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THE WEIGHTED g-DRAZIN INVERSE FOR OPERATORS

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Abstract

The paper introduces and studies the weighted g-Drazin inverse for bounded linear operators between Banach spaces, extending the concept of the weighted Drazin inverse of Rakočević and Wei (*Linear Algebra Appl.* **350** (2002), 25–39) and of Cline and Greville (*Linear Algebra Appl.* **29** (1980), 53–62). We use the Mbekhta decomposition to study the structure of an operator possessing the weighted g-Drazin inverse, give an operator matrix representation for the inverse, and study its continuity. An open problem of Rakočević and Wei is solved.

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1. Introduction

In recent papers [13, 14], Rakočević and Wei defined and investigated the weighted Drazin inverse for bounded linear operators between Banach and Hilbert spaces, extending the concept of a weighted Drazin inverse for rectangular matrices introduced by Cline and Greville [5]. The weighted Drazin inverse for operators was previously introduced and studied by Qiao in [12], and further investigated by Wang in [16, 17]. The main purpose of this paper is to introduce and study the weighted g-Drazin inverse for bounded linear operators between Banach spaces X and Y, thus further extending the above mentioned works.

Let $\mathcal{B}(X, Y)$ denote the set of all bounded linear operators between X and Y, and let W be a nonzero operator in $\mathcal{B}(Y, X)$. The W-weighted g-Drazin inverse (the Wg-Drazin inverse for short) can be studied in the framework of Banach algebras when we introduce on the space $\mathcal{B}(X, Y)$ the W-product $A \star B = AWB$, and the W-norm $\|A\|_W = \|A\|\|W\|$. This elegant approach which turns $\mathcal{B}(X, Y)$ into a Banach algebra was suggested to the authors of [13, 14] by an anonymous referee. Unless W is invertible (and this would require the spaces X and Y to be isomorphic and homeomorphic), the resulting algebra is without unit.

In our work we remove the restriction of finite polarity of the operator WA (and AW) adopted by Rakočević and Wei [13]. In addition, we solve an open problem posed in [13], and complete and extend the results of Buoni and Faires [3] on the ascent and descent of AB and BA.

In Section 2 we gather relevant results on the g-Drazin inverse in Banach algebras without unit in order to study the Wg-Drazin inverse within the space $\mathcal{B}(X, Y)$, without having to adjoin a unit. Section 3 introduces and studies the weighted g-Drazin inverse between two different Banach spaces. In Section 4 we explore some properties of the weighted g-Drazin inverse, including the core decomposition and an integral representation for the weighted inverse. The ascent and descent for WA and AW is studied in Section 5, and a solution to an open problem posed by Rakočević and Wei in [13] is given there. In Section 6 we compare the Mbekhta decomposition for the operators WA and AW and recover and sharpen a result of Yukhno [19] on rectangular matrices. In the remaining sections we give an operator matrix representation for the Wg-Drazin inverse, compare it with the Moore–Penrose inverse in Hilbert spaces, and give necessary and sufficient conditions for its continuity.

2. The g-Drazin inverse in Banach algebras without unit

Let $\mathcal{A}$ be a Banach algebra. We write $\mathcal{A}^{\mathrm{qnil}}$ for the set of all quasinilpotent elements in $\mathcal{A}$, that is, elements a satisfying $\lim_{n\to\infty} \|a^n\|^{1/n} = 0$; the set of all nilpotent elements is denoted by $\mathcal{A}^{\mathrm{nil}}$. If $\mathcal{A}$ is unital, we denote by $\mathcal{A}^{\mathrm{inv}}$ the group of all invertible elements in $\mathcal{A}$. An element $a \in \mathcal{A}$ is *quasipolar* if 0 is not an accumulation point of the spectrum of a. In an algebra without unit, this is equivalent to 0 being an isolated spectral point of a. The set of all quasipolar elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{\mathrm{qpol}}$. An element $a \in \mathcal{A}$ is *polar* if it is quasipolar and 0 is at most a pole of the resolvent of a. The set of all polar elements is denoted by $\mathcal{A}^{\mathrm{pol}}$.

The following holds [7, Theorems 4.2 and 5.1]:

LEMMA 2.1. *Let $\mathcal{A}$ be a unital Banach algebra. Then $a \in \mathcal{A}$ is quasipolar (polar) in $\mathcal{A}$ if and only there exists $p \in \mathcal{A}$ such that*

$$(2.1) \qquad p^2 = p, \quad ap = pa \in \mathcal{A}^{\mathrm{qnil}} \ (ap = pa \in \mathcal{A}^{\mathrm{nil}}), \quad a + p \in \mathcal{A}^{\mathrm{inv}}.$$

The resolvent $R(\lambda; a) = (\lambda 1 - a)^{-1}$ has a Laurent expansion in some punctured *neighbourhood* $0 < |\lambda| < r$ *of* 0 *given by*

$$R(\lambda; a) = \sum_{n=0}^{\infty} \lambda^{-n-1} a^n p - \sum_{n=1}^{\infty} \lambda^{n-1} b^n, \tag{2.2}$$

where $b = (a+p)^{-1}(1-p)$.

The element p is uniquely determined by the conditions of the theorem; it is called the *spectral idempotent* of a, and it double commutes with a. The element $q = 1 - p$ is the *support idempotent* of a. The support idempotent of a quasipolar element exists in an algebra without a unit, but not the spectral idempotent. The element $b = (a+p)^{-1}(1-p)$ defines the *g-Drazin inverse* of a in the case of a unital algebra; b also double commutes with a. We write a^π and a^σ for the spectral idempotent and the support idempotent of a quasipolar element a, respectively.

From now on we assume that $\mathcal{A}$ is a complex Banach algebra without unit.

The *unitisation* of $\mathcal{A}$ is the unital Banach algebra $\mathcal{A}_1 = \mathcal{A} \oplus \mathbb{C}$ containing $\mathcal{A}$ as a two sided ideal of codimension 1 [2, page 15]. Given $a \in \mathcal{A}$, we define the *spectrum* $\mathsf{Sp}(a)$ of a in $\mathcal{A}$ as the spectrum of a considered as an element of the unital Banach algebra $\mathcal{A}_1$, that is, the set of all $\lambda \in \mathbb{C}$ such that $\lambda 1 - a \notin \mathcal{A}_1^{\mathsf{inv}}$. Observe that 0 is always in the spectrum of any element of a Banach algebra without unit.

PROPOSITION 2.2. *Let* $\mathcal{A}$ *be a Banach algebra without unit. Then* $a \in \mathcal{A}^{\mathsf{qpol}}$ ($a \in \mathcal{A}^{\mathsf{pol}}$) *if and only if there exists* $b \in \mathcal{A}$ *such that*

$$ab = ba, \quad bab = b, \quad a - aba \in \mathcal{A}^{\mathsf{qnil}} \ (a - aba \in \mathcal{A}^{\mathsf{nil}}). \tag{2.3}$$

The element b*, if it exists, is unique.*

PROOF. We embed $\mathcal{A}$ into its unitisation $\mathcal{A}_1$.

If a is quasipolar in $\mathcal{A}$, then it is also quasipolar in $\mathcal{A}_1$. Let p be the spectral idempotent of a in $\mathcal{A}_1$, and $b = (a+p)^{-1}(1-p)$ the Drazin inverse of a in $\mathcal{A}_1$. Since $1-p$ is in $\mathcal{A}$, so is b ($\mathcal{A}$ is an ideal). The equations (2.3) are then easily verified.

Conversely, let equations (2.3) hold. Then $p = 1 - ab$ is the spectral idempotent of a in $\mathcal{A}_1$ [7, Theorem 4.2], and a is quasipolar, both in $\mathcal{A}_1$ and $\mathcal{A}$. From

$$(a+p)b = (a + 1 - ab)b = ab + b - bab = ab = 1 - p$$

and the invertibility of $a + p$ in $\mathcal{A}_1$, we get $b = (a+p)^{-1}(1-p)$ in $\mathcal{A}_1$ (and in $\mathcal{A}$). This proves the uniqueness of b satisfying (2.3). □

DEFINITION 2.3. Let $\mathcal{A}$ be a Banach algebra without unit and let $a \in \mathcal{A}^{\mathsf{qpol}}$. We define the *g-Drazin inverse* a^{D} of a to be the unique element b satisfying (2.3). The *Drazin index* of a quasipolar element a is defined by

$$i(a) = \inf \{k \in \mathbb{N} : (a - a^2 a^{\mathsf{D}})^k = 0\}$$

($\inf \emptyset = \infty$). The g-Drazin inverse of a polar element is called the *Drazin inverse*.

We observe that $a \in \mathcal{A}$ is polar if and only if it is quasipolar and has a finite Drazin index.

As in the unital case, any g-Drazin invertible element a of $\mathcal{A}$ has the 'core' decomposition.

PROPOSITION 2.4. *Let $\mathcal{A}$ be a Banach algebra without unit. Then $a \in \mathcal{A}^{\mathsf{qpol}}$ if and only if $a = c + u$, where c is simply polar, u quasinilpotent, and $cu = 0 = uc$. Such a decomposition is unique. In addition,*

$$a^{\mathsf{D}} = c^{\mathsf{D}}, \quad a^{\sigma} = c^{\sigma}, \quad \mathsf{Sp}(c) = \mathsf{Sp}(a). \tag{2.4}$$

We can show that $ua^{\sigma} = 0$ and that the element c, called the *core* of a, satisfies

$$c = aa^{\sigma} = (a^{\mathsf{D}})^{\mathsf{D}} = a^2 a^{\mathsf{D}}.$$

PROPOSITION 2.5. *Let $\mathcal{A}$ be a Banach algebra without unit and let $a \in \mathcal{A}^{\mathsf{qpol}}$. Then $a^{\mathsf{D}} = a$ if and only if $a^3 = a$.*

PROOF. Suppose that $a^3 = a$ and let $a = c + u$ be the core decomposition of a. We observe that $a^3 = c^3 + u^3$ is the core decomposition for $a^3 = a$. From the uniqueness, $c^3 = c$ and $u^3 = u$. Since $u^3 = u \in \mathcal{A}^{\mathsf{qnil}}$, we conclude that $u = 0$:

$$\lim_{n\to\infty} \|u\|^{1/3^n} = \lim_{n\to\infty} \|u^{3^n}\|^{1/3^n} = r(u) = 0.$$

Thus $a = c = aa^{\sigma}$ is simply polar, and

$$a^{\mathsf{D}} = (a^{\mathsf{D}})^2 a = (a^{\mathsf{D}})^2 a^3 = (a^{\mathsf{D}} a^2)(a^{\mathsf{D}} a) = aa^{\sigma} = a.$$

Conversely, if $a^{\mathsf{D}} = a$, then $a = (a^{\mathsf{D}})^2 a = a^3$. □

As an example of further properties of the g-Drazin inverse in Banach algebras without unit we prove the following result, which for matrices reduces to Theorem 7.8.4 of Campbell and Meyer [4].

PROPOSITION 2.6. *Let $\mathcal{A}$ be a Banach algebra without unit, and let $a, b \in \mathcal{A}$ be such that $(ba)^2 \in \mathcal{A}^{\mathsf{qpol}}$. Then both ab and ba are g-Drazin invertible, and*

$$(ab)^{\mathsf{D}} = a((ba)^2)^{\mathsf{D}} b. \tag{2.5}$$

PROOF. If $(ba)^2 \in \mathcal{A}^{\mathrm{qpol}}$, then also $(ab)^2$, ab and ba are quasipolar, and $w = ((ba)^2)^{\mathrm{D}} = ((ba)^{\mathrm{D}})^2$ commutes with ba. Set $c = a((ba)^2)^{\mathrm{D}}b = awb$. It is not difficult to show that $(ab)c = c(ab)$ and $(ab)c^2 = c$. The element $ab - (ab)^2c = (a - a(ba)^2w)b$ is quasinilpotent if and only if $x = b(a - a(ba)^2w) = ba - (ba)^3w$ is quasinilpotent. Imbedding $\mathcal{A}$ into its unitisation $\mathcal{A}_1$, we recall that $p = 1 - (ba)^2w$ is idempotent; hence $x = (ba)p \in \mathcal{A}^{\mathrm{qnil}}$ if and only if $x^2 = (ba)^2p \in \mathcal{A}^{\mathrm{qnil}}$ if and only if $(ba)^2 - (ba)^4w \in \mathcal{A}^{\mathrm{qnil}}$. This completes the proof. $\square$

3. The weighted g-Drazin inverse for operators

Throughout this section we assume that X, Y are nonzero complex Banach spaces and W is a fixed nonzero operator in $\mathcal{B}(Y, X)$, the set of all bounded linear operators on Y to X. First we turn $\mathcal{B}(X, Y)$ into a Banach algebra $\mathcal{B}_W(X, Y)$ (in general without a unit) by introducing a multiplication of elements of $\mathcal{B}(X, Y)$ facilitated by the operator W, and imposing a suitable norm on $\mathcal{B}(X, Y)$.

LEMMA 3.1. *Let $\mathcal{B}_W(X, Y)$ be the space $\mathcal{B}(X, Y)$ equipped with the multiplication*

$$A \star B = AWB, \tag{3.1}$$

and norm $\|A\|_W = \|A\|\|W\|$. Then $\mathcal{B}_W(X, Y)$ becomes a complex Banach algebra; $\mathcal{B}_W(X, Y)$ has a unit if and only if W is invertible, in which case W^{-1} is that unit.

PROOF. The verification of most Banach algebra axioms is straightforward. The positive definiteness of the norm is ensured by the fact that $W \neq 0$. We check the submultiplicativity of the norm. If $A, B \in \mathcal{B}(X, Y)$, then

$$\|A \star B\|_W = \|AWB\|\|W\| \le \|A\|\|W\|\|B\|\|W\| = \|A\|_W\|B\|_W. \tag{3.2}$$

If W is invertible, then $W^{-1} \in \mathcal{B}(X, Y)$ is the unit in $\mathcal{B}_W(X, Y)$. Conversely, assume that $P \in \mathcal{B}(X, Y)$ is the unit for $\mathcal{B}_W(X, Y)$. Then

$$AWP = A = PWA \quad \text{for all } A \in \mathcal{B}(X, Y). \tag{3.3}$$

For each $y \in Y$ and $f \in X^*$ define $f \otimes y : X \to Y$ by $(f \otimes y)x = f(x)y$ for all $x \in X$; then $f \otimes y \in \mathcal{B}(X, Y)$. From $PW(f \otimes y)x = (f \otimes y)x$ we get $f(x)PWy = f(x)y$. Selecting x and f so that $f(x) \neq 0$, we obtain $PWy = y$ for any $y \in Y$. From $(f \otimes y)WPx = (f \otimes y)x$ we get $f(WPx)y = f(x)y$. Selecting $y \neq 0$, yields $f(WPx) = f(x)$ for all $f \in X^*$, which implies $WPx = x$ for any $x \in X$. Then W is invertible; setting $A = W^{-1}$ in (3.3), we get $W^{-1} = PWW^{-1} = P$. $\square$

We observe that if $\mathcal{B}_W(X, Y)$ has the unit W^{-1}, the spaces X and Y are isomorphic and homeomorphic; in particular, X and Y are of the same dimension. Moreover, the norm of the unit in $\mathcal{B}_W(X, Y)$ is equal to $\|W^{-1}\|_W = \|W^{-1}\|\|W\| = \kappa(W)$, known as the condition number of W.

For any $n \in \mathbb{N}$ we write $A^{\star n} = A \star \cdots \star A$ (n factors). Observe that

$$A^{\star n} = (AW)^{n-1}A = A(WA)^{n-1}. \tag{3.4}$$

We write $r_W(\cdot)$ for the spectral radius of elements of $\mathcal{B}_W(X, Y)$. We show that

$$r_W(A) = r(AW) = r(WA), \tag{3.5}$$

where $r(\cdot)$ is the spectral radius in $\mathcal{B}(Y)$ or $\mathcal{B}(X)$. Indeed,

$$\begin{aligned} r(AW) &= \lim_{n\to\infty} \|(AW)^n\|^{1/n} \le \lim_{n\to\infty} (\|(AW)^{n-1}A\|\|W\|)^{1/n} \\ &= \lim_{n\to\infty} \|A^{\star n}\|_W^{1/n} = r_W(A). \end{aligned}$$

Conversely,

$$\begin{aligned} r_W(A) &= \lim_{n\to\infty} \|A^{\star n}\|_W^{1/n} = \lim_{n\to\infty} \|A^{\star n}\|^{1/n}\|W\|^{1/n} \\ &= \lim_{n\to\infty} \|(AW)^{n-1}A\|^{1/n}\|W\|^{1/n} \\ &\le \lim_{n\to\infty} \|(AW)^{n-1}\|^{1/n} \lim_{n\to\infty} (\|A\|\|W\|)^{1/n} \\ &= \lim_{n\to\infty} \|(AW)^{n-1}\|^{1/n} = r(AW) \end{aligned}$$

as $\lim_{n\to\infty} \|(AW)^{n-1}\|^{1/n} = \lim_{n\to\infty} \|(AW)^n\|^{1/n}$. The second equality in (3.5) follows by symmetry.

DEFINITION 3.2. Let W be a fixed nonzero operator in $\mathcal{B}(Y, X)$. An operator $A \in \mathcal{B}(X, Y)$ is called *Wg-Drazin invertible* if A is quasipolar in the Banach algebra $\mathcal{B}_W(X, Y)$. The *Wg-Drazin inverse* $A^{\mathsf{D},W}$ of A (or *W-weighted g-Drazin inverse*) is then defined as the g-Drazin inverse B of A in the Banach algebra $\mathcal{B}_W(X, Y)$; $i_W(A)$ is the Drazin index of A in $\mathcal{B}_W(X, Y)$. A polar element of $\mathcal{B}_W(X, Y)$ is called *W-Drazin invertible*, with the *W-Drazin inverse* $A^{\mathsf{D},W} = B$.

The Wg-Drazin inverse is unique if it exists (Proposition 2.2), and is characterised by the following theorem.

THEOREM 3.3. *Let W be a fixed nonzero operator in $\mathcal{B}(Y, X)$. Then $A \in \mathcal{B}(X, Y)$ is Wg-Drazin invertible with the Wg-Drazin inverse $A^{\mathsf{D},W} = B \in \mathcal{B}(X, Y)$ if and only if one of the following equivalent conditions holds*:

(i) *AW is quasipolar in $\mathcal{B}(Y)$ with $(AW)^{\mathsf{D}} = BW$;*

(ii) *WA is quasipolar in $\mathcal{B}(X)$ with $(WA)^{\mathsf{D}} = WB$;*
(iii) *There exists $B \in \mathcal{B}(X, Y)$ satisfying*

$$(AW)B = (BW)A, \quad (BW)^2A = B, \quad (AW)^2BW - AW \in \mathcal{B}(Y)^{\mathsf{qnil}};$$

(iv) *There exists $B \in \mathcal{B}(X, Y)$ satisfying*

$$A(WB) = B(WA), \quad A(WB)^2 = B, \quad WB(WA)^2 - WA \in \mathcal{B}(X)^{\mathsf{qnil}}.$$

The Wg-Drazin inverse $A^{\mathsf{D},W}$ of A then satisfies

$$A^{\mathsf{D},W} = ((AW)^{\mathsf{D}})^2A = A((WA)^{\mathsf{D}})^2. \tag{3.6}$$

PROOF. Suppose that A has the Wg-Drazin inverse B.
The conditions

$$A \star B = B \star A, \quad B \star A \star B = B, \quad A \star B \star A - A \in \mathcal{B}_W(X, Y)^{\mathsf{qnil}},$$

translate to

$$AWB = BWA, \quad (BW)^2A = B, \quad T = (AW)^2B - A \in \mathcal{B}_W(X, Y)^{\mathsf{qnil}}. \tag{3.7}$$

Let $C = BW$. Then $(AW)C = C(AW)$ and $C^2(AW) = C$ by (3.7). Finally, by (3.5), $r(TW) = r_W(T) = 0$. Hence $(AW)^2C - AW = TW$ is quasinilpotent in $\mathcal{B}(Y)$, and (i) is proved.

Condition (ii) follows from a symmetrical argument. Conditions (i) and (iii) (respectively (ii) and (iv)) are equivalent by the characterisation of the g-Drazin inverse given in Proposition 2.2.

Conversely, suppose that $AW \in \mathcal{B}(Y)$ has the g-Drazin inverse C. Let $B = C^2A$. The equations $(AW)C = C(AW)$ and $C^2(AW) = C$ imply

$$A \star B = AWC^2A = C^2AWA = B \star A, \quad \text{and}$$
$$B \star A \star B = (C^2AW)(AWC^2)A = C^2A = B.$$

Write $A \star B \star A - A = (AWC^2)AWA - A = CAWA - A = S$. Since $SW = C(AW)^2 - AW$ is quasinilpotent in $\mathcal{B}(Y)$, $r_W(S) = r(SW) = 0$, and S is quasinilpotent in $\mathcal{B}_W(X, Y)$. This proves that condition (i) implies that A is Wg-Drazin invertible with $A^{\mathsf{D},W} = C^2A$. The rest follows from Proposition 2.2 by symmetry. □

From (3.6) we find an expression for the support idempotent $A^{\sigma,W}$ of A in $\mathcal{B}_W(X, Y)$: $A^{\sigma,W} = A \star A^{\mathsf{D},W} = AW((AW)^{\mathsf{D}})^2A = (AW)^{\mathsf{D}}A$. By symmetry,

$$A^{\sigma,W} = (AW)^{\mathsf{D}}A = A(WA)^{\mathsf{D}}. \tag{3.8}$$

PROPOSITION 3.4. *If $A \in \mathcal{B}(X, Y)$ is Wg-Drazin invertible, then the Drazin indices $i_W(A)$, $i(WA)$, and $i(AW)$ are all finite or all infinite, and satisfy the inequalities*

$$\max\{i(AW), i(WA)\} \le i_W(A) \le \min\{i(AW), i(WA)\} + 1. \tag{3.9}$$

PROOF. Let $A^{\mathsf{D},W} = B$ be the Wg-Drazin inverse of A and let $T = (AW)^2B - A$. If $i_W(A) = k < \infty$, then $T^{\star k} = 0$. Consequently $(TW)^k = (TW)^{k-1}TW = T^{\star k}W = 0$ and hence $i(AW) \le i_W(A)$.

Let AW have the g-Drazin inverse C and let $S = CAWA - A$. If $i(AW) = k < \infty$, then $(SW)^k = 0$, and $S^{\star(k+1)} = (SW)^kS = 0$, that is, $i_W(A) \le k + 1$. This proves the inequality for $i(AW)$ in (3.9).

It is known that for any $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X)$,

$$\mathsf{Sp}(AW) \setminus \{0\} = \mathsf{Sp}(WA) \setminus \{0\}. \tag{3.10}$$

Hence AW is g-Drazin invertible in $\mathcal{B}(Y)$ if and only if WA is g-Drazin invertible in $\mathcal{B}(X)$. The inequality for $i(WA)$ in (3.9) is obtained by symmetry. □

EXAMPLE 3.5. The inequality $i(AW) \le i_W(A)$ (respectively $i(WA) \le i_W(A)$) in (3.9) can be strict. Let

$$W = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix},$$

and let $\mathcal{B}_W$ be the space $\mathcal{M}_{2,3}(\mathbb{C})$ of all complex 2×3 matrices with the multiplication (3.1). By the preceding theorem, every element $A \in \mathcal{M}_{2,3}(\mathbb{C})$ has a g-Drazin inverse of finite Drazin index in $\mathcal{B}_W$ since the matrix AW has the conventional Drazin inverse $(AW)^{\mathsf{D}}$ in $\mathcal{M}_{2,2}(\mathbb{C})$. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $AW = 0 = B$, where B is the g-Drazin inverse of A in $\mathcal{B}_W$, and $T = (AW)^2B - A = -A$. Since $T \ne 0$ and $T \star T = AWA = 0$, we have $i_W(A) = 2$. On the other hand, $i(AW) = i(0) = 1$. An example of a strict inequality between $i(WA)$ and $i_W(A)$ can be obtained from the present example and the following proposition involving the dual spaces X^* and Y^* of X and Y.

PROPOSITION 3.6. *$A \in \mathcal{B}(X, Y)$ is Wg-Drazin invertible if and only if the adjoint $A^* \in \mathcal{B}(Y^*, X^*)$ of A is W^*g-Drazin invertible. In this case*

$$(A^*)^{\mathsf{D},W^*} = (A^{\mathsf{D},W})^*. \tag{3.11}$$

PROOF. Since $\mathrm{Sp}((AW)^*) = \mathrm{Sp}(AW)$, $(AW)^*$ is quasipolar if and only if AW is quasipolar. Hence $W^*A^* = (AW)^*$ is g-Drazin invertible if and only if AW is. By Theorem 3.3, A^* is W^*g-Drazin invertible if and only if A is Wg-Drazin invertible. Equation (3.11) follows on application of Proposition 2.2. □

EXAMPLE 3.7 (Rakočević and Wei [13]). If $A \in \mathcal{B}(X, Y)$ is a finite rank operator, then A has a finite index Wg-Drazin inverse for any nonzero $W \in \mathcal{B}(Y, X)$. If $W \in \mathcal{B}(Y, X)$ is a nonzero operator of finite rank, then any $A \in \mathcal{B}(X, Y)$ has a finite index Wg-Drazin inverse.

4. Further properties of the Wg-Drazin inverse

First we briefly explore a duality between $A^{\mathsf{D},W}$ and $W^{\mathsf{D},A}$ provided $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X)$. From Theorem 3.3 we see that the weighted g-Drazin inverse $W^{\mathsf{D},A}$ exists if and only if $A^{\mathsf{D},W}$ exists. Equation (3.6) gives rise to the following relations:

$$W^{\mathsf{D},A}A = (WA)^{\mathsf{D}} = WA^{\mathsf{D},W},$$
$$AW^{\mathsf{D},A} = (AW)^{\mathsf{D}} = A^{\mathsf{D},W}W.$$

We can then express $W^{\mathsf{D},A}$ in terms of $A^{\mathsf{D},W}$ and vice versa:

$$W^{\mathsf{D},A} = WA^{\mathsf{D},W}WA^{\mathsf{D},W}W,$$
$$A^{\mathsf{D},W} = AW^{\mathsf{D},A}AW^{\mathsf{D},A}A. \tag{4.1}$$

We observe that in (4.1), the operators $AW^{\mathsf{D},A}$ and $W^{\mathsf{D},A}A$ are simply polar (that is, of index 1 or 0): for example, $AW^{\mathsf{D},A} = AW((AW)^{\mathsf{D}})^2 = (AW)^{\mathsf{D}}$. The simple polarity of the g-Drazin inverse of AW is well known (see [7]). Specialised to matrices, this proves the necessary part of Theorem 3 in [5].

PROPOSITION 4.1. *Let $A \in \mathcal{B}(X, Y)$ be Wg-Drazin invertible. Then the following are true*:

(i) $A = A^{\mathsf{D},W}$ *if and only if* $A = A^{\star 3} = AWAWA$.
(ii) $(A^{\mathsf{D},W})^{\mathsf{D},W} = (AW)^{\sigma}A = A(WA)^{\sigma}$.
(iii) $(A^{\mathsf{D},W})^{\sigma,W} = A^{\sigma,W}$.
(iv) *For any* $n \in \mathbb{N}$, $(A^{\mathsf{D},W})^{\star n} = ((AW)^{\mathsf{D}})^{n+1}A = A((WA)^{\mathsf{D}})^{n+1} = (A^{\star n})^{\mathsf{D},W}$.

PROOF. (i) This follows from Proposition 2.5 applied to $\mathcal{B}_W(X, Y)$.

(ii) Applying the results of [7] while working in the Banach algebra $\mathcal{B}_W(X, Y)$, we have $(A^{\mathsf{D},W})^{\mathsf{D},W} = A \star A^{\sigma,W} = AW(AW)^{\mathsf{D}}A = (AW)^{\sigma}A$.

(iii) In the proof of [7, Theorem 5.2] it is shown that a quasipolar element and its g-Drazin inverse have the same support idempotent.

(iv) This is shown via induction on n. □

Part (ii) of the preceding theorem implies that $(A^{\mathsf{D},W})^{\mathsf{D},W} = A$ if and only if $(AW)^{\sigma}A = A$ $(A(WA)^{\sigma} = A)$. This is equivalent to A being simply polar in $\mathcal{B}_W(X, Y)$.

From [7, Theorem 5.5] we can deduce the following result.

PROPOSITION 4.2. *Let $A, B \in \mathcal{B}(X, Y)$ be Wg-Drazin invertible. If $AWB = BWA$, then AWB is Wg-Drazin invertible with $(AWB)^{\mathsf{D},W} = A^{\mathsf{D},W}WB^{\mathsf{D},W}$.*

We now turn our attention to an analogue of the core decomposition for the weighted g-Drazin inverse.

THEOREM 4.3. *An operator $A \in \mathcal{B}(X, Y)$ is Wg-Drazin invertible if and only if there exist operators $C, U \in \mathcal{B}(X, Y)$ such that*

$$A = C + U, \quad CWU = 0, \quad UWC = 0, \tag{4.2}$$

$$(CW)^{\sigma}C = C, \quad UW \in \mathcal{B}(Y)^{\mathsf{qnil}}. \tag{4.3}$$

Such operators are uniquely determined, and $C = (A^{\mathsf{D},W})^{\mathsf{D},W} = (AW)^{\sigma}A$. Further,

$$(AW)^{\mathsf{D}} = (CW)^{\mathsf{D}}, \quad (AW)^{\sigma} = (CW)^{\sigma}, \quad \mathsf{Sp}(AW) \cup \{0\} = \mathsf{Sp}(CW). \tag{4.4}$$

PROOF. We apply Theorem 2.4 to $\mathcal{B}_W(X, Y)$. A is Wg-Drazin invertible if and only if there exist $C, U \in \mathcal{B}(X, Y)$ such that $A = C + U$, $C \star U = CWU = 0$, $U \star C = UWC = 0$, C is simply polar in $\mathcal{B}_W(X, Y)$, and U is quasinilpotent in $\mathcal{B}_W(X, Y)$. The element $C \in \mathcal{B}_W(X, Y)$ is simply polar if and only if $C \star C^{\sigma,W} = C$. From the equation

$$C \star C^{\sigma,W} = CW(CW)^{\mathsf{D}}C = (CW)^{\sigma}C$$

we conclude that the simple polarity of $C \in \mathcal{B}_W(X, Y)$ is equivalent to $(CW)^{\sigma}C = C$. Finally, $r_W(U) = r(UW)$, and UW is quasinilpotent in $\mathcal{B}(X, Y)$ if and only if U is quasinilpotent in $\mathcal{B}_W(X, Y)$. This proves the equivalence of (4.2) and (4.3) to the Wg-Drazin invertibility of A. Explicitly, $C = A \star A^{\sigma,W} = (AW)^{\sigma}A$.

Towards (4.4) in view of Theorem 2.4,

$$(AW)^{\mathsf{D}} = ((AW)^{\mathsf{D}})^2AW = A^{\mathsf{D},W}W = C^{\mathsf{D},W}W = ((CW)^{\mathsf{D}})^2CW = (CW)^{\mathsf{D}}.$$

Therefore

$$(CW)^{\sigma} = (CW)^{\mathsf{D}}C = (AW)^{\mathsf{D}}C = (AW)^{\mathsf{D}}(AW)^{\sigma}A = (AW)^{\mathsf{D}}A = (AW)^{\sigma}.$$

If $\mathsf{Sp}_W(A)$ denotes the spectrum of A as an element of the Banach algebra $\mathcal{B}_W(X, Y)$ without unit, then it can be shown that $\mathsf{Sp}_W(A) = \mathsf{Sp}(AW) \cup \{0\}$. Hence

$$\mathsf{Sp}(AW) \cup \{0\} = \mathsf{Sp}_W(A) = \mathsf{Sp}_W(C) = \mathsf{Sp}(CW) \quad (\text{as } 0 \in \mathsf{Sp}(CW)).$$

This completes the proof. $\square$

The statement of the theorem remains true when (4.3) is replaced by $C(WC)^{\sigma} = C$, $WU \in \mathcal{B}(X)^{\mathrm{qnil}}$, and AW, CW in (4.4) are replaced by WA, WC, respectively.

We close the section with an integral representation of the Wg-Drazin inverse. The representation of the g-Drazin inverse given by Castro *et al.* [6, Theorem 2.2] is valid also for Banach algebras without unit. Applying this result to $\mathcal{B}_W(X, Y)$, we get the integral representation

$$A^{\mathrm{D},W} = -\int_0^{\infty} \exp(tA) \star A^{\sigma,W}\, dt$$

provided A is Wg-Drazin invertible and the nonzero spectrum $\mathrm{Sp}_W(A) \setminus \{0\}$ lies in the open left half-plane. We express $\exp(tA) \star A^{\sigma,W}$ in terms of the usual multiplication of operators:

$$A^{\star n} \star A^{\sigma,W} = (AW)^{n-1} AWA^{\sigma,W} = (AW)^n A^{\sigma,W}.$$

Hence

$$\exp(tA) \star A^{\sigma,W} = \sum_{n=0}^{\infty} \frac{t^n}{n!} (AW)^n A^{\sigma,W} = \exp(tAW) A^{\sigma,W}.$$

Note that in general $\exp(tAW)$ belongs to the unitisation of $\mathcal{B}_W(X, Y)$ but not to $\mathcal{B}_W(X, Y)$, while $\exp(tAW)A^{\sigma,W}$ is in $\mathcal{B}_W(X, Y)$. We summarise our findings.

PROPOSITION 4.4. *Let $A \in \mathcal{B}(X, Y)$ be Wg-Drazin invertible such that* $\mathrm{Sp}(WA)\setminus\{0\}$ *lies in the left open half-plane. Then*

$$A^{\mathrm{D},W} = -\int_0^{\infty} \exp(tAW) A^{\sigma,W}\, dt. \tag{4.5}$$

If the Drazin index $i(AW)$ is finite and the set $\mathrm{Sp}((AW)^{m+1}) \setminus \{0\}$ *lies in the left open half-plane for some $m \geq \min\{i(AW), i(WA)\} + 1$, then*

$$A^{\mathrm{D},W} = -\int_0^{\infty} \exp(t(AW)^{m+1})(AW)^{m-1} A\, dt. \tag{4.6}$$

PROOF. Equation (4.5) follows from our calculations preceding the theorem. For (4.6) we find

$$(A^{\star(m+1)})^{\star n} \star A^{\star m} = A^{\star(m+1)n+m} = (AW)^{(m+1)n}(AW)^{m-1}A,$$

and

$$\exp(tA^{\star(m+1)}) \star A^{\star m} = \exp(t(AW)^{m+1})(AW)^{m-1}A.$$

Equation (4.6) then follows from [6, Theorem 2.4]. □

As expected from symmetry, there is also a WA version of the preceding theorem. If we specialise Equation (4.6) to matrices, we recover [18, Theorem 1]. The inequality $m \geq \min\{i(AW), i(WA)\}+1$ in the preceding theorem can be relaxed to $m \geq i_W(A)$.

Using the core decomposition of a Wg-Drazin invertible operator $A \in \mathcal{B}(X, Y)$, we obtain yet another integral representation for $A^{\mathrm{D},W}$.

COROLLARY 4.5. *Let $A \in \mathcal{B}(X, Y)$ be Wg-Drazin invertible such that* $\mathrm{Sp}((WA)^2)\setminus\{0\}$ *lies in the open left half-plane, and let $A = C + U$ be the core decomposition of A. Then*

$$A^{\mathrm{D},W} = C^{\mathrm{D},W} = -\int_0^\infty \exp(t(CW)^2)C\,dt.$$

PROOF. This follows from (4.6) when we note that $i_W(C) = 1$. □

5. Ascent and descent

We recall that the ascent and descent of an operator $T \in \mathcal{B}(X)$ are defined by

$$\begin{aligned}\mathrm{asc}\,(T) &= \inf\left\{k \in \mathbb{N} \cup \{0\} : N(T^{k+1}) = N(T^k)\right\},\\ \mathrm{des}\,(T) &= \inf\left\{k \in \mathbb{N} \cup \{0\} : R(T^{k+1}) = R(T^k)\right\}\end{aligned}$$

($\inf\emptyset = \infty$). Rakočević and Wei [13] ask whether the finiteness of $\mathrm{asc}\,(AW)$ and $\mathrm{des}\,(WA)$ is sufficient for A to have the W-weighted Drazin inverse. An equivalent question is whether $\mathrm{asc}\,(AW)$ and $\mathrm{asc}\,(WA)$ are always both finite or both infinite.

In this connection it is interesting to recall that Buoni and Faires [3] studied the ascent and descent for the operators $\lambda I - BA$ and $\lambda I - AB$, where $A, B \in \mathcal{B}(X)$, and proved, *inter alia*, that for any $\lambda \neq 0$,

$$\mathrm{asc}\,(AB - \lambda I) = \mathrm{asc}\,(BA - \lambda I), \qquad \mathrm{des}\,(AB - \lambda I) = \mathrm{des}\,(BA - \lambda I); \tag{5.1}$$

however, the case $\lambda = 0$ was left open. Later, Barnes [1] proved by different methods that the ascents of $I - RS$ and $I - SR$ are equal for $R \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, X)$. It can be shown that the arguments in [3] concerning descent are valid also when $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$. Thus (5.1) is valid for operators between different spaces. The following theorem, dealing with the ascent and descent in general, completes the results of Buoni and Faires in the case $\lambda = 0$.

THEOREM 5.1. *Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$. Then the ascents (descents) of AB and BA are both finite or both infinite, and satisfy the inequalities*

$$\begin{aligned}\mathrm{asc}\,(AB) - 1 \leq \mathrm{asc}\,(BA) \leq \mathrm{asc}\,(AB) + 1,\\ \mathrm{des}\,(AB) - 1 \leq \mathrm{des}\,(BA) \leq \mathrm{des}\,(AB) + 1.\end{aligned} \tag{5.2}$$

PROOF. Suppose that $\operatorname{asc}(AB) = p < \infty$. If there existed

$$x \in N((BA)^{p+2}) \setminus N((BA)^{p+1}),$$

we would have $(AB)^{p+2}Ax = A(BA)^{p+2}x = 0$, and $B(AB)^pAx = (BA)^{p+1}x \neq 0$, that is, $(AB)^pAx \neq 0$. Then Ax would belong to $N((AB)^{p+2}) \setminus N((AB)^p)$, which is empty by assumption. This contradiction proves $N((BA)^{p+1}) = N((BA)^{p+2})$, which shows that $\operatorname{asc}(BA) \leq p + 1 = \operatorname{asc}(AB) + 1$. A symmetrical argument gives $\operatorname{asc}(AB) \leq \operatorname{asc}(BA) + 1$. This proves the first inequality in (5.2).

Let $\operatorname{des}(AB) = p < \infty$. Suppose

$$x \in R((BA)^{p+1}) \setminus R((BA)^{p+2}). \tag{5.3}$$

Then there exists $x' \in X$ such that

$$x = (BA)^{p+1}x' = B(AB)^pAx' = By,$$

where $y = (AB)^pAx' \in R((AB)^p) = R((AB)^{p+2})$. Hence $y = (AB)^{p+2}y'$ for some $y' \in Y$, and $(BA)^{p+2}By' = B(AB)^{p+2}y' = By = x$ contrary to (5.3). This proves that $R((BA)^{p+1}) = R((BA)^{p+2})$, so that $\operatorname{des}(BA) \leq p + 1$. □

The inequalities in (5.2) can be strict; this follows from Example 3.5 since for matrices $i(AB) = \operatorname{asc}(AB) = \operatorname{des}(AB)$.

The following theorem gives a solution to the open problem of Rakočević and Wei [13, page 28].

THEOREM 5.2. *Let $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X) \setminus \{0\}$. Then A is W-Drazin invertible if and only if one of the following equivalent conditions hold*:

(i) *AW is polar in $\mathcal{B}(Y)$;*
(ii) *WA is polar in $\mathcal{B}(X)$;*
(iii) $\operatorname{asc}(AW)$ *and* $\operatorname{des}(WA)$ *are both finite;*
(iv) $\operatorname{asc}(WA)$ *and* $\operatorname{des}(AW)$ *are both finite.*

PROOF. Suppose that A is W-Drazin invertible. By Theorem 3.3, AW is quasipolar, and by (3.9) we have $i(AW) \leq i_W(A)$, which proves that AW is polar. Conversely, if AW is polar, then $i_W(A) \leq i(AW) + 1$, and A is W-Drazin invertible.

(i) implies (ii): Since AW is quasipolar, so is WA by (3.10). By (3.9) again, $i(WA) \leq i(AW) + 1$, and WA is polar.

(ii) implies (iii): It is well known that if WA is polar, then $\operatorname{asc}(WA)$ and $\operatorname{des}(WA)$ are both finite. However, $\operatorname{asc}(AW) \leq \operatorname{asc}(WA) + 1$ by Theorem 5.1 and (iii) follows.

(iii) implies (iv): This follows from Theorem 5.1 as $\operatorname{asc}(WA) \leq \operatorname{asc}(AW) + 1$ and $\operatorname{des}(AW) \leq \operatorname{des}(WA) + 1$.

(iv) implies (i): Since $\operatorname{asc}(AW) \leq \operatorname{asc}(WA) + 1$, both $\operatorname{asc}(AW)$ and $\operatorname{des}(AW)$ are finite; this implies that AW is polar. □

6. The Mbekhta decomposition for WA and AW

As before, X, Y are Banach spaces and W a nonzero operator in $\mathcal{B}(Y, X)$. In order to obtain an operator matrix representation for the weighted g-Drazin inverse of an operator $A \in \mathcal{B}(X, Y)$, we first recall the Mbekhta decomposition for a quasipolar operator. For any operator $T \in \mathcal{B}(X)$ we define spaces $H_0(T)$ and $K(T)$ as follows:

$$H_0(T) = \left\{ x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0 \right\},$$
$$K(T) = \left\{ x \in X : \exists\, x_n \in X,\ x_n = T x_{n+1},\ x_0 = x,\ \sup_{n\in\mathbb{N}} \|x_n\|^{1/n} < \infty \right\}.$$

Both spaces are hyperinvariant under T, $H_0(T) \supset N(T^n)$, and $K(T) \subset R(T^n)$ for all $n \in \mathbb{N}$. Further, $TK(T) = K(T)$ and $T^{-1}H_0(T) = H_0(T)$.

PROPOSITION 6.1 (See [8, 11]). *The following conditions on $T \in \mathcal{B}(X)$ are equivalent*:

(i) *T is quasipolar*;
(ii) *X is the topological direct sum $X = K(T) \oplus H_0(T)$*;
(iii) *$T = T_1 \oplus T_2$, where T_1 is invertible and T_2 quasinilpotent.*

Condition (ii) can be weakened to $X = K(T) \oplus H_0(T)$ being only an algebraic sum with at least one of the spaces closed (see [10] and [15]).

THEOREM 6.2. *Let $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X)$. If WA is quasipolar, then so is AW,*

$$\begin{aligned} A(K(WA)) &= K(AW), & A^{-1}(H_0(AW)) &= H_0(WA), \\ W(K(AW)) &= K(WA), & W^{-1}(H_0(WA)) &= H_0(AW), \end{aligned} \tag{6.1}$$

and the spaces $K(WA)$, $K(AW)$ are isomorphic and homeomorphic.

PROOF. The result on quasipolarity follows from (3.10). We introduce the following notation

$$X_1 = K(WA), \quad X_2 = H_0(WA), \quad Y_1 = K(AW), \quad Y_2 = H_0(AW). \tag{6.2}$$

Then X and Y are decomposed into the topological direct sums $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. The operator matrices

$$T = \begin{bmatrix} WA & 0 \\ 0 & AW \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \tag{6.3}$$

represent commuting operators in $\mathcal{B}(X \oplus Y)$ with T quasipolar. The support projection T^σ of T double commutes with T, that is, the matrix

$$T^\sigma = \begin{bmatrix} (WA)^\sigma & 0 \\ 0 & (AW)^\sigma \end{bmatrix}$$

commutes with the matrix S. This gives $A(WA)^\sigma = (AW)^\sigma A$. Since $(WA)^\sigma$ is the projection of X onto X_1 along X_2, and $(AW)^\sigma$ is the projection of Y onto Y_1 along Y_2, we have $A(X_i) \subset Y_i$ $(i = 1, 2)$. The inclusions $W(Y_i) \subset X_i$ $(i = 1, 2)$ are obtained by symmetry.

Note that $A(X_2) \subset Y_2$ is equivalent to $X_2 \subset A^{-1}(Y_2)$. In order to prove $A^{-1}(Y_2) \subset X_2$, assume that $Ax \in Y_2$. Then $x = k + h$ with $k \in X_1$ and $h \in X_2$, and $Ax = Ak + Ah \in Y_2$ implies that $Ak = 0$. From $k \in N(A) \subset N(WA) \subset X_2$, we obtain $k \in X_1 \cap X_2 = \{0\}$. Hence $x = h \in X_2$.

Let $A_0 : X_1 \to Y_1$ be the restriction of A. If $x \in X_1$ and $Ax = 0$, then $x = 0$ by the argument of the preceding paragraph. Hence A_0 is injective. Suppose that $y \in Y_1$. Since $AWY_1 = Y_1$, there exists $u \in Y_1$ such that $y = AWu$. But $WY_1 \subset X_1$, and so $Wu \in X_1$. This proves that A_0 is surjective. Therefore A_0 is a bounded linear bijection from X_1 to Y_1, and (6.1) is proved. □

In particular, if AW is quasipolar, then the spaces $K(AW)$ and $K(WA)$ have the same dimension being isomorphic.

If A and W are rectangular matrices of orders $m \times n$ and $n \times m$ respectively, we recover the result of Yukhno [19, Theorem]. For this the operator $T : \mathbb{C}^m \to \mathbb{C}^m$ with the matrix WA is polar, and $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 nilpotent; T_1 operates on $X_1 = K(T) = R(T^p)$, where p is the index of T. The eigenvalues of T_1 are the nonzero eigenvalues of WA. Let λ be a nonzero eigenvalue of WA, and $x_1, \dots, x_k$ a chain of generalised eigenvectors of WA corresponding to λ, that is,

$$WAx_1 = \lambda x_1 + x_2, \quad \dots, \quad WAx_{k-1} = \lambda x_{k-1} + x_k, \quad WAx_k = \lambda x_k.$$

In view of the decomposition of T as $T = T_1 \oplus T_2$, where T_1 operates on X_1, we can take $x_i \in X_1$ for all i. If $y_i = Ax_i$ $(i = 1, \dots, k)$, then $y_1, \dots, y_k$ is a chain of generalised eigenvectors of AW corresponding to λ (this follows from the bijectivity of the operator $x \mapsto Ax$ restricted from X_1 to Y_1). All chains corresponding to nonzero eigenvalues of WA are matched in this way. This leads to the following structure theorem for WA and AW.

PROPOSITION 6.3. *Let A and W be rectangular matrices of orders $m \times n$ and $n \times m$, respectively. The matrices WA and AW (of orders $n \times n$ and $m \times m$, respectively) have Jordan forms*

$$\begin{bmatrix} U & 0 \\ 0 & N_1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} U & 0 \\ 0 & N_2 \end{bmatrix},$$

where U is a matrix in Jordan form corresponding to the nonzero eigenvalues of WA (and AW), while N_1 and N_2 are nilpotent matrices in Jordan form, of different orders in general.

Recall that the entries of N_1 and N_2 are zero except for the superdiagonals, which consist of 0s and 1s.

7. An operator matrix representation of the Wg-Drazin inverse

From the Mbekhta decomposition theorem (Proposition 6.1), it follows that an operator $T \in \mathcal{B}(X)$ is quasipolar if and only if it can be expressed as the direct sum $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 quasinilpotent; the g-Drazin inverse of T is given by

$$T^{\mathsf{D}} = T_1^{-1} \oplus 0.$$

Our aim is to derive an analogous formula for the Wg-Drazin inverse using the results of the preceding section.

THEOREM 7.1. *Let $A \in \mathcal{B}(X, Y)$ and $W \in \mathcal{B}(Y, X) \setminus \{0\}$. Then A is Wg-Drazin invertible if and only if there exist topological direct sums $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ such that $A = A_1 \oplus A_2$ and $W = W_1 \oplus W_2$, where $A_i \in \mathcal{B}(X_i, Y_i)$, $W_i \in \mathcal{B}(Y_i, X_i)$, with A_1, W_1 invertible, and W_2A_2 and A_2W_2 quasinilpotent in $\mathcal{B}(X_2)$ and $\mathcal{B}(Y_2)$, respectively. The Wg-Drazin inverse of A is given by $A^{\mathsf{D},W} = (W_1A_1W_1)^{-1} \oplus 0$ with $(W_1A_1W_1)^{-1} \in \mathcal{B}(X_1, Y_1)$ and $0 \in \mathcal{B}(X_2, Y_2)$.*

PROOF. If WA is quasipolar, the decomposition exists with X_i and Y_i given by (6.2). By Theorem 6.2, A maps X_1 into Y_1, and X_2 into Y_2, that is, $A = A_1 \oplus A_2$, with $A_i \in \mathcal{B}(X_i, Y_i)$, $i = 1, 2$. Similarly, since W maps Y_1 into X_1 and Y_2 into X_2, $W = W_1 \oplus W_2$, where $W_i \in \mathcal{B}(Y_i, X_i)$, $i = 1, 2$. Hence

$$WA = W_1A_1 \oplus W_2A_2, \quad AW = A_1W_1 \oplus A_2W_2$$

relative to $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. Since WA and AW are quasipolar, W_1A_1 and A_1W_1 are invertible, and W_2A_2 and A_2W_2 are quasinilpotent. Hence A_1 and W_1 are invertible.

The Wg-Drazin inverse of A is equal to

$$A((WA)^{\mathsf{D}})^2 = (A_1 \oplus A_2)((W_1A_1)^{-2} \oplus 0) = (W_1A_1W_1)^{-1} \oplus 0.$$

Conversely, if the decompositions with the specified properties exist, then $AW = (A_1W_1) \oplus (A_2W_2)$ is quasipolar as A_1W_1 is invertible and A_2W_2 quasinilpotent. Then A is Wg-Drazin invertible. □

From the necessary part of the preceding theorem we recover [18, Theorem 2] when we specialise the operators to finite matrices. From Theorem 6.2 applied to finite matrices we deduce that the ranks of $(AW)^m$ and $(WA)^m$ are equal for any $m \geq \max\{\operatorname{ind}(AW), \operatorname{ind}(WA)\}$. (This is used, but not proved, in the derivation of [18, Theorem 2]).

From the commutativity of the operator matrices given in (6.3) and the double commutativity of the g-Drazin inverse we deduce that $(AW)^{\mathrm{D}}A = A(WA)^{\mathrm{D}}$, which leads to the new equality for $A^{\mathrm{D},W}$ derived from (3.6),

$$A^{\mathrm{D},W} = (AW)^{\mathrm{D}}A(WA)^{\mathrm{D}}.$$

8. Relation to the Moore–Penrose inverse

We briefly address the relation of the Wg-Drazin inverse to the Moore–Penrose inverse in Hilbert spaces (see [13, page 28]). Let H, K be Hilbert spaces and let $A \in \mathcal{B}(H, K)$. It is well known that $R(A)$ is closed if and only if $R(A^*)$ is closed, $R(A^*)$ is closed if and only if A^*A is simply polar, and A^*A is simply polar if and only if AA^* is simply polar. This means that $A \in \mathcal{B}(H, K)$ is A^*g-Drazin invertible if and only if the range of A is closed. We note that

$$(A^{\mathrm{D},A^*})^* = (A^*)^{\mathrm{D},A}. \tag{8.1}$$

We can then prove that the operator $A^\dagger = (A^*)^{\sigma,A} = A^*A^{\mathrm{D},A^*}A^*$ is the Moore–Penrose inverse characterised by the equations

$$A^\dagger A A^\dagger = A^\dagger, \quad AA^\dagger A = A, \quad (A^\dagger A)^* = A^\dagger A, \quad (AA^\dagger)^* = AA^\dagger. \tag{8.2}$$

We offer a sample of such proof

$$A^\dagger A A^\dagger = (A^*)^{\sigma,A}A(A^*)^{\sigma,A} = (A^*)^{\sigma,A} \circ (A^*)^{\sigma,A} = (A^*)^{\sigma,A} = A^\dagger,$$

where $T \circ S = TAS$, and

$$AA^\dagger A = AA^*A^{\mathrm{D},A^*}A^*A = A \star A^{\mathrm{D},A^*} \star A = A,$$

where $T \star S = TA^*S$. Other equations in (8.2) can be proved similarly.

9. Continuity of the Wg-Drazin inverse

THEOREM 9.1. *Let $A_n \to A_0$ in $\mathcal{B}(X, Y)$ and $W_n \to W_0 \neq 0$ in $\mathcal{B}(Y, X)$, where each A_n is W_ng-Drazin invertible, $n = 0, 1, 2, \ldots$ Then the following conditions are equivalent*:

(i) $A_n^{\mathsf{D},W_n} \to A_0^{\mathsf{D},W_0}$;
(ii) $\sup_n \|A_n^{\mathsf{D},W_n}\| < \infty$;
(iii) $(A_n W_n)^{\mathsf{D}} \to (A_0 W_0)^{\mathsf{D}}$;
(iv) $A_n^{\sigma,W_n} \to A_0^{\sigma,W_0}$.

PROOF. We rely on continuity results for the g-Drazin inverse obtained in [9]. Condition (i) clearly implies (ii). Suppose that (ii) holds. Since

$$(A_n W_n)^{\mathsf{D}} = ((A_n W_n)^{\mathsf{D}})^2 (A_n W_n) = A_n^{\mathsf{D},W_n} W_n,$$

we have $\sup_n \|(A_n W_n)^{\mathsf{D}}\| < \infty$. By [9, Theorem 2.4], $(A_n W_n)^{\mathsf{D}} \to (A_0 W_0)^{\mathsf{D}}$.

If (iii) holds, then $A_n^{\sigma,W_n} = (A_n W_n)^{\mathsf{D}} A_n \to (A_0 W_0)^{\mathsf{D}} A_0 = A_0^{\sigma,W_0}$.

Let (iv) hold. From the equation

$$(A_n W_n)^{\sigma} = (A_n W_n)^{\mathsf{D}} A_n W_n = A_n^{\sigma,W_n} W_n,$$

we deduce that $(A_n W_n)^{\sigma} \to (A_0 W_0)^{\sigma}$. Using [9, Theorem 2.4] again, we obtain $(A_n W_n)^{\mathsf{D}} \to (A_0 W_0)^{\mathsf{D}}$. Hence $A_n^{\mathsf{D},W_n} = ((A_n W_n)^{\mathsf{D}})^2 A_n \to ((A_0 W_0)^{\mathsf{D}})^2 A_0 = A_0^{\mathsf{D},W_0}$ and the theorem is proved. $\square$

From the preceding theorem we recover [13, Theorem 5.1] when we specialise the result to a finite index weighted Drazin inverse.

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