

## ON VON NEUMANN–JORDAN CONSTANTS

KAZUO HASHIMOTO<sup>✉</sup> and GEN NAKAMURA

(Received 7 June 2008; accepted 11 July 2009)

Communicated by G. A. Willis

### Abstract

In this note, we provide an example of a Banach space  $X$  for which  $\tilde{C}_{NJ}(X) = 1$  that is not isomorphic to any Hilbert space, where  $\tilde{C}_{NJ}(X)$  denotes the infimum of all von Neumann–Jordan constants for equivalent norms of  $X$ .

2000 *Mathematics subject classification*: primary 46B03; secondary 46B20.

*Keywords and phrases*: von Neumann–Jordan constant,  $n$ th von Neumann–Jordan constant, Hilbert space, isomorphism.

### 1. Introduction

Let  $(X, \|\cdot\|)$  be a real Banach space. The von Neumann–Jordan constant of  $X$ , denoted by  $C_{NJ}(X)$ , is the smallest constant  $C$  for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

for all  $x, y \in X$  such that  $\|x\|^2 + \|y\|^2 \neq 0$ . Classical results state that:

- (i)  $1 \leq C_{NJ}(X) \leq 2$  for any Banach space  $X$ , and  $X$  is a Hilbert space if and only if  $C_{NJ}(X) = 1$  (Jordan and von Neumann [2]);
- (ii)  $C_{NJ}(L_p) = 2^{2/t-1}$ , where  $t = \min\{p, p'\}$  and  $1/p + 1/p' = 1$  (see [1]).

The constant  $\tilde{C}_{NJ}(X)$  is defined by

$$\tilde{C}_{NJ}(X) = \inf\{C_{NJ}(X, |\cdot|) : |\cdot| \text{ is a norm equivalent to } \|\cdot\|\}.$$

Let  $Y$  be a subspace of  $X$ . It is easily checked that  $\tilde{C}_{NJ}(Y) \leq \tilde{C}_{NJ}(X)$ .

Many results on the constants  $C_{NJ}(X)$  and  $\tilde{C}_{NJ}(X)$  for various  $X$  have been proved by Kato *et al.* [3–8]. In particular, Kato and Takahashi [5] showed that  $\tilde{C}_{NJ}(X) < 2$

if and only if  $X$  is superreflexive. Moreover, in [8], they gave the following stronger result:  $C_{NJ}(X) < 2$  if and only if  $X$  is uniformly nonsquare.

We are concerned with the question whether a Banach space  $X$  with  $\tilde{C}_{NJ}(X) = 1$  is necessarily isomorphic to a Hilbert space. In this note, we provide a negative answer to this question, by giving an example of a Banach space  $X$  for which  $\tilde{C}_{NJ}(X) = 1$  that is not isomorphic to any Hilbert space.

We denote  $\ell_2$ -direct sums using the  $\oplus$  symbol: we write, for example, both  $\bigoplus_{n=1}^{\infty} X_n$  and  $X \oplus Y$ .

## 2. Main results

**DEFINITION 2.1** [7]. The  $n$ th von Neumann–Jordan constant, where  $n \geq 1$ , is defined by

$$C_{NJ}^{(n)}(X) := \sup \left\{ \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^2 / \left( 2^n \sum_{j=1}^n \|x_j\|^2 \right) : x_j \in X, \sum_{j=1}^n \|x_j\|^2 \neq 0 \right\}.$$

It is evident that  $C_{NJ}^{(2)}(X) = C_{NJ}(X)$ .

**THEOREM 2.2.** Let  $\{X_n\}_{n=1}^{\infty}$  be a sequence of Banach spaces satisfying the following conditions:

- (i) the dimension of each  $X_n$  is finite;
- (ii)  $\sup_{m,n} C_{NJ}^{(m)}(X_n) = \infty$ ;
- (iii)  $\lim_{n \rightarrow \infty} C_{NJ}(X_n) = 1$ .

Let  $X$  be the  $\ell_2$ -direct sum  $\bigoplus_{n=1}^{\infty} X_n$ . Then  $\tilde{C}_{NJ}(X) = 1$  and  $X$  is not isomorphic to any Hilbert space. In particular,  $\tilde{C}_{NJ}(X) < C_{NJ}(X)$ .

**EXAMPLE.** The following example satisfies conditions (i), (ii) and (iii) above. Suppose that  $1 \leq p < 2$ , and  $e_i$  are the unit coordinate vectors in  $\ell_p^n$ , where  $1 \leq i \leq n$  and  $n \in \mathbb{N}$ . Then

$$\frac{\sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j e_j \right\|^2}{2^n \sum_{j=1}^n \|e_j\|^2} = \frac{\sum_{\theta_j = \pm 1} n^{2/p}}{2^n n} = \frac{2^n n^{2/p}}{n 2^n} = n^{2/p-1}.$$

Hence,  $C_{NJ}^{(n)}(\ell_p^n) \geq n^{2/p-1}$ . When  $1 \leq p < 2$ , we have  $\lim_{n \rightarrow \infty} C_{NJ}^{(n)}(\ell_p^n) = \infty$ , and so we can take a sequence  $\{a_n\} \subseteq \mathbb{N}$  satisfying  $C_{NJ}^{(a_n)}(\ell_{2-1/n}^{a_n}) > n$ . We put  $X_n = \ell_{2-1/n}^{a_n}$ , then (i) and (ii) hold. As mentioned in the introduction, (iii) holds since  $C_{NJ}(X_n) = 2^{1/(2n-1)}$ .

**LEMMA 2.3.** If  $X$  is isomorphic to a Hilbert space, then

$$\sup_n C_{NJ}^{(n)}(X) < +\infty.$$

**PROOF.** We assume that the Banach space  $(X, \|\cdot\|)$  is isomorphic to a Hilbert space  $(X, |\cdot|)$ . Then there exists  $M \geq 1$  such that

$$\frac{1}{M}\|x\| \leq |x| \leq M\|x\| \quad \forall x \in X. \quad (2.1)$$

For all  $x_1, x_2, \dots, x_n \in X$  such that  $\sum_{j=1}^n \|x_j\|^2 \neq 0$ ,

$$\sum_{\theta_j=\pm 1} \left| \sum_{j=1}^n \theta_j x_j \right|^2 = 2^n \sum_{j=1}^n |x_j|^2,$$

by the parallelogram law in Hilbert space. Using this equality and inequality (2.1) above,

$$\sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^2 \leq M^2 \sum_{\theta_j=\pm 1} \left| \sum_{j=1}^n \theta_j x_j \right|^2 = M^2 2^n \sum_{j=1}^n |x_j|^2 \leq M^4 2^n \sum_{j=1}^n \|x_j\|^2.$$

Hence,

$$\sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^2 \bigg/ \left( 2^n \sum_{j=1}^n \|x_j\|^2 \right) \leq M^4,$$

and we conclude that  $C_{NJ}^{(n)}(X) \leq M^4$ . □

**LEMMA 2.4.** *Let  $\{X_n\}$  be a sequence of Banach spaces; then*

$$C_{NJ} \left( \bigoplus_{n=1}^{\infty} X_n \right) = \sup \{ C_{NJ}(X_n) \mid n \in \mathbb{N} \}.$$

**PROOF.** We first show that

$$C_{NJ} \left( \bigoplus_{n=1}^{\infty} X_n \right) \leq \sup \{ C_{NJ}(X_n) \mid n \in \mathbb{N} \}. \quad (2.2)$$

To prove this, it is sufficient to show that when  $C > 0$ ,

$$\sup \{ C_{NJ}(X_n) \mid n \in \mathbb{N} \} \leq C \implies C_{NJ} \left( \bigoplus_{n=1}^{\infty} X_n \right) \leq C.$$

Moreover, it suffices to show that this assertion holds for the case of two terms:

$$\max \{ C_{NJ}(X_1), C_{NJ}(X_2) \} \leq C \implies C_{NJ}(X_1 \oplus X_2) \leq C.$$

For all  $x, y \in X_1 \oplus X_2$ , we write  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$ , and then

$$\begin{aligned} \|x + y\|^2 + \|x - y\|^2 &= \|x_1 + y_1\|^2 + \|x_1 - y_1\|^2 + \|x_2 + y_2\|^2 + \|x_2 - y_2\|^2 \\ &\leq 2C(\|x_1\|^2 + \|y_1\|^2) + 2C(\|x_2\|^2 + \|y_2\|^2) \\ &= 2C(\|x_1\|^2 + \|x_2\|^2) + 2C(\|y_1\|^2 + \|y_2\|^2) \\ &= 2C(\|x\|^2 + \|y\|^2), \end{aligned}$$

and hence

$$\frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C.$$

In the same way,

$$\frac{1}{C} \leq \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)}.$$

Thus,  $C_{NJ}(X_1 \oplus X_2) \leq C$  and hence (2.2) holds.

The other inequality is obvious as  $\bigoplus_{k=1}^{\infty} X_k \supseteq X_n$  for all  $n \in \mathbb{N}$ . □

**COROLLARY 2.5.** *For any two Banach spaces  $X$  and  $Y$ ,*

$$\tilde{C}_{NJ}(X \oplus Y) = \max\{\tilde{C}_{NJ}(X), \tilde{C}_{NJ}(Y)\}.$$

**PROOF.** We first show that

$$\tilde{C}_{NJ}(X \oplus Y) \leq \max\{\tilde{C}_{NJ}(X), \tilde{C}_{NJ}(Y)\}. \quad (2.3)$$

By the definition of  $\tilde{C}_{NJ}$ , for any  $\varepsilon > 0$ , there exist Banach spaces  $X'$  and  $Y'$ , isomorphic to  $X$  and  $Y$ , such that

$$C_{NJ}(X') \leq \tilde{C}_{NJ}(X) + \varepsilon \quad \text{and} \quad C_{NJ}(Y') \leq \tilde{C}_{NJ}(Y) + \varepsilon.$$

Using Lemma 2.4,

$$\begin{aligned} \max\{\tilde{C}_{NJ}(X), \tilde{C}_{NJ}(Y)\} + \varepsilon &\geq \max\{C_{NJ}(X'), C_{NJ}(Y')\} \\ &= C_{NJ}(X' \oplus Y') \\ &\geq \tilde{C}_{NJ}(X \oplus Y). \end{aligned}$$

As  $\varepsilon > 0$  is arbitrary, (2.3) holds.

As mentioned in the introduction, the opposite inequality to (2.3) can easily be derived from the inclusion of both  $X$  and  $Y$  in  $X \oplus Y$ . □

**PROOF OF THEOREM 2.2.** By Corollary 2.5, for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} \tilde{C}_{NJ}(X) &= \tilde{C}_{NJ}\left(\bigoplus_{k=1}^n X_k \oplus \bigoplus_{k=n+1}^{\infty} X_k\right) \\ &= \max\left\{\tilde{C}_{NJ}\left(\bigoplus_{k=1}^n X_k\right), \tilde{C}_{NJ}\left(\bigoplus_{k=n+1}^{\infty} X_k\right)\right\} \\ &\leq \max\left\{\tilde{C}_{NJ}\left(\bigoplus_{k=1}^n X_k\right), C_{NJ}\left(\bigoplus_{k=n+1}^{\infty} X_k\right)\right\}. \end{aligned}$$

Since  $\bigoplus_{k=1}^n X_k$  is finite-dimensional, it is isomorphic to a Hilbert space and thus  $\tilde{C}_{NJ}(\bigoplus_{k=1}^n X_k) = 1$ . Further, by Lemma 2.4 and condition (iii),

$$\lim_{n \rightarrow \infty} C_{NJ} \left( \bigoplus_{k=n+1}^{\infty} X_k \right) = 1.$$

Hence  $\tilde{C}_{NJ}(X) \leq 1$  and so  $\tilde{C}_{NJ}(X) = 1$ .

On the other hand,  $X_n \subseteq X$  for each  $n \in \mathbb{N}$  and condition (ii) holds, so

$$\sup\{C_{NJ}^{(m)}(X) \mid m \in \mathbb{N}\} \geq \sup\{C_{NJ}^{(m)}(X_n) \mid m, n \in \mathbb{N}\} = \infty.$$

Thus from Lemma 2.3,  $X$  is not isomorphic to any Hilbert space.  $\square$

## References

- [1] J. A. Clarkson, ‘The von Neumann–Jordan constant for the Lebesgue space’, *Ann. of Math.* (2) **38** (1937), 114–115.
- [2] P. Jordan and J. von Neumann, ‘On inner products in linear metric spaces’, *Ann. of Math.* (2) **36** (1935), 719–723.
- [3] M. Kato, L. Maligranda and Y. Takahashi, ‘On James and Jordan–von Neumann constants and normal structure coefficient of Banach spaces’, *Studia Math.* **144**(2) (2001), 275–295.
- [4] M. Kato and Y. Takahashi, ‘Uniform convexity, uniform non-squareness and von Neumann–Jordan constant for Banach spaces’, *RIMS Kokyuroku* **939** (1996), 87–96.
- [5] M. Kato and Y. Takahashi, ‘On the von Neumann–Jordan constant for Banach spaces’, *Proc. Amer. Math. Soc.* **125** (1997), 1055–1062.
- [6] M. Kato and Y. Takahashi, ‘Von Neumann–Jordan constant for Lebesgue–Bochner spaces’, *J. Inequal. Appl.* **2** (1998), 89–97.
- [7] M. Kato, Y. Takahashi and K. Hashimoto, ‘On  $n$ -th von Neumann–Jordan constants for Banach spaces’, *Bull. Kyushu Inst. Technol. Pure Appl. Math.* **45** (1998), 25–33.
- [8] Y. Takahashi and M. Kato, ‘Von Neumann–Jordan constant and uniformly non-square Banach spaces’, *Nihonkai Math. J.* **9** (1998), 155–169.

KAZUO HASHIMOTO, Hiroshima Jogakuin University, 4-13-1 Ushita Higashi  
Higashi-ku, Hiroshima 732-0063, Japan  
e-mail: [hasimoto@gaines.hju.ac.jp](mailto:hasimoto@gaines.hju.ac.jp)

GEN NAKAMURA, Matsue College of Technology, 14-4 Nishi-ikuma, Matsue,  
Shimane 690-8518, Japan  
e-mail: [nakamura@matsue-ct.ac.jp](mailto:nakamura@matsue-ct.ac.jp)