J. Aust. Math. Soc. **81** (2006), 387–404

RESOLUTION IN HÖLDER SPACES OF AN ELLIPTIC PROBLEM IN AN UNBOUNDED DOMAIN

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(Received 23 July 2004; revised 4 July 2005)

Communicated by A. J. Pryde

Abstract

In this paper we give new results concerning the maximal regularity of the strict solution of an abstract second-order differential equation, with non-homogeneous boundary conditions of Dirichlet type, and set in an unbounded interval. The right-hand term of the equation is a Hölder continuous function.

2000 *Mathematics subject classification*: primary 12H20, 34G10; secondary 35J25, 44A45.
Keywords and phrases: Hölder Spaces, abstract elliptic equation.

1. Introduction

This work is devoted to the study of the second-order abstract differential equation

$$u''(t) + Au(t) = f(t), \quad t \in (0, +\infty), \tag{1.1}$$

under non-homogeneous boundary conditions of Dirichlet type given by

$$u(0) = \varphi, \quad u(+\infty) = 0, \tag{1.2}$$

where φ and $f(t)$ belong to a complex Banach space E and A is a closed linear operator with domain $D(A)$ not necessarily dense in E.

For $l \in \mathbb{N}$, we denote by $BUC^l([0, +\infty[; E)$ the space of vector-valued functions with uniformly continuous and bounded derivatives up to order l in $[0, +\infty[$ and by $C^\sigma([0, +\infty[; E)$ for $\sigma \in]0, 1[$, the space of bounded and σ-Hölder continuous functions $f : [0, +\infty[\to E$, such that $\sup_{t\in[0,+\infty[} \|f(t)\|_E < \infty$ and there exists $C > 0$ such that for all $t, \tau \in [0, +\infty[$,

$$\|f(t) - f(\tau)\|_E \leqslant C|t - \tau|^\sigma,$$

endowed with the norm

$$\begin{aligned}\|f\|_{C^\sigma([0,+\infty[;E)} &= \sup_{t\in[0,+\infty[} \|f(t)\|_E + \sup_{t\neq\tau}\frac{\|f(t)-f(\tau)\|_E}{|t-\tau|^\sigma} \\ &= \|f\|_\infty + [f]_{C^\sigma([0,+\infty[;E)}.\end{aligned}$$

For simplicity, we shall write $C^\sigma(E)$ instead of $C^\sigma([0,+\infty[\,;E)$.

In the present study, we are interested in the existence, the uniqueness and the maximal regularity of the strict solution u in the Banach space $X = BUC([0,+\infty[\,;E)$, when the right-hand term f is regular (Hölder continuous function).

We recall that $u \in BUC([0,+\infty[\,;E)$ is a *strict solution* of Problem (1.1)–(1.2) if

$$u \in BUC([0,+\infty[\,;E) \cap BUC([0,+\infty[\,;D(A)),$$

and u satisfies (1.1) and (1.2).

Throughout this paper we assume that the resolvent of A verifies the hypothesis that there exists $K > 0$ such that for all $\lambda \geqslant 0$

$$\left\|(A-\lambda I)^{-1}\right\|_{L(E)} \leqslant \frac{K}{1+\lambda}. \tag{1.3}$$

Equations (1.1)–(1.2) can be illustrated by the following example of a Laplacian problem

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}(t,x) + \dfrac{\partial^2 u}{\partial x^2}(t,x) = f(t,x), & (t,x)\in(0,+\infty)\times(0,1), \\ u(0,x) = \varphi(x), \quad u(+\infty,x) = 0, & x\in(0,1), \\ u(t,0) = u(t,1) = 0. \end{cases}$$

Indeed, in $E = C([0,1])$, we can choose the operator A as follows:

$$\begin{aligned} D(A) &= \{v \in C^2([0,1]) : v(0) = v(1) = 0\} \subset E, \\ (Av)(x) &= v''(x), \quad v\in D(A). \end{aligned}$$

Several authors have studied equation (1.1) under various (homogeneous or nonhomogeneous) boundary conditions, as well as in the case of variable operators, but on a finite interval. See, for instance, Krein [5], Sobolevskii [11], Véron [13], Kuyazyuk [6], Da Prato-Grisvard [9], Labbas [7].

Our approach is based on the direct use of the operational calculus and of Dunford's integrals as in Da Prato-Grisvard [9]. We make use of the real Banach interpolation spaces $D_A(\theta,+\infty)$, between $D(A)$ and E, which are well-known in many concrete cases and can be characterized by

$$D_A(\theta,+\infty) = \left\{\xi\in E : \sup_{t>0}\left\|t^\theta A(A-tI)^{-1}\xi\right\|_E < \infty\right\},$$

where $\theta \in (0, 1)$ (see Grisvard [4]).

Assumption (1.3) does not imply that A is an infinitesimal generator of an analytical semigroup. However it allows us to define $(-A)^{1/2}$, (for details, see Balakrishnan [1]). We do not use this fractional power of the operator nor the techniques of semigroups estimates generated by them as in Krein [5].

Our main results are the following.

THEOREM 1.1. *Let $\varphi \in D(A)$ and $f \in C^{2\theta}(E)$, with $\theta \in (0, 1/2)$ such that*

$$f(0) - A\varphi \in \overline{D(A)}.$$

Then Problem (1.1)–(1.2) *admits a unique strict solution.*

THEOREM 1.2. *Let $\varphi \in D(A)$ and $f \in C^{2\theta}(E)$, with $\theta \in (0, 1/2)$ such that*

$$f(0) - A\varphi \in D_A(\theta, +\infty).$$

Then the unique strict solution of (1.1)–(1.2) *satisfies the property of maximal regularity $Au(\cdot)$, $u''(\cdot) \in C^{2\theta}(E)$.*

If f is in a L^p-Lebesgue space, we can prove that the same representation given in (2.1) leads to the existence of a strict solution, because Lebesgue spaces satisfy the so-called UMD property. This no longer holds true for Hölder spaces; so the function needs more regularity (see, for instance, Favini *et al.* [3]).

The paper is organized as follows. In Section 2, we give the natural representation of the solution u to Problem (1.1)–(1.2) by using the operational calculus and Dunford's integral. In Section 3, we give necessary and sufficient conditions on φ and $A\varphi - f(0)$ to obtain the maximal smoothness of the solution u given by Dunford's integral when f is Hölderian. In Section 4, we present an example, to which our abstract results can be applied.

2. Construction of the solution

If A is a complex scalar z such that $\operatorname{Re}\sqrt{-z}$ is positive, then the solution of (1.1)–(1.2) is given by

$$u(t) = e^{-\sqrt{-z}t}\varphi - \int_0^{+\infty} k(t,s) f(s)\, ds,$$

where

$$k(t,s) = \begin{cases} \dfrac{e^{-\sqrt{-z}t}\sinh\sqrt{-z}s}{\sqrt{-z}} & 0 \leqslant s \leqslant t, \\ \dfrac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}t}{\sqrt{-z}} & s \geqslant t. \end{cases}$$

Here $\sqrt{-z}$ is the analytic determination defined by $\operatorname{Re}\sqrt{-z} > 0$. In the general case, it is well known that Hypothesis (1.3) implies the existence of $\delta_0 \in (0, \pi/2)$ and $\epsilon_0 > 0$ such that the resolvent set of A contains a sectorial domain of the form

$$S_{\epsilon_0,\delta_0} = S = \left\{\lambda \in \mathbb{C}^* : |\arg\lambda| \leqslant \delta_0\right\} \cup \overline{B(O, \epsilon_0)},$$

where $B(O, \epsilon_0)$ is an open ball of radius ϵ_0. If γ denotes the sectorial boundary curve of S_{ϵ_0,δ_0} oriented positively, remaining in $\rho(A)\backslash\mathbb{R}_+$, and defined by

$$\gamma = \left\{z \in \mathbb{C} : |z| \geqslant \epsilon_0 \text{ and } |\arg z| = \delta_0\right\} \cup \left\{z = \epsilon_0 e^{iv} : \delta_0 \leqslant v \leqslant 2\pi - \delta_0\right\},$$

then the natural representation of the solution of (1.1)–(1.2), in the abstract case, is given by Dunford's integral

$$\begin{aligned} u(t) = {}& \frac{1}{2i\pi}\int_\gamma e^{-\sqrt{-z}t}(A - zI)^{-1}\varphi\,dz \\ & - \frac{1}{2i\pi}\int_\gamma\int_0^{+\infty} k(t,s)(A - zI)^{-1}f(s)\,ds\,dz. \end{aligned} \tag{2.1}$$

These integrals converge absolutely for every $t \in (0, +\infty)$. Indeed, we have for $z \in \gamma$, $\left|e^{-\sqrt{-z}t}\right| \leqslant e^{-\operatorname{Re}\sqrt{-z}t} = e^{-c_0|z|^{1/2}t}$, where $c_0 = \cos((\pi - \delta_0)/2)$. On the other hand, for any $f \in X$, we see that

$$\left\|\int_0^{+\infty} k(t,s)(A - zI)^{-1}f(s)\,ds\right\|_E \leqslant \frac{1}{c_0|z|^2}\|f\|_X.$$

Set, for $\varphi \in E$ and $t \in (0, +\infty)$,

$$B(t, A)\varphi = \frac{1}{2i\pi}\int_\gamma e^{-\sqrt{-z}t}(A - zI)^{-1}\varphi\,dz.$$

Then we have the following result, which allows us to study the properties of the solution u.

PROPOSITION 2.1. *Under Assumption* (1.3) *we have*

(1) *that there exists* $K > 0$ *depending only on* γ *such that for all* $\varphi \in E$ *and for all* $t > 0$, $\|B(t, A)\varphi\|_E \leqslant K\|\varphi\|_E$;
(2) *for all* $\varphi \in E$ *and for all* $t > 0$, $B(t, A)\varphi \in D(A)$;
(3) $B(\cdot, A)\varphi \in X$ *if and only if* $\varphi \in \overline{D(A)}$.

PROOF. (1) For $t > 0$, we can write

$$\begin{aligned} B(t, A)\varphi &= \frac{1}{2i\pi}\int_{\gamma_t^+} e^{-\sqrt{-z}t}(A - zI)^{-1}\varphi\,dz + \frac{1}{2i\pi}\int_{\gamma_t^-} e^{-\sqrt{-z}t}(A - zI)^{-1}\varphi\,dz \\ &= I_+ + I_-, \end{aligned}$$

where

$$\gamma_t^+ = \{z \in \gamma : |z| \geqslant 1/t^2\}, \quad \gamma_t^- = \{z \in \gamma : |z| \leqslant 1/t^2\}. \tag{2.2}$$

Then

$$\|I_+\|_E \leqslant K \int_{\gamma_t^+} e^{-(\mathrm{Re}\sqrt{-z})t} \frac{|dz|}{|z|} \|\varphi\|_E \leqslant K \int_1^{+\infty} \frac{e^{-c_0\sigma}}{\sigma^2/t^2} \frac{2\sigma}{t^2} \, d\sigma \, \|\varphi\|_E \leqslant K \|\varphi\|_E,$$

and

$$I_- = \frac{1}{2i\pi} \int_{\gamma_t^-} (e^{-\sqrt{-z}t} - 1)(A - zI)^{-1}\varphi \, dz - \frac{1}{2i\pi} \int_{C_t} (A - zI)^{-1}\varphi \, dz = I_-' + I_-'',$$

where $C_t = \{z = e^{iv}/t^2 : -\delta_0 \leqslant v \leqslant \delta_0\}$. For I_-', we write

$$\begin{aligned} I_-' = {} & \frac{1}{2i\pi} \int_{z \in \gamma_t^-,\ \epsilon_0 \leqslant |z| \leqslant 1/t^2} (e^{-\sqrt{-z}t} - 1)(A - zI)^{-1}\varphi \, dz \\ & + \frac{1}{2i\pi} \int_{z \in \gamma_t^-,\ z = \epsilon_0 e^{iv}} (e^{-\sqrt{-z}t} - 1)(A - zI)^{-1}\varphi \, dz = J_-^1 + J_-^2, \end{aligned}$$

then

$$\|J_-^1\|_E \leqslant K \int_{\epsilon_0}^{1/t^2} \frac{|e^{-\sqrt{-z}t} - 1|}{|z|} \, d|z| \|\varphi\|_E \leqslant K \int_0^{1/t^2} |z|^{1/2} t \frac{d|z|}{|z|} \|\varphi\|_E \leqslant K |\varphi\|_E,$$

$$\|J_-^2\|_E \leqslant \frac{2\epsilon_0}{2\pi} \int_{\delta_0}^{2\pi - \delta_0} \left\| (A - \epsilon_0 e^{iv} I)^{-1}\varphi \right\|_E dv \leqslant K \|\varphi\|_E.$$

For I_-'' , we have

$$\|I_-''\|_E \leqslant K \int_{-\delta_0}^{+\delta_0} \left\| \left(A - \frac{1}{t^2} e^{iv} \right)^{-1} \varphi \right\|_E \frac{1}{t^2} \, dv \leqslant K \|\varphi\|_E.$$

(2) This rises from the convergence of the integral

$$\frac{1}{2i\pi} \int_\gamma e^{-\sqrt{-z}t} A(A - zI)^{-1}\varphi \, dz,$$

and from the fact that $e^{-\sqrt{-z}t}(A - zI)^{-1}\varphi \in D(A)$ for all $\varphi \in E$ and $t > 0$.

(3) Fix $\epsilon > 0$ and let $\varphi \in \overline{D(A)}$, then there exists $\psi \in D(A)$ such that

$$\|\varphi - \psi\|_E \leqslant \epsilon. \tag{2.3}$$

Using the resolvent's identity $(A - zI)^{-1}A\psi = \psi + z(A - zI)^{-1}\psi$, we obtain

$$B(t, A)\psi = \frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z}(A - zI)^{-1}A\psi dz.$$

Thanks to the inequality

$$\left\| \frac{e^{-\sqrt{-z}\,t}}{z}(A - zI)^{-1}A\psi \right\|_E \leqslant \frac{K}{|z|^2}\|A\psi\|_E,$$

Lebesgue's and Cauchy's theorems give us

$$\lim_{t\to 0^+} B(t, A)\psi = \frac{1}{2i\pi}\int_\gamma \frac{(A - zI)^{-1}}{z}A\psi\, dz = \psi.$$

Now from the equality

$$B(t, A)\varphi - \varphi = (B(t, A)\varphi - B(t, A)\psi) + (B(t, A)\psi - \psi) + (\psi - \varphi),$$

and from the estimate (2.3), we deduce that $B(t, A)\varphi - \varphi \to 0$ as $t \to 0^+$. The uniform continuity in $t > 0$ is easily verified.

Conversely, if $B(\cdot, A)\varphi \in X$, then for $z \in \gamma$ one has

$$\begin{aligned}(A - zI)^{-1}\lim_{t\to 0^+} B(t, A)\varphi &= \lim_{t\to 0^+}(A - zI)^{-1}B(t, A)\varphi \\ &= \lim_{t\to 0^+} B(t, A)(A - zI)^{-1}\varphi \\ &= (A - zI)^{-1}\varphi,\end{aligned}$$

wich implies that $\varphi = \lim_{t\to 0^+} B(t, A)\varphi \in \overline{D(A)}$. □

PROPOSITION 2.2. *Under Assumption* (1.3) *and for* $\theta \in (0, 1/2)$ *we have*

$$B(\cdot, A)\varphi \in C^{2\theta}(E) \quad \textit{if and only if} \quad \varphi \in D_A(\theta, +\infty).$$

PROOF. Let $\varphi \in D_A(\theta, +\infty)$ and $0 \leqslant \tau < t$. Thus

$$\begin{aligned}(\tau, A)\varphi - B(t, A)\varphi &= \frac{1}{2i\pi}\int_\gamma \left(e^{-\sqrt{-z}\,\tau} - e^{-\sqrt{-z}\,t}\right)\frac{A(A - zI)^{-1}}{z}\varphi\, dz \\ &= \frac{1}{2i\pi}\int_{z\in\gamma, |z|\geqslant(t-\tau)^{-2}} \left(e^{-\sqrt{-z}\,\tau} - e^{-\sqrt{-z}\,t}\right)\frac{A(A - zI)^{-1}}{z}\varphi\, dz \\ &\quad + \frac{1}{2i\pi}\int_{z\in\gamma, |z|\leqslant(t-\tau)^{-2}} \left(e^{-\sqrt{-z}\,\tau} - e^{-\sqrt{-z}\,t}\right)\frac{A(A - zI)^{-1}}{z}\varphi\, dz,\end{aligned}$$

and

$$
\begin{aligned}
&\left\| B(\tau, A)\varphi - B(t, A)\varphi \right\|_E \\
&\quad \leqslant K \int_{z\in\gamma, |z|\geqslant (t-\tau)^{-2}} \frac{|dz|}{|z|^{\theta+1}} \|\varphi\|_{D_A(\theta,+\infty)} \\
&\qquad + K' \int_{z\in\gamma, |z|\leqslant (t-\tau)^{-2}} \frac{e^{-(\mathrm{Re}\sqrt{-z})\tau} |z|^{1/2}(t-\tau)}{|z|^{\theta+1}} |dz| \|\varphi\|_{D_A(\theta,+\infty)} \\
&\quad \leqslant K \int_{z\in\gamma, |z|\geqslant (t-\tau)^{-2}} \frac{|dz|}{|z|^{\theta+1}} \|\varphi\|_{D_A(\theta,+\infty)} \\
&\qquad + K' \int_{z\in\gamma, |z|\leqslant (t-\tau)^{-2}} \frac{|z|^{1/2}(t-\tau)}{|z|^{\theta+1}} |dz| \|\varphi\|_{D_A(\theta,+\infty)} \\
&\quad \leqslant \max(K, K')(t-\tau)^{2\theta} \|\varphi\|_{D_A(\theta,+\infty)}.
\end{aligned}
$$

For the proof of the direct sense, we know (see Sinestrari [10]) that if B is the infinitesimal generator of an analytic semigroup $V(t)$, then $V(t)\varphi - \varphi = O(t^\alpha)$, as $t \to 0^+$, if and only if $\varphi \in D_B(\alpha, +\infty)$. Observe that $-(-A)^{1/2}$ is the infinitesimal generator of the analytic semigroup $V(t) = B(t, A)$ (see Krein [5]). Therefore,

$$\varphi \in D_{(-A)^{1/2}}(2\theta, +\infty) = D_A(\theta, +\infty).$$

The last equality holds by using the reiteration theorem in interpolation theory (see Lions-Peetre [8]). $\square$

3. Smoothness of the solution

Now we can state some regularity properties of u and Au.

PROPOSITION 3.1. *Let $\varphi \in D(A)$ and $f \in C^{2\theta}(E)$, with $\theta \in (0, 1/2)$. Then*

(1) *for all $t \geqslant 0$, $u(t) \in D(A)$;*
(2) *$u(\cdot)$ and $u'(\cdot) \in X$;*
(3) *$S(\cdot) = Au(\cdot) - B(\cdot, A)(A\varphi - f(0)) \in C^{2\theta}(E)$;*
(4) *$Au(\cdot) \in X$ if and only if $A\varphi - f(0) \in \overline{D(A)}$;*
(5) *$Au(\cdot) \in C^{2\theta}(E)$ if and only if $A\varphi - f(0) \in D_A(\theta, +\infty)$.*

PROOF. (1) In the second integral in (2.1) we write $f(s) = (f(s) - f(t)) + f(t)$. Then, after a calculation of the integral in $f(t)$, we get

$$u(t) = B(t, A)\varphi - \frac{1}{2i\pi} \int_\gamma \int_0^{+\infty} k(t, s)(A - zI)^{-1}\big(f(s) - f(t)\big)\, ds\, dz$$

$$
\begin{aligned}
&- \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z} (A - zI)^{-1} f(t)\, dz \\
&+ \frac{1}{2i\pi} \int_\gamma \frac{1}{z} (A - zI)^{-1} f(t)\, dz \\
= {}& I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

From Proposition 2.1, we deduce that $I_1 \in D(A)$. The convergence of the integral

$$
\frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z} A(A - zI)^{-1} f(t)\, dz
$$

implies that $I_3 \in D(A)$. For I_4 we use the Cauchy theorem, which gives

$$
I_4 = A^{-1} f(t) \in D(A).
$$

Finally for I_2, it is sufficient to prove the convergence of the integral

$$
\frac{1}{2i\pi} \int_\gamma \int_0^{+\infty} k(t, s) A(A - zI)^{-1} (f(s) - f(t))\, ds\, dz.
$$

In fact, we have

$$
\begin{aligned}
&\left\| \frac{1}{2i\pi} \int_\gamma \int_0^{+\infty} k(t, s) A(A - zI)^{-1} (f(s) - f(t))\, ds\, dz \right\|_E \\
&\quad \leqslant K \int_{\epsilon_0}^{+\infty} \left(\sup_{t>0} \int_0^{+\infty} |k(t, s)| |t - s|^{2\theta}\, ds \right) d|z| \|f\|_{C^{2\theta}(E)} \\
&\quad \leqslant K \int_{\epsilon_0}^{+\infty} \frac{d|z|}{|z|^{1+\theta}} \|f\|_{C^{2\theta}(E)},
\end{aligned}
$$

where the last estimate holds by Hölder's inequality.

(2) Let us prove that $u(\cdot) \in X$ (the same techniques give the result $u'(\cdot) \in X$). For $0 \leqslant \tau < t$, we get $u(t) - u(\tau) = I + \Delta_1 + \Delta_2 + J$, where

$$
\begin{aligned}
I &= B(t, A)\varphi - B(\tau, A)\varphi, \qquad J = A^{-1}\big(f(t) - f(\tau)\big), \\
\Delta_1 &= -\frac{1}{2i\pi} \int_\gamma \int_0^{+\infty} k(t, s)(A - zI)^{-1} (f(s) - f(t))\, ds\, dz \\
&\quad + \frac{1}{2i\pi} \int_\gamma \int_0^{+\infty} k(\tau, s)(A - zI)^{-1} (f(s) - f(\tau))\, ds\, dz, \\
\Delta_2 &= -\frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z} (A - zI)^{-1} f(t)\, dz + \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}\,\tau}}{z} (A - zI)^{-1} f(\tau)\, dz.
\end{aligned}
$$

The result is obvious for I, since $\varphi \in D(A)$. For J we use Assumption (1.3).

On the other hand, we have

$$\begin{aligned}\Delta_2 &= \frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}t}}{z}(A-zI)^{-1}(f(\tau)-f(t))\,dz \\ &\quad - \frac{1}{2i\pi}\int_\gamma \frac{\left(e^{-\sqrt{-z}t}-e^{-\sqrt{-z}\tau}\right)}{z}(A-zI)^{-1}f(\tau)\,dz \\ &= \Delta_2' + \Delta_2'',\end{aligned}$$

where

$$\left\|\Delta_2'\right\|_E \leqslant \left(\frac{1}{2\pi}\int_{\epsilon_0}^{+\infty}\frac{|dz|}{|z|^2}\right)(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)} \leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)},$$

and writing Δ_2'' as

$$\Delta_2'' = \frac{1}{2i\pi}\int_\gamma \frac{1}{z}\left(\int_\tau^t \sqrt{-z}e^{-\sqrt{-z}\xi}\,d\xi\right)(A-zI)^{-1}f(\tau)\,dz,$$

we obtain

$$\left\|\Delta_2''\right\|_E \leqslant \left(\frac{1}{2\pi}\int_{\epsilon_0}^{+\infty}\frac{|dz|}{|z|^{3/2}}\right)(t-\tau)\|f\|_X \leqslant K(t-\tau)\|f\|_X.$$

Now for the quantity Δ_1, one has

$$\begin{aligned}\Delta_1 &= -\frac{1}{2i\pi}\int_\gamma\int_\tau^t \frac{e^{-\sqrt{-z}t}\sinh\sqrt{-z}\,s}{\sqrt{-z}}(A-zI)^{-1}(f(s)-f(t))\,ds\,dz \\ &\quad + \frac{1}{2i\pi}\int_\gamma\int_\tau^t \frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}\,\tau}{\sqrt{-z}}(A-zI)^{-1}(f(s)-f(\tau))\,ds\,dz \\ &\quad - \frac{1}{2i\pi}\int_\gamma\int_0^\tau \frac{e^{-\sqrt{-z}t}\sinh\sqrt{-z}\,s}{\sqrt{-z}}(A-zI)^{-1}(f(\tau)-f(t))\,ds\,dz \\ &\quad - \frac{1}{2i\pi}\int_\gamma\int_0^\tau \frac{e^{-\sqrt{-z}t}-e^{-\sqrt{-z}\tau}}{\sqrt{-z}}\sinh\sqrt{-z}\,s(A-zI)^{-1}(f(s)-f(\tau))\,ds\,dz \\ &\quad - \frac{1}{2i\pi}\int_\gamma\int_t^{+\infty} \frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}\,\tau}{\sqrt{-z}}(A-zI)^{-1}(f(\tau)-f(t))\,ds\,dz \\ &\quad - \frac{1}{2i\pi}\int_\gamma\int_t^{+\infty} \frac{\left(\sinh\sqrt{-z}\,t-\sinh\sqrt{-z}\,\tau\right)}{e^{\sqrt{-z}s}\sqrt{-z}}(A-zI)^{-1}(f(s)-f(t))\,ds\,dz \\ &= \sum_{i=1}^{i=6} I_i.\end{aligned}$$

Then

$$\|I_1\|_E \leqslant K \int_\tau^t \left(\int_{\epsilon_0}^{+\infty} \frac{|dz|}{|z|^{3/2}} \right) (t-s)^{2\theta}\, ds \|f\|_{C^{2\theta}(E)} \leqslant K(t-\tau)^{2\theta+1} \|f\|_{C^{2\theta}(E)},$$

and

$$\|I_2\|_E \leqslant K \int_\tau^t \left(\int_{\epsilon_0}^{+\infty} \frac{|dz|}{|z|^{3/2}} \right) (s-\tau)^{2\theta}\, ds \|f\|_{C^{2\theta}(E)} \leqslant K(t-\tau)^{2\theta+1} \|f\|_{C^{2\theta}(E)}.$$

Writing I_4 and I_6 as

$$I_4 = \frac{1}{2i\pi} \int_\gamma \int_0^\tau \left(\int_\tau^t e^{-\sqrt{-z}\xi}\, d\xi \right) \sinh \sqrt{-z} s (A - zI)^{-1} (f(s) - f(\tau))\, ds\, dz,$$

$$I_6 = -\frac{1}{2i\pi} \int_\gamma \int_t^{+\infty} e^{-\sqrt{-z}s} \left(\int_\tau^t \cosh \sqrt{-z}\, \xi\, d\xi \right) (A - zI)^{-1} (f(s) - f(t))\, ds\, dz,$$

we get

$$\begin{aligned} \|I_4\|_E &\leqslant \frac{1}{2\pi} \int_{\epsilon_0}^{+\infty} \left(\int_0^\tau \int_\tau^t e^{-\operatorname{Re}\sqrt{-z}(\xi - s)} (\tau - s)^{2\theta}\, d\xi\, ds \right) \frac{|dz|}{1+|z|} \|f\|_{C^{2\theta}(E)} \\ &\leqslant K \int_0^\tau \int_{\tau-s}^{t-s} \frac{(\tau-s)^{2\theta}}{\eta^2} \left(\int_0^{+\infty} e^{-c_0\sigma} 2\sigma\, d\sigma \right) d\eta\, ds \|f\|_{C^{2\theta}(E)} \\ &\leqslant K \int_0^\tau (\tau-s)^{2\theta} \left(\frac{1}{\tau-s} - \frac{1}{t-s} \right) ds \|f\|_{C^{2\theta}(E)} \\ &\leqslant K(t-\tau) \int_0^\tau \frac{(\tau-s)^{2\theta-1}}{(t-\tau+\tau-s)}\, ds \|f\|_{C^{2\theta}(E)}, \end{aligned}$$

by making the change of variable $(\tau - s) = \xi(t-\tau)$, we obtain

$$\|I_4\|_E \leqslant K(t-\tau)^{2\theta} \left(\int_0^{+\infty} \frac{\xi^{2\theta-1}}{(1+\xi)}\, d\xi \right) \|f\|_{C^{2\theta}(E)} \leqslant K(t-\tau)^{2\theta} \|f\|_{C^{2\theta}(E)}.$$

For I_6, we have

$$\begin{aligned} \|I_6\|_E &\leqslant K \int_{\epsilon_0}^{+\infty} \int_t^{+\infty} \int_\tau^t e^{-\operatorname{Re}\sqrt{-z}(s-\xi)} (s-t)^{2\theta} d\xi\, ds \frac{|dz|}{1+|z|} \|f\|_{C^{2\theta}(E)} \\ &\leqslant K \int_t^{+\infty} \int_\tau^t \left(\int_0^{+\infty} e^{-c_0\sigma} \sigma\, d\sigma \right) \frac{(s-t)^{2\theta}}{(s-\xi)^2} d\xi ds \|f\|_{C^{2\theta}(E)} \\ &\leqslant K \int_t^{+\infty} (s-t)^{2\theta} \left(\int_{s-t}^{s-\tau} \frac{d\eta}{\eta^2} \right) ds \|f\|_{C^{2\theta}(E)} \\ &\leqslant K(t-\tau) \int_t^{+\infty} \frac{(s-t)^{2\theta-1}}{(s-t+t-\tau)}\, ds \|f\|_{C^{2\theta}(E)} \end{aligned}$$

$$
\begin{aligned}
&\leqslant K(t-\tau)^{2\theta}\left(\int_0^{+\infty}\frac{\rho^{2\theta-1}}{(1+\rho)}\,d\rho\right)\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)}.
\end{aligned}
$$

A direct calculation of the integrals in $(f(\tau)-f(t))$ implies that

$$
\begin{aligned}
I_3+I_5 &= \frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}(t-\tau)}}{z}(A-zI)^{-1}(f(\tau)-f(t))\,dz\\
&\quad-\frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z}(A-zI)^{-1}(f(\tau)-f(t))\,dz\\
&= J-\Delta_2'.
\end{aligned}
$$

We write J as

$$
\begin{aligned}
J &= \frac{1}{2i\pi}\int_{\gamma^+_{(t-\tau)}} \frac{e^{-\sqrt{-z}(t-\tau)}}{z}(A-zI)^{-1}(f(\tau)-f(t))\,dz\\
&\quad+\frac{1}{2i\pi}\int_{\gamma^-_{(t-\tau)}} \frac{e^{-\sqrt{-z}(t-\tau)}}{z}(A-zI)^{-1}(f(\tau)-f(t))\,dz\\
&= J^+ + J^-,
\end{aligned}
$$

(where $\gamma^+_{(t-\tau)}$ and $\gamma^-_{(t-\tau)}$ are defined in (2.2)), then

$$
\begin{aligned}
\|J^+\|_E &\leqslant K\int_1^{+\infty}\frac{e^{-\operatorname{Re}\sqrt{-z}(t-\tau)}}{|z|}\,\frac{(t-\tau)^{2\theta}}{1+|z|}\,|dz|\,\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)^{2\theta}\left(\int_1^{+\infty}\frac{e^{-\operatorname{Re}\sqrt{-z}(t-\tau)}}{|z|}\,|dz|\right)\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)^{2\theta}\left(\int_1^{+\infty}\frac{e^{-c_0\sigma}}{\sigma}\,d\sigma\right)\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)},
\end{aligned}
$$

and for J^- we have

$$
\begin{aligned}
J^- &= \frac{1}{2i\pi}\int_{\gamma^-_{(t-\tau)}}\left(e^{-\sqrt{-z}(t-\tau)}-1\right)\frac{(A-zI)^{-1}(f(\tau)-f(t))}{z}\,dz\\
&\quad+A^{-1}(f(\tau)-f(t))-\frac{1}{2i\pi}\int_{C_{(t-\tau)}}\frac{(A-zI)^{-1}(f(\tau)-f(t))}{z}\,dz\\
&= J_1^- + A^{-1}(f(\tau)-f(t)) + J_2^-,
\end{aligned}
$$

with $C_{(t-\tau)}=\left\{z=(t-\tau)^{-2}e^{i\nu} : -\delta_0\leqslant\nu\leqslant\delta_0\right\}$.

So we obtain

$$\begin{aligned}\|J_1^-\|_E &\leqslant K\int_0^{1/(t-\tau)^2}\frac{|z|^{1/2}(t-\tau)}{|z|}\frac{(t-\tau)^{2\theta}}{1+|z|}|dz|\|f\|_{C^{2\theta}(E)}\\&\leqslant K(t-\tau)^{2\theta+1}\int_0^{1/(t-\tau)^2}\frac{|z|^{1/2}}{|z|}|dz|\|f\|_{C^{2\theta}(E)}\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)},\end{aligned}$$

and

$$\|J_2^-\|_E\leqslant K\frac{(t-\tau)^{2\theta}}{1+(t-\tau)^{-2}}\int_{-\delta_0}^{\delta_0}(t-\tau)^2\frac{dv\|f\|_{C^{2\theta}(E)}}{(t-\tau)^2}\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)}.$$

(3) We have

$$\begin{aligned}S(t)&=\frac{1}{2i\pi}\int_\gamma\frac{e^{-\sqrt{-z}t}}{z}A(A-zI)^{-1}A\varphi\,dz\\&\quad-\frac{1}{2i\pi}\int_\gamma\int_0^{+\infty}k(t,s)A(A-zI)^{-1}(f(s)-f(t))\,ds\,dz\\&\quad-\frac{1}{2i\pi}\int_\gamma\frac{e^{-\sqrt{-z}t}}{z}A(A-zI)^{-1}f(t)\,dz+f(t)\\&\quad-\frac{1}{2i\pi}\int_\gamma\frac{e^{-\sqrt{-z}t}}{z}A(A-zI)^{-1}(A\varphi-f(0))\,dz,\end{aligned}$$

thus

$$\begin{aligned}S(t)&=\frac{1}{2i\pi}\int_\gamma\frac{e^{-\sqrt{-z}t}}{z}A(A-zI)^{-1}f(0)\,dz\\&\quad-\frac{1}{2i\pi}\int_\gamma\int_0^{+\infty}k(t,s)A(A-zI)^{-1}(f(s)-f(t))\,ds\,dz\\&\quad-\frac{1}{2i\pi}\int_\gamma\frac{e^{-\sqrt{-z}t}}{z}A(A-zI)^{-1}f(t)\,dz+f(t).\end{aligned}$$

Let $0\leqslant\tau<t$, then $S(t)-S(\tau)=f(t)-f(\tau)+\Lambda+\Pi$, where

$$\begin{aligned}\Lambda&=\frac{1}{2i\pi}\int_\gamma\int_0^{+\infty}k(\tau,s)A(A-zI)^{-1}(f(s)-f(\tau))\,dsdz\\&\quad-\frac{1}{2i\pi}\int_\gamma\int_0^{+\infty}k(t,s)A(A-zI)^{-1}(f(s)-f(t))\,dsdz,\end{aligned}$$

and

$$\Pi=\frac{1}{2i\pi}\left(\int_\gamma\frac{e^{-\sqrt{-z}t}}{z}A(A-zI)^{-1}f(0)\,dz-\int_\gamma\frac{e^{-\sqrt{-z}t}}{z}A(A-zI)^{-1}f(t)\,dz\right)$$

$$-\frac{1}{2i\pi}\left(\int_\gamma \frac{e^{-\sqrt{-z}\,\tau}}{z}A(A-zI)^{-1}f(0)\,dz-\int_\gamma \frac{e^{-\sqrt{-z}\,\tau}}{z}A(A-zI)^{-1}f(\tau)\,dz\right).$$

Regarding the quantity Π, we have

$$\begin{aligned}
\Pi &= \frac{1}{2i\pi}\int_\gamma \frac{\left(e^{-\sqrt{-z}\,t}-e^{-\sqrt{-z}\tau}\right)}{z}A(A-zI)^{-1}f(0)\,dz\\
&\quad+\frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}\,\tau}}{z}A(A-zI)^{-1}f(\tau)\,dz\\
&\quad+\frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z}A(A-zI)^{-1}f(\tau)\,dz\\
&\quad-\frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z}A(A-zI)^{-1}f(\tau)\,dz\\
&\quad-\frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z}A(A-zI)^{-1}f(t)\,dz\\
&= -\frac{1}{2i\pi}\int_\gamma \frac{\left(e^{-\sqrt{-z}\,t}-e^{-\sqrt{-z}\tau}\right)}{z}A(A-zI)^{-1}\left(f(\tau)-f(0)\right)dz\\
&\quad+\frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z}A(A-zI)^{-1}\left(f(\tau)-f(t)\right)dz\\
&= \Pi_1+\Pi_2.
\end{aligned}$$

From Proposition 2.1, we deduce that

$$\|\Pi_2\|_E = \|B(t,A)(f(\tau)-f(t))\|_E \leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)}.$$

For Π_1, we get

$$\|\Pi_1\|_E \leqslant K\int_\tau^t\int_{\epsilon_0}^{+\infty}\frac{|z|^{1/2}e^{-c_0|z|^{1/2}s}\tau^{2\theta}}{|z|}\,|dz|\,ds\,\|f\|_{C^{2\theta}(E)},$$

and setting $|z|^{1/2}s=\sigma$ in this last inequality, we obtain

$$\begin{aligned}
\|\Pi_1\|_E &\leqslant K\int_\tau^t\frac{\tau^{2\theta}}{s}\left(\int_0^{+\infty}e^{-c_0\sigma}\,d\sigma\right)ds\,\|f\|_{C^{2\theta}(E)} \leqslant K\left(\int_\tau^t s^{2\theta-1}\,ds\right)\|f\|_{C^{2\theta}(E)}\\
&\leqslant K\left(t^{2\theta}-\tau^{2\theta}\right)\|f\|_{C^{2\theta}(E)} \leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)}.
\end{aligned}$$

Λ may be written as

$$\Lambda = -\frac{1}{2i\pi}\int_\gamma\int_\tau^t\frac{e^{-\sqrt{-z}\,t}\sinh\sqrt{-z}\,s}{\sqrt{-z}}A(A-zI)^{-1}(f(s)-f(t))\,ds\,dz$$

$$
\begin{aligned}
&+\frac{1}{2i\pi}\int_\gamma\int_\tau^t \frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}\,\tau}{\sqrt{-z}}A(A-zI)^{-1}(f(s)-f(\tau))\,ds\,dz\\
&-\frac{1}{2i\pi}\int_\gamma\int_0^\tau \frac{e^{-\sqrt{-z}t}\sinh\sqrt{-z}\,s}{\sqrt{-z}}A(A-zI)^{-1}(f(\tau)-f(t))\,ds\,dz\\
&-\frac{1}{2i\pi}\int_\gamma\int_0^\tau \frac{\left(e^{-\sqrt{-z}t}-e^{-\sqrt{-z}\tau}\right)}{\sqrt{-z}}\sinh\sqrt{-z}\,s\\
&\quad\times A(A-zI)^{-1}(f(s)-f(\tau))\,ds\,dz\\
&-\frac{1}{2i\pi}\int_\gamma\int_t^{+\infty} \frac{e^{-\sqrt{-z}s}\sinh\sqrt{-z}\,\tau}{\sqrt{-z}}A(A-zI)^{-1}(f(\tau)-f(t))\,ds\,dz\\
&-\frac{1}{2i\pi}\int_\gamma\int_t^{+\infty} \frac{e^{-\sqrt{-z}s}\left(\sinh\sqrt{-z}\,t-\sinh\sqrt{-z}\,\tau\right)}{\sqrt{-z}}\\
&\quad\times A(A-zI)^{-1}(f(s)-f(t))\,ds\,dz\\
=&\sum_{i=1}^{i=6}\Lambda_i.
\end{aligned}
$$

For Λ_1 and Λ_2, we prove that

$$
\begin{aligned}
\|\Lambda_1\|_E &\leqslant K\int_{\epsilon_0}^{+\infty}\int_\tau^t \frac{e^{-\operatorname{Re}\sqrt{-z}(t-s)}}{|z|^{1/2}}|t-s|^{2\theta}\,ds|dz|\|f\|_{C^{2\theta}(E)}\\
&\leqslant K\int_\tau^t (t-s)^{2\theta-1}\left(\int_0^{+\infty}e^{-c_0\sigma}\,d\sigma\right)ds\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)},\\
\|\Lambda_2\|_E &\leqslant K\int_{\epsilon_0}^{+\infty}\int_\tau^t \frac{e^{-\operatorname{Re}\sqrt{-z}(s-\tau)}}{|z|^{1/2}}|s-\tau|^{2\theta}\,ds|dz|\|f\|_{C^{2\theta}(E)}\\
&\leqslant K\int_\tau^t (s-\tau)^{2\theta-1}\left(\int_0^{+\infty}e^{-c_0\sigma}\,d\sigma\right)ds\|f\|_{C^{2\theta}(E)}\\
&\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)}.
\end{aligned}
$$

Writing Λ_4 and Λ_6 as

$$
\begin{aligned}
\Lambda_4 &= \frac{1}{2i\pi}\int_\gamma\int_0^\tau\left(\int_\tau^t e^{-\sqrt{-z}\xi}\,d\xi\right)\sinh\sqrt{-z}s\,A(A-zI)^{-1}(f(s)-f(\tau))\,ds\,dz,\\
\Lambda_6 &= -\frac{1}{2i\pi}\int_\gamma\int_t^{+\infty}e^{-\sqrt{-z}s}\left(\int_\tau^t\cosh\sqrt{-z}\xi\,d\xi\right)A(A-zI)^{-1}(f(s)-f(t))\,ds\,dz,
\end{aligned}
$$

we obtain

$$
\|\Lambda_4\|_E \leqslant K\int_{\epsilon_0}^{+\infty}\int_0^\tau\int_\tau^t e^{-\operatorname{Re}\sqrt{-z}(\xi-s)}(\tau-s)^{2\theta}\,d\xi\,ds|dz|\|f\|_{C^{2\theta}(E)}
$$

$$
\begin{aligned}
&\leqslant K \int_0^\tau \int_{\tau-s}^{t-s} \frac{(\tau-s)^{2\theta}}{\eta^2}\left(\int_0^{+\infty} e^{-c_0\sigma} 2\sigma \, d\sigma\right) d\eta \, ds \|f\|_{C^{2\theta}(E)} \\
&\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)},
\end{aligned}
$$

and

$$
\begin{aligned}
\|\Lambda_6\|_E &\leqslant \frac{1}{2\pi}\int_{\epsilon_0}^{+\infty}\int_t^{+\infty}\int_\tau^t e^{-\operatorname{Re}\sqrt{-z}(s-\xi)}(s-t)^{2\theta}\, d\xi \, ds|dz|\|f\|_{C^{2\theta}(E)} \\
&\leqslant K \int_t^{+\infty}\int_\tau^t \left(\int_0^{+\infty} e^{-c_0\sigma}\sigma \, d\sigma\right)\frac{(s-t)^{2\theta}}{(s-\xi)^2} d\xi \, ds \|f\|_{C^{2\theta}(E)} \\
&\leqslant K \int_t^{+\infty} (s-t)^{2\theta}\left(\int_{s-t}^{s-\tau}\frac{d\eta}{\eta^2}\right) ds \|f\|_{C^{2\theta}(E)} \\
&\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)}.
\end{aligned}
$$

By calculating the integrals in $(f(\tau)-f(t))$, we get

$$
\begin{aligned}
\Lambda_3+\Lambda_5 &= \frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}(t-\tau)}}{z} A(A-zI)^{-1}(f(\tau)-f(t))\, dz \\
&\quad - B(t, A)(f(\tau)-f(t)) \\
&= Q - \Pi_2,
\end{aligned}
$$

where $Q = B(t-\tau, A)(f(\tau)-f(t))$. Proposition 2.1 implies that

$$
\|Q\|_E \leqslant K(t-\tau)^{2\theta}\,\|f\|_{C^{2\theta}(E)}\,.
$$

(4) Assume that $A\varphi - f(0) \in \overline{D(A)}$. Then, by Proposition 2.1

$$
Au(\cdot) = S(\cdot) + B(\cdot, A)(A\varphi - f(0)) \in X.
$$

Conversely, if $Au(\cdot) \in X$, the function $B(\cdot, A)(A\varphi - f(0)) = Au(\cdot) - S(\cdot)$, is in X. Now using Proposition 2.1, we obtain $A\varphi - f(0) \in \overline{D(A)}$.

(5) Let us suppose that $A\varphi - f(0) \in D_A(\theta, +\infty)$. By Proposition 2.1, $Au(\cdot) \in C^{2\theta}(E)$. Conversely, if $Au(\cdot) \in C^{2\theta}(E)$, then $B(\cdot, A)(A\varphi - f(0)) \in C^{2\theta}(E)$, and $A\varphi - f(0) \in D_A(\theta, +\infty)$. □

REMARK. By using the same methods and techniques of calculation, we have a similar result to Proposition 3.1, when the right-hand term of the equation has a spatial smoothness, that is, for all $t \geqslant 0$, $f(t) \in D_A(\theta, +\infty)$, $\sup_{t\geq 0}\|f(t)\|_{D_A(\theta,+\infty)} < \infty$, with $\theta \in (0, 1/2)$. See [2] for details.

Now, we can deduce our main results concerning the regularity of u.

THEOREM 3.2. *Let* $\varphi \in D(A)$ *and* $f \in C^{2\theta}(E)$, *with* $\theta \in (0, 1/2)$ *such that*

$$f(0) - A\varphi \in \overline{D(A)}.$$

Then u, *given in* (2.1), *is a strict solution of* (1.1)–(1.2).

By the continuity of $B(\cdot, A)\varphi$ and Lebesgue's theorem we can verify that $u(0) = \varphi$ and $u(+\infty) = 0$. On the other hand, we prove that u verifies (1.1) by using Dunford's operational calculus.

Finally, by Proposition 3.1, the solution u belongs to

$$BUC^2([0, +\infty[\,; E) \cap BUC([0, +\infty[\,; D(A)).$$

Furthermore, if $f(0) - A\varphi \in D_A(\theta, +\infty)$, then we have more regularity on $Au(\cdot)$ and $u''(\cdot)$.

THEOREM 3.3. *Let* $\varphi \in D(A)$ *and* $f \in C^{2\theta}(E)$, *with* $\theta \in (0, 1/2)$ *such that*

$$f(0) - A\varphi \in D_A(\theta, +\infty).$$

Then the unique strict solution of (1.1)–(1.2) *satisfies the property of maximal regularity* $Au(\cdot), u''(\cdot) \in C^{2\theta}(E)$.

PROOF. It suffices to apply the previous results, using the fact that $D_A(\theta, +\infty) \subset \overline{D(A)}$. □

4. Example

We give an example governed by (1.1)–(1.2). Consider $E = C([0, 1])$ and

$$\begin{aligned} D(A) &= \left\{ v \in C^2([0, 1]) : v(0) = v(1) = 0 \right\}, \\ Av &= v''. \end{aligned}$$

It is easy to check that A satisfies Assumption (1.3). We can thus apply our results to Laplacian problem in an infinite interval, given by

$$\text{(4.1)} \quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t, x) + \dfrac{\partial^2 u}{\partial x^2}(t, x) = f(t, x), & (t, x) \in (0, +\infty) \times (0, 1), \\ u(0, x) = \varphi(x), & x \in (0, 1), \\ u(+\infty, x) = 0, & x \in (0, 1), \\ u(t, 0) = u(t, 1) = 0. \end{cases}$$

Observe that conditions $\varphi \in D(A)$ and $f(0) - A\varphi \in \overline{D(A)}$, become

$$
\begin{aligned}
&\varphi \in C^2([0,1]) : \varphi(0) = \varphi(1) = 0, \\
(4.2) \qquad &f(0,\cdot) - \varphi''(\cdot) \in C([0,1]) \quad \text{and} \\
&f(0,0) - \varphi''(0) = f(0,1) - \varphi''(1) = 0.
\end{aligned}
$$

The interpolation space $D_A(\theta, +\infty)$ is given by

$$
D_A(\theta, +\infty) = \begin{cases} \{v \in C^{2\theta}([0,1]) : v(0) = v(1) = 0\} & \text{if } 2\theta < 1, \\ C^{1,*}([0,1]), & \text{if } 2\theta = 1, \\ \{v \in C^{1,2\theta-1}([0,1]) : v(0) = v(1) = 0\} & \text{if } 2\theta > 1, \end{cases}
$$

where $C^{1,*}([0,1])$ is a Zigmund space (see, for example, [12]). Applying Theorem 3.2 and Theorem 3.3 we have the following results.

THEOREM 4.1. *Let* $f \in C^{2\theta}([0,+\infty[\,; C([0,1]))$, *with* $2\theta \in (0,1)$, *be such that the conditions* (4.2) *are satisfied. Then Problem* (4.1) *has a unique solution u satisfying* $u \in BUC^2([0,+\infty[\,; C([0,1])) \cap BUC([0,+\infty[\,; C^2([0,1]))$.

THEOREM 4.2. *Let* $f \in C^{2\theta}([0,+\infty[\,; C([0,1])$, *with* $2\theta \in (0,1)$, *be such that*

$$
\begin{cases} \varphi \in C^2([0,1]) : \varphi(0) = \varphi(1) = 0, \\ f(0,\cdot) - \varphi''(\cdot) \in C^{2\theta}([0,1]) \quad \text{and} \\ f(0,0) - \varphi''(0) = f(0,1) - \varphi''(1) = 0. \end{cases}
$$

Then Problem (4.1) *has a unique solution*

$$
u \in BUC^2([0,+\infty[\,; C([0,1])) \cap BUC([0,+\infty[\,; C^2([0,1])).
$$

Moreover, u satisfies the maximal regularity

$$
\partial_x^2 u(\cdot, x), \partial_t^2 u(\cdot, x) \in C^{2\theta}([0,+\infty[\,; C([0,1])).
$$

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