R(v).$$

PROOF. Consider any finite families $(f_k)_k$ in $C(K)$ and $(z_k)_k$ in Z and observe that

$$\Big\|\sum_k f_k \otimes v(z_k)\Big\|_R = R\bigg(\bigg\{v\bigg(\sum_k f_k(t)z_k\bigg) : t \in K\bigg\}\bigg) \leq R(v)\Big\|\sum_k f_k \otimes z_k\Big\|_\infty.$$

Then, from Theorem 2.6 and the assumption that v takes values in E_u, we find that

$$\Big\|\sum_k u(f_k)v(z_k)\Big\| \leq \|u\|^2 R(v)\Big\|\sum_k f_k \otimes z_k\Big\|_\infty,$$

which proves the result. □

REMARK 2.9. As a special case of Corollary 2.8, we obtain the following result due to de Pagter and Ricker [8, Proposition 2.27]: let K_1, K_2 be two compact sets, and let

$$u\colon C(K_1) \longrightarrow B(X) \quad \text{and} \quad v\colon C(K_2) \longrightarrow B(X)$$

be two bounded homomorphisms which commute, that is, $u(f)v(g) = v(g)u(f)$ for all $f \in C(K_1)$ and $g \in C(K_2)$. Assume further that $R(v) < \infty$. Then there exists a bounded homomorphism

$$w\colon C(K_1 \times K_2) \longrightarrow B(X)$$

such that $w_{|C(K_1)} = u$ and $w_{|C(K_2)} = v$, where $C(K_j)$ is regarded to be a subalgebra of $C(K_1 \times K_2)$ in the natural way.

3. Uniformly bounded H^∞-calculus

We briefly recall the basic notions on H^∞-calculus for sectorial operators. For more information, we refer, for example, to [6, 21, 23, 24].

For all $\theta \in (0, 2\pi)$, we define

$$\Sigma_\theta = \{re^{i\phi} : r > 0, |\phi| < \theta\}$$

and $H^\infty(\Sigma_\theta)$ to be the set of all bounded analytic functions from Σ_θ to $\mathbb{C}$. This space is equipped with the norm $\|f\|_{\infty,\theta} = \sup_{\lambda\in\Sigma_\theta} |f(\lambda)|$ and is a Banach algebra. We consider the auxiliary space $H_0^\infty(\Sigma_\theta)$ consisting of all functions f in $H^\infty(\Sigma_\theta)$ for which there exist positive constants ϵ and C such that

$$|f(\lambda)| \leq C \min |\lambda|^\epsilon, |\lambda|^{-\epsilon} \quad \forall\lambda \in \Sigma_\theta.$$

A closed linear operator $A : D(A) \subseteq X \to X$ is said to be ω-sectorial, where $\omega \in (0, 2\pi)$, if its domain $D(A)$ is dense in X, its spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\omega}$, and for all $\theta > \omega$ there is a constant $C_\theta > 0$ such that

$$\|\lambda(\lambda - A)^{-1}\| \leq C_\theta \quad \forall\lambda \in \mathbb{C}\backslash\overline{\Sigma_\theta}.$$

In this case, we define

$$\omega(A) = \inf\{\omega : A \text{ is } \omega\text{-sectorial}\}.$$

For all $\theta \in (\omega(A), \pi)$ and all $f \in H_0^\infty(\Sigma_\theta)$, we define

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} f(\lambda)(\lambda - A)^{-1}\, d\lambda, \tag{3.1}$$

where $\omega(A) < \gamma < \theta$ and Γ_γ is the boundary $\partial\Sigma_\gamma$ oriented counterclockwise. This definition does not depend on γ and the resulting mapping $f \mapsto f(A)$ is an algebra homomorphism from $H_0^\infty(\Sigma_\theta)$ into $B(X)$. We say that A has a bounded $H^\infty(\Sigma_\theta)$-calculus if the latter homomorphism is bounded, that is, if there exists a constant $C > 0$ such that $\|f(A)\| \leq C\|f\|_{\infty,\theta}$ for all $f \in H_0^\infty(\Sigma_\theta)$. If, in addition, A is one-to-one and has a dense range, then this homomorphism extends to a bounded homomorphism $H^\infty(\Sigma_\theta) \to B(X)$.

We will now focus on the sectorial operators A such that $\omega(A) = 0$.

DEFINITION 3.1. We say that a sectorial operator A with $\omega(A) = 0$ has a uniformly bounded H^∞-calculus if there exists a constant $C > 0$ such that $\|f(A)\| \leq C\|f\|_{\infty,\theta}$ for all $\theta > 0$ and $f \in H_0^\infty(\Sigma_\theta)$.

The space $C_\ell([0, \infty))$, consisting of all continuous functions $f : [0, \infty) \to \mathbb{C}$ for which $\lim_{\lambda\to\infty} f(\lambda)$ exists, is a unital commutative C^*-algebra when equipped with the natural norm

$$\|f\|_{\infty,0} = \sup\{|f(t)| : t \geq 0\}$$

and involution. For all $\theta > 0$, we can regard $H_0^\infty(\Sigma_\theta)$ as a subalgebra of $C_\ell([0, \infty))$, by identifying $f \in H_0^\infty(\Sigma_\theta)$ with its restriction $f_{|[0,\infty)}$.

For all $\lambda \in \mathbb{C}\backslash[0, \infty)$, we let $R_\lambda \in C_\ell([0, \infty))$ be defined by $R_\lambda(t) = (\lambda - t)^{-1}$. Then we let $\mathcal{R}$ be the unital algebra generated by the R_λ. Equivalently, $\mathcal{R}$ is the algebra of all rational functions of nonpositive degree, whose poles lie outside the half line $[0, \infty)$. We recall that, for all $f \in H_0^\infty(\Sigma_\theta) \cap \mathcal{R}$, the definition of $f(A)$ given by (3.1) coincides with the usual rational functional calculus.

The following lemma is closely related to [22, Corollary 6.9].

LEMMA 3.2. *Let A be a sectorial operator on X with $\omega(A) = 0$. The following assertions are equivalent.*

(a) *A has a uniformly bounded H^∞-calculus.*

(b) *There exists a (necessarily unique) bounded unital homomorphism*

$$u\colon C_\ell([0, \infty)) \longrightarrow B(X)$$

such that $u(R_\lambda) = (\lambda - A)^{-1}$ for all $\lambda \in \mathbb{C}\backslash[0, \infty)$.

PROOF. Assume (a). We claim that, for all $\theta > 0$ and all $f \in H_0^\infty(\Sigma_\theta)$,

$$\|f(A)\| \leq C\|f\|_{\infty,0}.$$

Indeed, if $0 \neq f \in H_0^\infty(\Sigma_{\theta_0})$ for some $\theta_0 > 0$, then there exists some $t_0 > 0$ such that $f(t_0) \neq 0$. Now take r and R such that $r < R$ and $|f(z)| < |f(t_0)|$ when $|z| < r$ or $|z| > R$. Choose, for every $n \in \mathbb{N}$, a $t_n \in \Sigma_{\theta_0/n}$ such that $|f(t_n)| = \|f\|_{\infty,\theta_0/n}$. Necessarily, $|t_n| \in [r, R]$, and there exists a convergent subsequence t_{n_k} whose limit t_∞ is real. Then

$$\|f\|_{\infty,0} \geq |f(t_\infty)| \geq \liminf_{\theta\to 0} \|f\|_{\infty,\theta} \geq C^{-1}\|f(A)\|.$$

This readily implies that the rational functional calculus $(\mathcal{R}, \|\cdot\|_{\infty,0}) \to B(X)$ is bounded. By the Stone–Weierstrass theorem, this extends continuously to $C_\ell([0, \infty))$, which yields (b). The uniqueness property is clear.

Assume (b). Then for all $\theta \in (0, \pi)$ and all $f \in H_0^\infty(\Sigma_\theta) \cap \mathcal{R}$,

$$\|f(A)\| \leq \|u\|\|f\|_{\infty,\theta}.$$

By [24, Proposition 2.10] and its proof, this implies that A has a bounded $H^\infty(\Sigma_\theta)$-calculus, with a boundedness constant uniform in θ. □

REMARK 3.3. An operator A which admits a bounded $H^\infty(\Sigma_\theta)$-calculus for all $\theta > 0$ does not necessarily have a uniformly bounded H^∞-calculus. To get a simple example, consider

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \ell_2^2 \longrightarrow \ell_2^2.$$

Then $\sigma(A) = \{1\}$ and, for all $\theta > 0$ and all $f \in H_0^\infty(\Sigma_\theta)$,

$$f(A) = \begin{pmatrix} f(1) & f'(1) \\ 0 & f(1) \end{pmatrix}.$$

Assume that $\theta < \pi/2$. Using Cauchy's formula, it is easy to see that $|f'(1)| \leq (\sin(\theta))^{-1}\|f\|_{\infty,\theta}$ for all $f \in H_0^\infty(\Sigma_\theta)$. Thus A admits a bounded $H^\infty(\Sigma_\theta)$-calculus.

Now let h be a fixed function in $H_0^\infty(\Sigma_{\pi/2})$ such that $h(1) = 1$, set $g_s(\lambda) = \lambda^{is}$ for all $s > 0$, and let $f_s = hg_s$. Then $\|g_s\|_{\infty,0} = 1$, and hence $\|f_s\|_{\infty,0} \leq \|h\|_{\infty,0}$ for all $s > 0$. Furthermore, $g_s'(\lambda) = is\lambda^{is-1}$ and $f_s' = h'g_s + hg_s'$. Hence $f_s'(1) = h'(1) + is$. Thus

$$\|f_s(A)\|\|f_s\|_{\infty,0}^{-1} \geq |f_s'(1)|\|h_s\|_{\infty,0}^{-1} \longrightarrow \infty$$

when $s \to \infty$. Hence A does not have a uniformly bounded H^∞-calculus.

The above result can also be deduced from Proposition 3.7 below. In fact we will show in that proposition and in Corollary 3.11 that operators with a uniformly bounded H^∞-calculus are 'rare'.

We now turn to the so-called generalized (or operator-valued) H^∞-calculus. Throughout, we let A be a sectorial operator. We let $E_A \subseteq B(X)$ denote the commutant of A, defined as the subalgebra of all bounded operators $T : X \to X$ such that $T(\lambda - A)^{-1} = (\lambda - A)^{-1}T$ for all λ belonging to the resolvent set of A. We let $H_0^\infty(\Sigma_\theta; B(X))$ be the algebra of all bounded analytic functions $F : \Sigma_\theta \to B(X)$ for which there exist $\epsilon, C > 0$ such that $\|F(\lambda)\| \leq C \min(|\lambda|^\epsilon, |\lambda|^{-\epsilon})$ for all $\lambda \in \Sigma_\theta$. Also, we let $H_0^\infty(\Sigma_\theta; E_A)$ denote the space of all E_A-valued functions belonging to $H_0^\infty(\Sigma_\theta; B(X))$. The generalized H^∞-calculus of A is an extension of (3.1) to this class of functions. Namely, for all $F \in H_0^\infty(\Sigma_\theta; E_A)$, we set

$$F(A) = \frac{1}{2\pi i}\int_{\Gamma_\gamma} F(\lambda)(\lambda - A)^{-1}\, d\lambda,$$

where $\gamma \in (\omega(A), \pi)$. Again, this definition does not depend on γ and the mapping $F \mapsto F(A)$ is an algebra homomorphism. The following fundamental result is due to Kalton and Weis.

THEOREM 3.4 [21, Theorem 4.4], [23, Theorem 12.7]. *Let $\omega_0 \geq \omega(A)$ and assume that A has a bounded $H^\infty(\Sigma_\theta)$-calculus for all $\theta > \omega_0$. Then, for all $\theta > \omega_0$, there exists a constant $C_\theta > 0$ such that, for all $F \in H_0^\infty(\Sigma_\theta; E_A)$,*

$$\|F(A)\| \leq C_\theta R(\{F(z) : z \in \Sigma_\theta\}). \tag{3.2}$$

Our aim is to prove a version of this result in the case when A has a uniformly bounded H^∞-calculus. We will find in Theorem 3.6 that in this case the constant C_θ in (3.2) can be taken to be independent of θ.

The algebra $C_\ell([0, \infty))$ is a $C(K)$-space and we will apply the results of Section 2 to the bounded homomophism u appearing in Lemma 3.2. We recall Remark 2.5.

LEMMA 3.5. *Let $J : C_\ell([0, \infty)) \otimes^R B(X) \to C_\ell([0, \infty); B(X))$ be the canonical embedding. Let $\theta \in (0, \pi)$, let $F \in H_0^\infty(\Sigma_\theta; B(X))$, and let $\gamma \in (0, \theta)$.*

(1) *The integral*

$$\varphi_F = \frac{1}{2\pi i} \int_{\Gamma_\gamma} R_\lambda \otimes F(\lambda)\, d\lambda \tag{3.3}$$

is absolutely convergent in $C_\ell([0, \infty)) \otimes^R B(X)$, *and* $J(\varphi_F)$ *is equal to the restriction of* F *to* $[0, \infty)$.

(2) *The set* $\{F(t) : t > 0\}$ *is* R*-bounded.*

PROOF. Part (2) readily follows from part (1) and Remark 2.5. To prove part (1), observe that, for all $\lambda \in \partial\Sigma_\gamma$,

$$\|R_\lambda \otimes F(\lambda)\|_R \leq 2\|R_\lambda\|_{\infty,0}\|F(\lambda)\| \leq \frac{2}{\sin(\gamma)|\lambda|}\|F(\lambda)\|$$

by (2.2). Thus, for appropriate constants ϵ, $C > 0$,

$$\|R_\lambda \otimes F(\lambda)\|_R \leq \frac{2C}{\sin(\gamma)} \min(|\lambda|^{\epsilon-1}, |\lambda|^{-\epsilon-1}).$$

This shows that the integral defining φ_F is absolutely convergent. Next, for all $t > 0$,

$$[J(\varphi_F)](t) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} (R_\lambda \otimes F(\lambda))(t)\, d\lambda = \frac{1}{2\pi i} \int_{\Gamma_\gamma} \frac{F(\lambda)}{\lambda - t}\, d\lambda = F(t)$$

by Cauchy's theorem. □

THEOREM 3.6. *Let A be a sectorial operator such that $\omega(A) = 0$ and assume that A has a uniformly bounded H^∞-calculus. Then there exists a constant $C > 0$ such that, for all $\theta > 0$ and all $F \in H_0^\infty(\Sigma_\theta; E_A)$,*

$$\|F(A)\| \leq C R(\{F(t) : t > 0\}).$$

PROOF. Let $u : C_\ell([0, \infty)) \to B(X)$ be the representation given by Lemma 3.2. It is plain that $E_u = E_A$. Then we let

$$\widehat{u} : C_\ell([0, \infty)) \overset{R}{\otimes} E_A \longrightarrow B(X)$$

be the associated bounded map provided by Theorem 2.6.

Let $F \in H_0^\infty(\Sigma_\theta; E_A)$ for some $\theta > 0$, and let $\varphi_F \in C_\ell([0, \infty)) \otimes^R E_A$ be defined by (3.3). We claim that

$$F(A) = \widehat{u}(\varphi_F).$$

Indeed, for all $\lambda \in \partial\Sigma_\gamma$, we have $u(R_\lambda) = (\lambda - A)^{-1}$, and hence $\widehat{u}(R_\lambda \otimes F(\lambda)) = (\lambda - A)^{-1}F(\lambda)$. Thus according to the definition of φ_F and the continuity of $\widehat{u}$,

$$\widehat{u}(\varphi_F) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} \widehat{u}(R_\lambda \otimes F(\lambda))\, d\lambda \;= \frac{1}{2\pi i} \int_{\Gamma_\gamma} (\lambda - A)^{-1}F(\lambda)\, d\lambda \;= F(A).$$

Consequently,

$$\|F(A)\| \leq \|\widehat{u}\|\|\varphi_F\|_R \leq \|u\|^2\|\varphi_F\|_R.$$

It follows from Lemma 3.5 and Remark 2.5 that $\|\varphi_F\|_R = R(\{F(t) : t > 0\})$, and the result follows at once. □

In the rest of this section we will investigate further the operators with a uniformly bounded H^∞-calculus. We start with the case when X is a Hilbert space.

PROPOSITION 3.7. *Let H be a Hilbert space and let A be a sectorial operator on H, such that $\omega(A)=0$. Then A admits a uniformly bounded H^∞-calculus if and only if there exists an isomorphism $S\colon H\to H$ such that $S^{-1}AS$ is self-adjoint.*

PROOF. Assume that A admits a uniformly bounded H^∞-calculus and denote the associated representation by $u\colon C_\ell([0,\infty))\to B(H)$. According to [28, Theorems 9.1 and 9.7], there exists an isomorphism $S\colon H\to H$ such that the unital homomorphism $u_S\colon C_\ell([0,\infty))\to B(H)$ defined by $u_S(f)=S^{-1}u(f)S$ satisfies $\|u_S\|\leq 1$. We let $B=S^{-1}AS$. For each $s\in\mathbb{R}^*$, we have $\|R_{is}\|_{\infty,0}=|s|$ and furthermore $u_S(R_{is})=S^{-1}(is-A)^{-1}S=(is-B)^{-1}$. Hence

$$\|(is-B)^{-1}\|\leq|s|\quad\forall s\in\mathbb{R}^*.$$

By the Hille–Yosida theorem, this implies that iB and $-iB$ both generate contractive c_0-semigroups on H. Thus iB generates a unitary c_0-group. By Stone's theorem, this implies that B is self-adjoint.

The converse implication is clear. □

In the non-Hilbertian setting, we will first show that operators with a uniformly bounded H^∞-calculus satisfy a spectral mapping theorem with respect to continuous functions defined on the one-point compactification of $\sigma(A)$. Then we will discuss the connections with spectral measures and scalar-type operators. We mainly refer to [13, Chs. 5–7] for this topic.

For any compact set K and any closed subset $F\subseteq K$, we let

$$I_F=\{f\in C(K): f_{|F}=0\}.$$

We recall that the restriction map $f\mapsto f_{|F}$ induces a $*$-isomorphism $C(K)/I_F\to C(F)$.

LEMMA 3.8. *Let $K\subset\mathbb{C}$ be a compact set and let $u\colon C(K)\to B(X)$ be a representation. Let $\kappa\in C(K)$ be the function defined by $\kappa(z)=z$ and take $T=u(\kappa)$.*

(1) *Then $\sigma(T)\subseteq K$ and u vanishes on $I_{\sigma(T)}$.*

Let $v\colon C(\sigma(T))\simeq C(K)/I_{\sigma(T)}\longrightarrow B(X)$ be the representation induced by u.

(2) *For any $f\in C(\sigma(T))$, we have $\sigma(v(f))=f(\sigma(T))$.*
(3) *v is an isomorphism onto its range.*

PROOF. The inclusion $\sigma(T)\subseteq K$ is clear. Indeed, for all $\lambda\notin K$, we have that $(\lambda-T)^{-1}$ is equal to $u((\lambda-\cdot\,)^{-1})$. We will now show that u vanishes on $I_{\sigma(T)}$.

Define $w\colon C(K)\to B(X^*)$ by $w(f)=[u(f)]^*$, and let $\widetilde{w}\colon C(K)^{**}\to B(X^*)$ be its w^*-extension. Since w takes values in $w^*B(X^*)\simeq B(X)$, this is a representation (see Lemma 2.3). Let Δ_K be the set of all Borel subsets of K. It is easy to check that the mapping

$$P\colon \Delta_K \longrightarrow B(X^*), \qquad P(B) = \widetilde{w}(\chi_B),$$

is a spectral measure of class (Δ_K, X) in the sense of [13, p. 119]. According to [13, Proposition 5.8], the operator T^* is prespectral of class X (in the sense of [13, Definition 5.5]) and the above mapping P is its resolution of the identity. Applying [13, Lemma 5.6] and the equality $\sigma(T^*) = \sigma(T)$, we find that $\widetilde{w}(\chi_{\sigma(T)}) = P(\sigma(T)) = I_{X^*}$. Therefore, for all $f \in I_{\sigma(T)}$,

$$u(f)^* = \widetilde{w}(f(1-\chi_{\sigma(T)})) = \widetilde{w}(f)\widetilde{w}(1-\chi_{\sigma(T)}) = 0.$$

Hence u vanishes on $I_{\sigma(T)}$.

The proofs of parts (2) and (3) now follow from [13, Proposition 5.9] and the above proof. □

In what follows we consider a sectorial operator A such that $\omega(A) = 0$. This implies that $\sigma(A) \subseteq [0, \infty)$. By $C_\ell(\sigma(A))$, we denote either the space $C(\sigma(A))$ if A is bounded, or the space $\{f : \sigma(A) \to \mathbb{C} \mid f$ is continuous and $\lim_{t\to\infty} f(t)$ exists$\}$ if A is unbounded. In this case, $C_\ell(\sigma(A))$ coincides with the space of continuous functions on the one-point compactification of $\sigma(A)$. The following strengthens Lemma 3.2.

PROPOSITION 3.9. *Let A be a sectorial operator on X with $\omega(A) = 0$. The following assertions are equivalent.*

(1) *A has a uniformly bounded H^∞-calculus.*

(2) *There exists a (necessarily unique) bounded unital homomorphism*

$$\Psi\colon C_\ell(\sigma(A)) \longrightarrow B(X)$$

such that $\Psi((\lambda - \cdot\,)^{-1}) = (\lambda - A)^{-1}$ for all $\lambda \in \mathbb{C}\backslash\sigma(A)$.

In this case, Ψ is an isomorphism onto its range and, for all $f \in C_\ell(\sigma(A))$,

$$\sigma(\Psi(f)) = f(\sigma(A)) \cup f_\infty, \tag{3.4}$$

where $f_\infty = \emptyset$ if A is bounded and $f_\infty = \lim_{t\to\infty} f(t)$ if A is unbounded.

PROOF. Assume part (1) and let $u\colon C_\ell([0,\infty)) \to B(X)$ be given by Lemma 3.2. We introduce the particular function $\phi \in C_\ell([0,\infty))$ defined by $\phi(t) = (1+t)^{-1}$. Consider the $*$-isomorphism

$$\tau\colon C([0,1]) \longrightarrow C_\ell([0,\infty)), \qquad \tau(g) = g \circ \phi,$$

and set $T = (1+A)^{-1}$. We define $\kappa(z) = z$ as in Lemma 3.8, and so $(u \circ \tau)(\kappa) = T$. Let $v\colon C(\sigma(T)) \to B(X)$ be the resulting factorization of $u \circ \tau$. The spectral mapping theorem gives $\sigma(A) = \phi^{-1}(\sigma(T)\backslash\{0\})$ and $0 \in \sigma(T)$ if and only if A is unbounded. Thus the mapping

$$\tau_A\colon C(\sigma(T)) \longrightarrow C_\ell(\sigma(A))$$

defined by $\tau_A(g) = g \circ \phi$ is also a $*$-isomorphism. Take $\Psi \colon C_\ell(\sigma(A)) \to B(X)$ to be $r \circ \tau_A^{-1}$. This is a unital bounded homomorphism. Note that $\phi^{-1}(z) = (1-z)/z$ for all $z \in (0, 1]$. Then, for all $\lambda \in \mathbb{C} \setminus \sigma(A)$,

$$\begin{aligned}\Psi((\lambda - \cdot\,)^{-1}) = v((\lambda - \cdot\,)^{-1} \circ \phi^{-1}) &= v\Big(z \mapsto \Big(\lambda - \frac{1-z}{z}\Big)^{-1}\Big)\\ &= v\Big(z \mapsto \frac{z}{(\lambda+1)z - 1}\Big)\\ &= T((\lambda+1)T - 1)^{-1} = (\lambda - A)^{-1}.\end{aligned}$$

Hence Ψ satisfies part (2). Its uniqueness follows from Lemma 3.2. The fact that Ψ is an isomorphism onto its range and the spectral property (3.4) follow from the above construction and Lemma 3.8. Lemma 3.2 shows that (2) implies (1). □

REMARK 3.10. Let A be a sectorial operator with a uniformly bounded H^∞-calculus, and let $T = (1 + A)^{-1}$. It follows from Lemma 3.8 and the proof of Proposition 3.9 that there exists a representation

$$v \colon C(\sigma(T)) \longrightarrow B(X)$$

satisfying $v(\kappa) = T$ (where $\kappa(z) = z$), such that $\sigma(v(f)) = f(\sigma(T))$ for all $f \in C(\sigma(T))$ and v is an isomorphism onto its range. Also, it follows from the proof of Lemma 3.8 that T^* is a scalar-type operator of class X, in the sense of [13, Definition 5.14].

Next, according to [13, Theorem 6.24], the operator T (and hence A) is a scalar-type spectral operator if and only if, for all $x \in X$, the mapping $C(\sigma(T)) \to X$ taking f to $v(f)x$ for all $f \in C(\sigma(T))$ is weakly compact.

COROLLARY 3.11. *Let A be a sectorial operator on X, with $\omega(A) = 0$, and assume that X does not contain a copy of c_0. Then A admits a uniformly bounded H^∞-calculus if and only if it is a scalar-type spectral operator.*

PROOF. The 'only if' part follows from the previous remark. Indeed, if X does not contain a copy of c_0, then any bounded map $C(K) \to X$ is weakly compact [10, VI, Theorem 15]. (See also [8, 31] for related approaches.) The 'if' part follows from [16, Proposition 2.7] and its proof. □

REMARK 3.12.

(1) The hypothesis on X in Corollary 3.11 is necessary. Namely, it follows from [11, Theorem 3.2] and its proof that if $c_0 \subseteq X$, then there is a sectorial operator A with a uniformly bounded H^∞-calculus on X which is not scalar-type spectral.
(2) An operator on a Hilbert space is scalar-type spectral if and only if it is similar to a normal operator (see [13, Ch. 7]). Thus, when X is a Hilbert space, the above corollary reduces to Proposition 3.7.

4. Matricial R-boundedness

For all integers $n \geq 1$ and all vector spaces E, we denote by $M_n(E)$ the space of $n \times n$ matrices with entries in E. We will be concerned mostly with the cases $E = C(K)$ or $E = B(X)$. As mentioned in the introduction, we identify $M_n(C(K))$ with the space $C(K; M_n)$ in the usual way. We now introduce a specific norm on $M_n(B(X))$. Namely, for all $[T_{ij}] \in M_n(B(X))$, we set

$$\|[T_{ij}]\|_R = \sup\left\{\left\|\sum_{i,j=1}^n \epsilon_i \otimes T_{ij}(x_j)\right\|_{\mathrm{Rad}(X)} : x_1, \ldots, x_n \in X,\right.$$
$$\left.\left\|\sum_{j=1}^n \epsilon_j \otimes x_j\right\|_{\mathrm{Rad}(X)} \leq 1\right\}.$$

Clearly $\|\cdot\|_R$ is a norm on $M_n(B(X))$. Moreover, if we consider any element of $M_n(B(X))$ as an operator on $\ell_n^2 \otimes X$ in the natural way, and if we equip the latter tensor product with the norm of $\mathrm{Rad}_n(X)$, we obtain an isometric identification

$$(M_n(B(X)), \|\cdot\|_R) = B(\mathrm{Rad}_n(X)). \tag{4.1}$$

Definition 4.1. Let $u : C(K) \to B(X)$ be a bounded linear mapping. We say that u is matricially R-bounded if there is a constant $C \geq 0$ such that, for all $n \geq 1$ and all $[f_{ij}] \in M_n(C(K))$,

$$\|[u(f_{ij})]\|_R \leq C\|[f_{ij}]\|_{C(K;M_n)}. \tag{4.2}$$

Remark 4.2. The above definition obviously extends to any bounded map $E \to B(X)$ defined on an operator space E, or more generally on any matricially normed space (see [14, 15]). The basic observations below apply to this general case as well.

(1) In the case when $X = H$ is a Hilbert space,

$$\left\|\sum_{j=1}^n \epsilon_j \otimes x_j\right\|_{\mathrm{Rad}(H)} = \left(\sum_{j=1}^n \|x_j\|^2\right)^{1/2}$$

for all $x_1, \ldots, x_n \in H$. Consequently, writing that a mapping $u : C(K) \to B(H)$ is matricially R-bounded is equivalent to writing that u is completely bounded (see, for example, [28]). See Section 5 for the case when X is an L^p-space.

(2) The notation $\|\cdot\|_R$ introduced above is consistent with that considered so far in Section 2. Indeed, let $b_1, \ldots, b_n$ in $B(X)$. Then the diagonal matrix $\mathrm{Diag}\{b_1, \ldots, b_n\} \in M_n(B(X))$ and the tensor element $\sum_{k=1}^n e_k \otimes b_k \in \ell_n^\infty \otimes B(X)$ satisfy

$$\|\mathrm{Diag}\{b_1, \ldots, b_n\}\|_R = R(\{b_1, \ldots, b_n\}) = \left\|\sum_{k=1}^n e_k \otimes b_k\right\|_R.$$

(3) If $u : C(K) \to B(X)$ is matricially R-bounded (with the estimate (4.2)), then u is R-bounded and $R(u) \leq C$. Indeed, consider $f_1, \ldots, f_n$ in the unit ball of $C(K)$.

Then we have $\|\mathrm{Diag}\{f_1, \ldots, f_n\}\|_{C(K;M_n)} \leq 1$. Hence, for all $x_1, \ldots, x_n$ in X,

$$\Big\|\sum_k \epsilon_k \otimes u(f_k)x_k\Big\|_{\mathrm{Rad}(X)} \leq \|\mathrm{Diag}\{u(f_1), \ldots, u(f_n)\}\|_R \Big\|\sum_k \epsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)}$$
$$\leq C\Big\|\sum_k \epsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)}.$$

Let $(g_k)_{k\geq 1}$ be a sequence of complex-valued, independent, standard Gaussian random variables on some probability space Ω_G. For all $x_1, \ldots, x_n$ in X let

$$\Big\|\sum_k g_k \otimes x_k\Big\|_{G(X)} = \bigg(\int_{\Omega_G}\Big\|\sum_k g_k(\lambda)x_k\Big\|_X^2 \, d\lambda\bigg)^{1/2}.$$

It is well known that for each scalar-valued matrix $a = [a_{ij}] \in M_n$,

$$\Big\|\sum_{i,j=1}^n a_{ij} g_i \otimes x_j\Big\|_{G(X)} \leq \|a\|_{M_n}\Big\|\sum_{j=1}^n g_j \otimes x_j\Big\|_{G(X)}, \tag{4.3}$$

see, for example, [9, Corollary 12.17]. For all $n \geq 1$, introduce $\sigma_{n,X} \colon M_n \to B(\mathrm{Rad}_n(X))$ by letting

$$\sigma_{n,X}([a_{ij}]) = [a_{ij} I_X].$$

If X has finite cotype, then we have a uniform equivalence

$$\Big\|\sum_k \epsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)} \asymp \Big\|\sum_k g_k \otimes x_k\Big\|_{G(X)} \tag{4.4}$$

between Rademacher and Gaussian averages on X (see, for example, [9, Theorem 12.27]). In combination with (4.3), this implies that

$$\sup_{n\geq 1}\|\sigma_{n,X}\| < \infty.$$

Following [29] we say that X has property (α) if there is a constant $C \geq 1$ such that, for each finite family (x_{ij}) in X and each finite family (t_{ij}) of complex numbers,

$$\Big\|\sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes t_{ij}x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \leq C \sup_{i,j}|t_{ij}|\Big\|\sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}. \tag{4.5}$$

Equivalently, X has property (α) if and only if we have a uniform equivalence

$$\Big\|\sum_{i,j} \epsilon_i \otimes \epsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \asymp \Big\|\sum_{i,j} \epsilon_{ij} \otimes x_{ij}\Big\|_{\mathrm{Rad}(X)},$$

where $(\epsilon_{ij})_{i,j\geq 1}$ is a doubly indexed family of independent Rademacher variables.

The following is a characterization of property (α) in terms of the R-boundedness of $\sigma_{n,X}$.

LEMMA 4.3. *A Banach space X has property (α) if and only if*

$$\sup_{n\geq 1} R(\sigma_{n,X}) < \infty.$$

PROOF. Assume that X has property (α). This implies that X has finite cotype, and hence X satisfies the equivalence property (4.4). Let $a(1), \ldots, a(N)$ be in M_n and let $z_1, \ldots, z_N$ be in $\mathrm{Rad}_n(X)$. Let x_{jk} be in X such that $z_k = \sum_j \epsilon_j \otimes x_{jk}$ for all k. We consider a doubly indexed family $(\epsilon_{ik})_{i,k\geq 1}$ as above, as well as a doubly indexed family $(g_{ik})_{i,k\geq 1}$ of independent standard Gaussian variables. Then

$$\sum_k \epsilon_k \otimes \sigma_{n,X}(a(k))z_k = \sum_{k,i,j} \epsilon_k \otimes \epsilon_i \otimes a(k)_{ij} x_{jk}. \tag{4.6}$$

Hence, using the properties reviewed above,

$$\begin{aligned}
&\Big\|\sum_k \epsilon_k \otimes \sigma_{n,X}(a(k))z_k\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}\\
&\quad \asymp \Big\|\sum_{k,i,j} \epsilon_{ik} \otimes a(k)_{ij} x_{jk}\Big\|_{\mathrm{Rad}(X)} \asymp \Big\|\sum_{k,i,j} g_{ik} \otimes a(k)_{ij} x_{jk}\Big\|_{G(X)}\\
&\quad \lesssim \left\|\begin{pmatrix} a(1) & 0\ldots & 0\\ 0 & \ddots & 0\\ 0 & \ldots 0 & a(N)\end{pmatrix}\right\|_{M_{Nn}} \Big\|\sum_{k,j} g_{jk} \otimes x_{jk}\Big\|_{G(X)}\\
&\quad \lesssim \max_k \|a(k)\|_{M_n} \Big\|\sum_{k,j} \epsilon_{jk} \otimes x_{jk}\Big\|_{\mathrm{Rad}(X)}\\
&\quad \lesssim \max_k \|a(k)\|_{M_n} \Big\|\sum_{k,j} \epsilon_k \otimes \epsilon_j \otimes x_{jk}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}\\
&\quad \lesssim \max_k \|a(k)\|_{M_n} \Big\|\sum_{k} \epsilon_k \otimes z_k\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}.
\end{aligned}$$

This shows that the $\sigma_{n,X}$ are uniformly R-bounded.

Conversely, assume that for some constant $C \geq 1$ we have $R(\sigma_{n,X}) \leq C$ for all $n \geq 1$. Let $(t_{jk})_{j,k} \in \mathbb{C}^{n^2}$ where $|t_{jk}| \leq 1$ and, for all $k = 1, \ldots, n$, let $a(k) \in M_n$ be the diagonal matrix with entries $t_{1k}, \ldots, t_{nk}$ on the diagonal. Then $\|a(k)\| \leq 1$ for all k. Hence, applying (4.6), we find that, for all $(x_{jk})_{j,k}$ in X^{n^2},

$$\begin{aligned}
&\Big\|\sum_{j,k} \epsilon_k \otimes \epsilon_j \otimes t_{jk} x_{jk}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}\\
&\quad \leq R(\{a(1), \ldots, a(n)\}) \Big\|\sum_{j,k} \epsilon_k \otimes \epsilon_j \otimes x_{jk}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}\\
&\quad \leq C \Big\|\sum_{j,k} \epsilon_k \otimes \epsilon_j \otimes x_{jk}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}.
\end{aligned}$$

This means that X has property (α). □

PROPOSITION 4.4. *Assume that X has property (α). Then any bounded homomorphism $u\colon C(K)\to B(X)$ is matricially R-bounded.*

PROOF. Let $u\colon C(K)\to B(X)$ be a bounded homomorphism and let $w\colon C(K)\to B(\mathrm{Rad}_n(X))$ be defined by

$$w(f)=I_{\mathrm{Rad}_n}\otimes u(f).$$

Clearly w is also a bounded homomorphism, with $\|w\|=\|u\|$. Recall the identification (4.1) and note that $w(f)=\mathrm{Diag}\{u(f),\ldots,u(f)\}$ for all $f\in C(K)$. Then, for all $a=[a_{ij}]\in M_n$,

$$w(f)\sigma_{n,X}(a)=[a_{ij}u(f)]=\sigma_{n,X}(a)w(f).$$

By Corollary 2.8 and Lemma 4.3, the resulting mapping $w\cdot\sigma_{n,X}$ satisfies

$$\|w\cdot\sigma_{n,X}\colon C(K;M_n)\longrightarrow B(\mathrm{Rad}_n(X))\|\leq C\|u\|^2$$

where C does not depend on n. Let E_{ij} denote the canonical matrix units of M_n, for $i,j=1,\ldots,n$. Consider $[f_{ij}]\in C(K;M_n)\simeq M_n(C(K))$ and write this matrix as $\sum_{i,j}E_{ij}\otimes f_{ij}$. Then

$$w\cdot\sigma_{n,X}([f_{ij}])=\sum_{i,j=1}^{n}w(f_{ij})\sigma_{n,X}(E_{ij})=\sum_{i,j=1}^{n}u(f_{ij})\otimes E_{ij}=[u(f_{ij})].$$

Hence $\|[u(f_{ij})]\|_R\leq C\|u\|^2\|[f_{ij}]\|_{C(K;M_n)}$, which proves that u is matricially R-bounded. □

When $X=H$ is a Hibert space, it follows from Remark 4.2(1) that the above proposition reduces to the fact that any bounded homomorphism $C(K)\to B(H)$ is completely bounded.

We also observe that by applying the above proposition together with Remark 4.2(3) we obtain the following corollary originally due to de Pagter and Ricker [8, Corollary 2.19]. Indeed, Proposition 4.4 should be regarded as a strengthening of their result.

COROLLARY 4.5. *Assume that X has property (α). Then any bounded homomorphism $u\colon C(K)\to B(X)$ is R-bounded.*

REMARK 4.6. The above corollary is nearly optimal. Indeed, we claim that if X does not have property (α) and if K is any infinite compact set, then there exists a unital bounded homomorphism

$$u\colon C(K)\longrightarrow B(\mathrm{Rad}(X))$$

which is not R-bounded.

To prove this, let $(z_n)_{n\geq 1}$ be an infinite sequence of distinct points in K and let u be defined by

$$u(f)\Big(\sum_{k\geq 1}\epsilon_k\otimes x_k\Big)=\sum_{k\geq 1}f(z_k)\epsilon_k\otimes x_k.$$

According to (1.4), this is a bounded unital homomorphism satisfying $\|u\|\leq 2$. Assume now that u is R-bounded. Let $n\geq 1$ be an integer and consider families $(t_{ij})_{i,j}$ in $\mathbb{C}^{n^2}$ and $(x_{ij})_{i,j}$ in X^{n^2}. For all $i=1,\ldots,n$, there exists $f_i\in C(K)$ such that $\|f_i\|=\sup_j|t_{ij}|$ and $f_i(z_j)=t_{ij}$ for all $j=1,\ldots,n$. Then

$$\sum_i\epsilon_i\otimes u(f_i)\Big(\sum_j\epsilon_j\otimes x_{ij}\Big)=\sum_{i,j}t_{ij}\epsilon_i\otimes\epsilon_j\otimes x_{ij},$$

and hence

$$\begin{aligned}\Big\|\sum_{i,j}t_{ij}\epsilon_i\otimes\epsilon_j\otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}&\leq R(u)\sup_i\|f_i\|\Big\|\sum_{i,j}\epsilon_i\otimes\epsilon_j\otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}\\&\leq R(u)\sup_{i,j}|t_{ij}|\Big\|\sum_{i,j}\epsilon_i\otimes\epsilon_j\otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}.\end{aligned}$$

This shows (4.5).

5. Application to L^p-spaces and unconditional bases

Let X be a Banach lattice with finite cotype. A classical theorem of Maurey asserts that, in addition to (4.4), we have a uniform equivalence

$$\Big\|\sum_k\epsilon_k\otimes x_k\Big\|_{\mathrm{Rad}(X)}\asymp\Big\|\Big(\sum_k|x_k|^2\Big)^{1/2}\Big\|$$

for finite families $(x_k)_k$ of X (see, for example, [9, Theorem 16.18]). Thus a bounded linear mapping $u\colon C(K)\to B(X)$ is matricially R-bounded if there is a constant $C\geq 0$ such that, for all $n\geq 1$, for all matrices $[f_{ij}]\in M_n(C(K))$ and for all $x_1,\ldots,x_n\in X$,

$$\Big\|\Big(\sum_i\Big|\sum_j u(f_{ij})x_j\Big|^2\Big)^{1/2}\Big\|\leq C\|[f_{ij}]\|_{C(K;M_n)}\Big\|\Big(\sum_j|x_j|^2\Big)^{1/2}\Big\|.$$

Mappings satisfying this property were introduced by Simard in [32] under the name of ℓ^2-cb maps. In this section we will apply a factorization property of ℓ^2-cb maps established in [32], in the case when X is merely an L^p-space.

Throughout this section, we let (Ω,μ) be a σ-finite measure space. By definition, a density on that space is a measurable function $g\colon\Omega\to(0,\infty)$ such that $\|g\|_1=1$. For all such functions and all $1\leq p<\infty$, we consider the linear mapping

$$\phi_{p,g}\colon L^p(\Omega,\mu)\longrightarrow L^p(\Omega,g\mu),\quad \phi_{p,g}(h)=g^{-1/p}h,$$

which is an isometric isomorphism. Note that $(\Omega, g\mu)$ is a probability space. Passing from (Ω, μ) to $(\Omega, g\mu)$ by means of the maps $\phi_{p,g}$ is usually called a change of density. A classical theorem of Johnson and Jones [18] asserts that, for all bounded operators $T: L^p(\mu) \to L^p(\mu)$, there is a density g on Ω such that $\phi_{p,g} \circ T \circ \phi_{p,g}^{-1}$, initially defined on $L^p(g\mu)$, extends to a bounded operator on $L^2(g\mu)$. The next statement is an analog of that result for $C(K)$-representations.

PROPOSITION 5.1. *Let $1 \leq p < \infty$ and let $u: C(K) \to B(L^p(\mu))$ be a bounded homomorphism. Then there exists a density $g: \Omega \to (0, \infty)$ and a bounded homomorphism $w: C(K) \to B(L^2(g\mu))$ such that*

$$\phi_{p,g} \circ u(f) \circ \phi_{p,g}^{-1} = w(f) \quad \textit{for } f \in C(K),$$

where equality holds on $L^2(g\mu) \cap L^p(g\mu)$.

PROOF. Since $X = L^p(\mu)$ has property (α), the mapping u is matricially R-bounded by Proposition 4.4. According to the above discussion, this means that u is ℓ^2-cb in the sense of [32, Definition 2]. The result therefore follows from [32, Theorems 3.4 and 3.6]. □

We will now focus on Schauder bases on separable L^p-spaces. We refer to [27, Ch. 1] for general information on this topic. We simply recall that a sequence $(e_k)_{k\geq 1}$ in a Banach space X is a basis if, for every $x \in X$, there exists a unique scalar sequence $(a_k)_{k\geq 1}$ such that $\sum_k a_k e_k$ converges to x. A basis $(e_k)_{k\geq 1}$ is said to be unconditional if this convergence is unconditional for all $x \in X$. We record the following standard characterization.

LEMMA 5.2. *A sequence $(e_k)_{k\geq 1} \subset X$ of nonzero vectors is an unconditional basis of X if and only if $X = \overline{\mathrm{Span}}\{e_k : k \geq 1\}$ and there exists a constant $C \geq 1$ such that, for all bounded scalar sequences $(\lambda_k)_{k\geq 1}$ and for all finite scalar sequences $(a_k)_{k\geq 1}$,*

$$\Big\| \sum_k \lambda_k a_k e_k \Big\| \leq C \sup_k |\lambda_k| \Big\| \sum_k a_k e_k \Big\|. \tag{5.1}$$

We will need the following elementary lemma.

LEMMA 5.3. *Let (Ω, ν) be a σ-finite measure space, let $1 \leq p < \infty$ and let $Q: L^p(\nu) \to L^p(\nu)$ be a finite rank bounded operator such that $Q_{|L^2(\nu)\cap L^p(\nu)}$ extends to a bounded operator $L^2(\nu) \to L^2(\nu)$. Then $Q(L^p(\nu)) \subset L^2(\nu)$.*

PROOF. Let $E = Q(L^p(\nu) \cap L^2(\nu))$. By assumption, E is a finite-dimensional subspace of $L^p(\nu) \cap L^2(\nu)$. Since E is automatically closed under the L^p-norm and Q is continuous, we find that $Q(L^p(\nu)) = E$. □

THEOREM 5.4. *Let $1 \leq p < \infty$ and assume that $(e_k)_{k\geq 1}$ is an unconditional basis of $L^p(\Omega, \mu)$. Then there exists a density g on Ω such that $\phi_{p,g}(e_k) \in L^2(g\mu)$ for all $k \geq 1$, and the sequence $(\phi_{p,g}(e_k))_{k\geq 1}$ is an unconditional basis of $L^2(g\mu)$.*

PROOF. Property (5.1) implies that, for all $\lambda = (\lambda_k)_{k\geq 1} \in \ell^\infty$, there exists a (necessarily unique) bounded operator $T_\lambda : L^p(\mu) \to L^p(\mu)$ such that $T_\lambda(e_k) = \lambda_k e_k$ for all $k \geq 1$. Moreover, $\|T_\lambda\| \leq C\|\lambda\|_\infty$. We can therefore consider the mapping

$$u : \ell^\infty \longrightarrow B(L^p(\mu)), \quad u(\lambda) = T_\lambda,$$

and u is a bounded homomorphism. By Proposition 5.1, there is a constant $C_1 > 0$ and a density g on Ω such that the mapping

$$\phi T_\lambda \phi^{-1} : L^p(g\mu) \longrightarrow L^p(g\mu)$$

(where $\phi = \phi_{p,g}$) extends to a bounded operator

$$S_\lambda : L^2(g\mu) \longrightarrow L^2(g\mu)$$

for all $\lambda \in \ell^\infty$, where $\|S_\lambda\| \leq C_1\|\lambda\|_\infty$.

Assume first that $p \geq 2$, so that $L^p(g\mu) \subset L^2(g\mu)$. Let $\lambda = (\lambda_k)_{k\geq 1}$ in ℓ^∞ and let $(a_k)_{k\geq 1}$ be a finite scalar sequence. Then $S_\lambda(\phi(e_k)) = \phi T_\lambda \phi^{-1}(\phi(e_k)) = \lambda_k \phi(e_k)$ for all $k \geq 1$, and hence

$$\begin{aligned} \Big\| \sum_k \lambda_k a_k \phi(e_k) \Big\|_{L^2(g\mu)} &= \Big\| S_\lambda\Big(\sum_k a_k \phi(e_k)\Big) \Big\|_{L^2(g\mu)} \\ &\leq C_1 \|\lambda\|_\infty \Big\| \sum_k a_k \phi(e_k) \Big\|_{L^2(g\mu)}. \end{aligned}$$

Moreover, the linear span of the $\phi(e_k)$ is dense in $L^p(g\mu)$, and hence in $L^2(g\mu)$. By Lemma 5.2, this shows that $(\phi(e_k))_{k\geq 1}$ is an unconditional basis of $L^2(g\mu)$.

Assume now that $1 \leq p < 2$. For all $n \geq 1$, let $f_n \in \ell^\infty$ be defined by $(f_n)_k = \delta_{n,k}$ for all $k \geq 1$, and let $Q_n : L^p(g\mu) \to L^p(g\mu)$ be the projection defined by

$$Q_n\Big(\sum_k a_k \phi(e_k)\Big) = a_n \phi(e_n).$$

Then $Q_n = \phi T_{f_n} \phi^{-1}$ and hence Q_n extends to an L^2 operator. Therefore, $\phi(e_n)$ belongs to $L^2(g\mu)$ by Lemma 5.3.

Let $p' = p/(p-1)$ be the conjugate number of p, let $(e_k^*)_{k\geq 1}$ be the bi-orthogonal system of $(e_k)_{k\geq 1}$, and let $\phi' = \phi^{*-1}$. (It is easy to check that $\phi' = \phi_{p',g}$, but we will not use this point.) The linear span of the e_k^* is w^*-dense in $L^{p'}(\mu)$. Equivalently, the linear span of the $\phi'(e_k^*)$ is w^*-dense in $L^{p'}(g\mu)$, and hence it is dense in $L^2(g\mu)$. Moreover, for all $\lambda \in \ell^\infty$ and for all $k \geq 1$, we have $T_\lambda^*(e_k^*) = \lambda_k e_k^*$. Thus, for all finite scalar sequences $(a_k)_{k\geq 1}$,

$$\sum_k \lambda_k a_k \phi'(e_k^*) = (\phi T_\lambda \phi^{-1})^* \Big(\sum_k a_k \phi'(e_k^*)\Big) = S_\lambda^* \Big(\sum_k a_k \phi'(e_k^*)\Big).$$

Hence

$$\Big\|\sum_k \lambda_k a_k \phi'(e_k^*)\Big\|_{L^2(g\mu)} \leq C_1 \Big\|\sum_k a_k \phi'(e_k^*)\Big\|_{L^2(g\mu)}.$$

According to Lemma 5.2, this shows that $(\phi'(e_k^*))_{k\geq 1}$ is an unconditional basis of $L^2(g\mu)$. It is plain that $(\phi(e_k))_{k\geq 1} \subset L^2(g\mu)$ is the bi-orthogonal system of $(\phi'(e_k^*))_{k\geq 1} \subset L^2(g\mu)$. This shows that, in turn, $(\phi(e_k))_{k\geq 1}$ is an unconditional basis of $L^2(g\mu)$. □

We will now establish a variant of Theorem 5.4 for conditional bases. Recall that if $(e_k)_{n\geq 1}$ is a basis on some Banach space X, then the projections $P_N : X \to X$ defined by

$$P_N\Big(\sum_k a_k e_k\Big) = \sum_{k=1}^{N} a_k e_k$$

are uniformly bounded. We will say that $(e_k)_{k\geq 1}$ is an R-basis if the set $\{P_N : N \geq 1\}$ is actually R-bounded. It follows from [4, Corollary 3.15] that any unconditional basis on L^p is an R-basis. See Remark 5.6(2) for more details on this.

PROPOSITION 5.5. *Let $1 \leq p < \infty$ and let $(e_k)_{k\geq 1}$ be an R-basis of $L^p(\Omega, \mu)$. Then there exists a density g on Ω such that $\phi_{p,g}(e_k) \in L^2(g\mu)$ for all $k \geq 1$, and the sequence $(\phi_{p,g}(e_k))_{k\geq 1}$ is a basis of $L^2(g\mu)$.*

PROOF. According to [26, Theorem 2.1], there exists a constant $C \geq 1$ and a density g on Ω such that, taking $\phi = \phi_{p,g}$,

$$\|\phi P_N \phi^{-1} h\|_2 \leq C\|h\|_2 \quad \forall N \geq 1,\ h \in L^2(g\mu) \cap L^p(g\mu).$$

Then the proof is similar to that of Theorem 5.4, using [27, Proposition 1.a.3] instead of Lemma 5.2. We skip the details. □

REMARK 5.6. (1) Theorem 5.4 and Proposition 5.5 can be easily extended to finite-dimensional Schauder decompositions. We refer to [27, Section 1.g] for general information on this notion. Given a Schauder decomposition $(X_k)_{k\geq 1}$ of a Banach space X, let P_N be the associated projections; namely, for all $N \geq 1$, $P_N : X \to X$ is the bounded projection onto $X_1 \oplus \cdots \oplus X_N$ vanishing on X_k for all $k \geq N + 1$. We say that $(X_k)_{k\geq 1}$ is an R-Schauder decomposition if the set $\{P_N : N \geq 1\}$ is R-bounded. Then we find that, for all $1 < p < \infty$ and for all finite-dimensional R-Schauder (respectively unconditional) decompositions $(X_k)_{k\geq 1}$ of $L^p(\mu)$, there exists a density g on Ω such that $\phi_{p,g}(X_k) \subset L^2(g\mu)$ for all $k \geq 1$, and $(\phi_{p,g}(X_k))_{k\geq 1}$ is a Schauder (respectively unconditional) decomposition of $L^2(g\mu)$.

(2) The concept of R-Schauder decompositions can be tracked down to [2], and it played a key role in [4] and in various works on L^p-maximal regularity and H^∞-calculus; see, in particular, [20, 21]. Let C_p denote the Schatten spaces. For $1 < p \neq 2 < \infty$, an explicit example of a Schauder decomposition on $L^2([0, 1]; C_p)$

which is not R-Schauder is given in [4, Section 5]. More generally, it follows from [20] that whenever a reflexive Banach space X has an unconditional basis and is not isomorphic to ℓ^2, then X admits a finite-dimensional Schauder decomposition which is not R-Schauder. This applies, in particular, to $X = L^p([0, 1])$, for all $1 < p \neq 2 < \infty$. However, whether $L^p([0, 1])$ admits a Schauder basis that is not R-Schauder is apparently an open question.

We finally mention that, according to [21, Theorem 3.3], any unconditional decomposition on a Banach space X with property (Δ) is an R-Schauder decomposition.

Acknowledgement

We are grateful to the referee for various remarks and references which improved the presentation of this paper.

References

[1] W. Arendt and S. Bu, 'The operator-valued Marcinkiewicz multiplier theorem and maximal regularity', *Math. Z.* **240** (2002), 311–343.

[2] E. Berkson and T. A. Gillespie, 'Spectral decompositions and harmonic analysis on UMD Banach spaces', *Studia Math.* **112** (1994), 13–49.

[3] J. W. Bunce, 'Representations of strongly amenable C^*-algebras', *Proc. Amer. Math. Soc.* **32** (1972), 241–246.

[4] P. Clément, B. de Pagter, F. A. Sukochev and H. Witvliet, 'Schauder decompositions and multiplier theorems', *Studia Math.* **138** (2000), 135–163.

[5] J. B. Conway, *A Course in Operator Theory*, Graduate Studies in Mathematics, 21 (American Mathematical Society, Providence, RI, 2000).

[6] M. Cowling, I. Doust, A. McIntosh and A. Yagi, 'Banach space operators with a bounded H^∞ functional calculus', *J. Aust. Math. Soc., Ser. A* **60** (1996), 51–89.

[7] G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs, New Series, 24 (Oxford University Press, Oxford, 2000).

[8] B. de Pagter and W. J. Ricker, '$C(K)$-representations and R-boundedness', *J. London Math. Soc.* **76** (2007), 498–512.

[9] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics, 43 (Cambridge University Press, Cambridge, 1995).

[10] J. Diestel and J. J. Uhl, *Vector Measures*, Mathematical Surveys and Monographs, 15 (American Mathematical Society, Providence, RI, 1977).

[11] I. Doust and R. deLaubenfels, 'Functional calculus, integral representations, and Banach space geometry', *Quaestiones Math.* **17** (1994), 161–171.

[12] I. Doust and T. A. Gillespie, 'Well-boundedness of sums and products of operators', *J. London Math. Soc.* **68** (2003), 183–192.

[13] H. R. Dowson, *Spectral Theory of Linear Operators*, London Mathematical Society Monographs, 12 (Academic Press, London, 1978).

[14] E. Effros and Z.-J. Ruan, 'On matricially normed spaces', *Pacific J. Math.* **132** (1988), 243–264.

[15] E. Effros and Z.-J. Ruan, *Operator Spaces*, London Mathematical Society Monographs, New Series, 23 (Oxford University Press, Oxford, 2000).

[16] A. M. Fröhlich and L. Weis, 'H^∞ calculus and dilations', *Bull. Soc. Math. France* **134** (2006), 487–508.

[17] M. Hoffmann, N. J. Kalton and T. Kucherenko, 'R-bounded approximating sequences and applications to semigroups', *J. Math. Anal. Appl.* **294** (2004), 373–386.

[18] W. B. Johnson and L. Jones, 'Every L^p operator is an L^2 operator', *Proc. Amer. Math. Soc.* **72** (1978), 309–312.

[19] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras, I and II*, Graduate Studies in Mathematics, 16 (American Mathematical Society, Providence, RI, 1997).
[20] N. J. Kalton and G. Lancien, 'A solution to the problem of L^p-maximal regularity', *Math. Z.* **235** (2000), 559–568.
[21] N. J. Kalton and L. Weis, 'The H^∞-calculus and sums of closed operators', *Math. Ann.* **321** (2001), 319–345.
[22] N. J. Kalton and L. Weis, 'The H^∞ functional calculus and square function estimates', Preprint.
[23] P. C. Kunstmann and L. Weis, 'Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus', in: *Functional Analytic Methods for Evolution Equations*, Lecture Notes in Mathematics, 1855 (Springer, New York, 2004), pp. 65–311.
[24] C. Le Merdy, 'H^∞-functional calculus and applications to maximal regularity', *Publ. Math. Besançon* **16** (1998), 41–77.
[25] C. Le Merdy, 'A strong similarity property of nuclear C^*-algebras', *Rocky Mountain J. Math.* **30** (2000), 279–292.
[26] C. Le Merdy and A. Simard, 'A factorization property of R-bounded sets of operators on L^p-spaces', *Math. Nachr.* **243** (2002), 146–155.
[27] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I* (Springer, Berlin, 1977).
[28] V. I. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Studies in Advanced Mathematics, 78 (Cambridge University Press, Cambridge, 2002).
[29] G. Pisier, 'Some results on Banach spaces without local unconditional structure', *Compositio Math.* **37** (1978), 3–19.
[30] G. Pisier, *Similarity Problems and Completely Bounded Maps (Second, expanded version)*, Lecture Notes in Mathematics, 1618 (Springer, New York, 2001).
[31] W. J. Ricker, *Operator Algebras Generated by Commuting Projections: A Vector Measure Approach*, Lecture Notes in Mathematics, 1711 (Springer, New York, 1999).
[32] A. Simard, 'Factorization of sectorial operators with bounded H^∞-functional calculus', *Houston J. Math.* **25** (1999), 351–370.

CHRISTOPH KRIEGLER, Institut für Analysis, Karlsruhe Institute of Technology,
Kaiserstrasse 89, 76133 Karlsruhe, Germany
and
Laboratoire de Mathématiques, Université de Franche-Comté,
25030 Besançon Cedex, France
e-mail: christoph.kriegler@univ-fcomte.fr

CHRISTIAN LE MERDY, Laboratoire de Mathématiques,
Université de Franche-Comté, 25030 Besançon Cedex, France
e-mail: clemerdy@univ-fcomte.fr