

THE MULTIDIRECTIONAL MEAN VALUE INEQUALITIES WITH SECOND ORDER INFORMATION

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Abstract

We give a multidirectional mean value inequality with second order information. This result extends the classical Clarke-Ledyaev's inequality to the second order. As application, we give the uniqueness of viscosity solution of second order Hamilton-Jacobi equations in finite dimensions.

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1. Introduction

In 1994, Clarke and Ledyaev proved a multidirectional mean value inequality for the Fréchet differentiable functions in Banach spaces [1], and for the lower semicontinuous functions in Hilbert spaces [2]. Using a similar technique as in [2], Clarke and Radulescu [3] extended the multidirectional mean value inequality for the locally Lipschitz continuous functions in smooth Banach spaces. These authors considered bounded sets of constraints. Recently, Zhu [10] generalized the result of Clarke and Radulescu to a non necessarily bounded set of constraints, where the functions are assumed to be lower semicontinuous (lsc) on smooth Banach spaces.

The main result of this paper is Theorem 1.3. It gives a second order generalization to the multidirectional mean value inequality of Clarke and Ledyaev. The results of this paper recover the mean value inequality establishes by Zhu in [10] and extend some work of Deville and Ivanov in [9]. On the other hand, our extension will permit to give the uniqueness of viscosity solution of second order Hamilton-Jacobi

equations in finite dimensions by a simple proof. Note that the notion of viscosity solution has been introduced by Crandall and Lions in [5]. In this paper, we develop our conclusions from a smooth variational principle due to Deville, Godefroy and Zizler in [6].

Let X be a real Banach space, we denote by X^* the set of all continuous linear forms on X , by $B_X(x, r)$ the closed ball with center x and radius r and by B_X the closed unit ball. For a point $x \in X$ and a subset C of X , we denote by $d(x, C) := \inf\{\|x - c\| : c \in C\}$ and $[x, C] := \{x + t(c - x) : c \in C, t \in [0, 1]\}$. We say that a Banach space X satisfies property (H) if there exists a C^2 bump function b on X such that b' is Lipschitz continuous. We denote by $\mathcal{B}(X)$ the space of all symmetric bilinear forms on X . Let Y be a closed subspace of X , we denote by X/Y the quotient space.

REMARK 1.1. Since property (H) is clearly hereditary and X/Y is isomorphic to a subspace of X when the complementation takes place, the space X/Y satisfies (H) . The Hilbert space situation is more trivial. However, property (H) fails the three-space property (see [7, Remark V.1.10]).

DEFINITION 1.2. Let X be a Banach space and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function. Suppose that $x \in X$ is such that $f(x) < +\infty$. The *viscosity (Fréchet) subdifferential of f at x* is defined as follows:

$$D^- f(x) := \{\phi'(x); \phi : X \rightarrow \mathbb{R} \text{ is } C^1 \text{ and } f - \phi \text{ has a local minimum at } x\}.$$

The *viscosity (Fréchet) subdifferential of second order of f at x* is defined as follows:

$$D^{2^-} f(x) := \{(\phi'(x), \phi''(x)); \phi : X \rightarrow \mathbb{R} \text{ is } C^2 \text{ and } f - \phi \text{ has a local minimum at } x\}.$$

Let X be a Banach space and let $(x^*, x^\mathcal{B}) \in X^* \times \mathcal{B}(X)$. We use the following notation: $\|x^*\| := \sup\{|x^*(x)| : x \in B_X\}$ and $\|x^\mathcal{B}\| := \sup\{|x^\mathcal{B}(x, x)| : x \in B_X\}$. For a closed subspace Y of X , we use the following notation: $\|x^*\|_{Y^*} := \sup\{|x^*(y)| : y \in B_Y\}$ and $\|x^\mathcal{B}\|_{\mathcal{B}(Y)} := \sup\{|x^\mathcal{B}(y, y)| : y \in B_Y\}$.

THEOREM 1.3. Let X be a Banach space satisfying (H) and Y be a closed subspace of X such that X/Y satisfies also (H) . There exists a constant $a_{X/Y} > 0$ satisfying the following result: Let $\hat{x} \in X$, $r \in \mathbb{R}$. Set $H := Y + C$, where C is a closed convex (not necessarily bounded) subset of X and $\Delta := [\hat{x}, H]$. Suppose that $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is lsc bounded below on $\Delta + hB_X$ for some $h > 0$ and that

$$\lim_{\eta \rightarrow 0} \inf_{y \in H + \eta B_X} f(y) > f(\hat{x}) + r.$$

Then, for all $\varepsilon > 0$, there exists $x_0 \in X$ and $(x_0^*, x_0^\mathcal{B}) \in D^{2^-} f(x_0)$ such that

- (i) $\|x_0^*\|_{Y^*} < \varepsilon$, $\|x_0^{\mathcal{B}}\|_{\mathcal{B}(Y)} < \varepsilon$ and $d(x_0, \Delta) < \varepsilon$;
- (ii) $r < \langle x_0^*, y - \hat{x} \rangle + \varepsilon \|y - \hat{x}\|$, $\forall y \in H$;
- (iii) $f(x_0) < \lim_{\eta \rightarrow 0} \inf_{y \in \Delta + \eta B_X} f(y) + |r| + \varepsilon$;
- (iv) $\|x_0^*\| < \varepsilon + \frac{a_{X/Y}}{\varepsilon} (\inf_{\Delta} f - \inf_{\Delta + h B_X} f)$; $\|x_0^{\mathcal{B}}\| < \varepsilon^2 + \frac{a_{X/Y}}{\varepsilon^2} (\inf_{\Delta} f - \inf_{\Delta + h B_X} f)$.

REMARK 1.4. (i) If we replace property (H) by the existence of a Lipschitz and C^1 bump function b on X , and if we set $Y = \{0\}$ in Theorem 1.3, then we recover the result of Zhu in [10].

(ii) If we suppose, in Theorem 1.3, that $\sup_{\delta} \inf_{\Delta + \delta B_X} f = \inf_{\Delta} f$, then (iv) can be replaced by: $d(x_0, \Delta) \|x_0^*\| < \varepsilon$ and $d^2(x_0, \Delta) \|x_0^{\mathcal{B}}\| < \varepsilon^2$.

For a subset S of X , we define the indicator function δ_S by

$$\delta_S(x) = \begin{cases} 0 & \text{if } x \in S; \\ +\infty & \text{otherwise.} \end{cases}$$

We denote by $\text{dom } f := \{x \in X : f(x) < +\infty\}$.

Let f be a convex function on a Banach space X and $x \in X$ be such that $f(x) < +\infty$, then the subdifferential of f at x is the set

$$\partial f(x) = \{p \in X^*; f - p \text{ has a minimum at } x\}.$$

When f is lsc convex, the Fréchet subdifferential of f coincides with the subdifferential in the sense of convex analysis, that is, $D^- f(x) = \partial f(x)$. We shall say that a function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ admits a strong minimum at some point x if:

- (i) $f(x) = \inf\{f(y), y \in X\}$ and
- (ii) (y_n) converges to x for every sequence $(y_n) \subset X$ satisfying $\lim_n f(y_n) = f(x)$.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.3 and in Section 3, we prove the uniqueness of second order viscosity solutions of certain Hamilton-Jacobi equations in finite dimensions.

2. The multidirectional mean value inequalities of second order

The variational principle below (Theorem 2.1), was proved by Deville, Godefroy and Zizler (see [6]). In this statement, we denote by $\|g\|_{\infty} = \sup\{|g(x)|; x \in X\}$, $\|g'\|_{\infty} = \sup\{\|g'(x)\|; x \in X\}$ and $\|g''\|_{\infty} = \sup\{\|g''(x)\|; x \in X\}$, where $\|g'(x)\| := \sup\{\|g'(x)(h)\|; h \in X, \|h\| \leq 1\}$, and $\|g''(x)\| = \{\sup\{g''(x)(h, h)\}; h \in X, \|h\| \leq 1\}$.

THEOREM 2.1. *Let X be a Banach space satisfying (H) and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a bounded below, lsc function, which is not identically equal to $+\infty$. Then for all $\varepsilon > 0$, there exists a C^2 function $\phi : X \rightarrow \mathbb{R}$ such that*

- (i) $f + \phi$ admits a strong minimum,
- (ii) $\max(\|\phi\|_\infty, \|\phi'\|_\infty, \|\phi''\|_\infty) < \varepsilon$.

For the proof of Theorem 1.3, we need the following three lemmas.

LEMMA 2.2. *Let X be a Banach space satisfying (H). Let $g : X \times X \rightarrow \mathbb{R} \cup \{+\infty\}$ be an lsc bounded below function which is not identically equal to $+\infty$. Let C be a non empty subset of X . For all integer $m > 0$, let us denote $A_m(x, z) := g(x, z) + md(z, C)$ for all $(x, z) \in X \times X$. Then, for all $m > 0$, there exist $(x_m, z_m) \in X \times X$ and a C^2 function $\phi_m : X \times X \rightarrow \mathbb{R}$ such that*

- (i) $md(z_m, C) \rightarrow 0$ when $m \rightarrow \infty$.
- (ii) $\max(\|\phi_m\|_\infty, \|\phi'_m\|_\infty, \|\phi''_m\|_\infty) < 1/m$ and $A_m - \phi_m$ admits a strong minimum at (x_m, z_m) .
- (iii) $\liminf_{m \rightarrow \infty} g(x_m, z_m) = \lim_{\eta \rightarrow 0} \inf_{x \in X, d(z, C) \leq \eta} g(x, z)$.

PROOF. Let $a_m := \inf_{(x, z) \in X \times X} A_m(x, z)$, then $(a_m)_m$ is an increasing sequence which is bounded by $\lim_{\eta \rightarrow 0} \inf\{g(x, z) : x \in X, d(z, C) \leq \eta\}$. So $(a_m)_m$ converges to some real number $a \in \mathbb{R}$ such that

$$a \leq \liminf_{\eta \rightarrow 0} \{g(x, z) : x \in X, d(z, C) \leq \eta\}.$$

Since X satisfies (H), then $X \times X$ also satisfies (H). In fact, if $b : X \rightarrow \mathbb{R}$ is an C^2 bump function with Lipschitz derivative, then the function $B : X \times X \rightarrow \mathbb{R}$, defined by $B(x, z) := b(x)b(z)$, is also a C^2 bump function with Lipschitz derivative. By applying Theorem 2.1 to A_m for all m , we obtain a C^2 function ϕ_m defined on $X \times X$ such that $\max(\|\phi_m\|_\infty, \|\phi'_m\|_\infty, \|\phi''_m\|_\infty) < 1/m$, and $(x_m, z_m) \in X \times X$ such that $A_m - \phi_m$ has a strong minimum at (x_m, z_m) and

$$(1) \quad A_m(x_m, z_m) < \inf_{x, z \in X} A_m(x, z) + 1/m = a_m + 1/m.$$

This implies (ii).

By the definition of a_m and (1), we have

$$a_m \leq A_m(x_m, z_m) = A_{2m}(x_m, z_m) - md(z_m, C) \leq a_{2m} + \frac{1}{2m} - md(z_m, C)$$

Thus $md(z_m, C) \leq a_{2m} + 1/(2m) - a_m$ and it follows that $md(z_m, C) \rightarrow 0$ when $m \rightarrow \infty$. This gives (i).

Now let us prove (iii):

$$\begin{aligned} a &\leq \liminf_{\eta \rightarrow 0} \{g(x, z) : x \in X, d(z, C) \leq \eta\} \\ &\leq \liminf_{m \rightarrow \infty} g(x_m, z_m) = \liminf_{m \rightarrow \infty} A_m(x_m, z_m) \leq a. \end{aligned}$$

So $\liminf_{m \rightarrow \infty} g(x_m, z_m) = \lim_{\eta \rightarrow 0} \inf\{g(x, z) : x \in X, d(z, C) \leq \eta\}$. □

In [9], Deville and Ivanov proved a variational principle of constraints of second order, with finite dimensional spaces of constraints. Lemma 2.3 extends their result to infinite dimensional spaces of constraints.

LEMMA 2.3. *Let X be a Banach space satisfying (H) and Y be a closed subspace of X such that the space X/Y also satisfies (H). Then there exists a constant $a_{X/Y} > 0$ with the following property: for every lsc bounded below function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$, every subset Δ of X ($\Delta \cap \text{dom } f \neq \emptyset$) and every $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}^*$, $x_0, z_0 \in X$, $(x_0^*, x_0^{\mathcal{B}}) \in D^{2^-} f(x_0)$ and $(z_0^*, z_0^{\mathcal{B}}) \in D^{2^-} m_0 d(\cdot, \Delta)(z_0)$ such that*

- (i) $\|x_0^*\|_{Y^*} < \varepsilon$, $\|x_0^{\mathcal{B}}\|_{\mathcal{B}(Y)} < \varepsilon$, $d(x_0 - z_0, Y) < \varepsilon$ and $d(z_0, \Delta) < \varepsilon$;
- (ii) $\|x_0^* + z_0^*\| < \varepsilon$;
- (iii) $f(x_0) < \lim_{\eta \rightarrow 0} \inf_{y \in \Delta + \eta B_X} f(y) + \varepsilon$;
- (iv) $\|x_0^*\| < \varepsilon + \frac{a_{X/Y}}{\varepsilon}(\inf_{\Delta} f - \inf_X f)$; $\|x_0^{\mathcal{B}}\| < \varepsilon^2 + \frac{a_{X/Y}}{\varepsilon^2}(\inf_{\Delta} f - \inf_X f)$.

PROOF. Let us denote by $\pi : X \rightarrow X/Y$ the canonical surjection and by $b : X/Y \rightarrow \mathbb{R}$ a bump function of class C^2 on X/Y with support in the unit ball $B_{X/Y}$ of X/Y such that $\max(b) = b(\pi(0)) = 1$. Set $a_{X/Y} := \max(\|b'\|_{\infty}, \|b''\|_{\infty})$. The function $b \circ \pi : X \rightarrow \mathbb{R}$ is also of class C^2 . Let $\varepsilon > 0$. We can suppose without loss of generality that $0 < \varepsilon < \min\{1, 1/2a_{X/Y}\}$. Since f is bounded below, the constant $\beta := \inf_{\Delta} f - \inf_X f \geq 0$ is well defined. We set $\lambda := -(\varepsilon^4 + \beta) < 0$. Now, for every integer $m > 2/\varepsilon^4$, we consider the following function

$$A_m(z, x) := f(x) + \lambda b \circ \pi \left(\frac{x - z}{\varepsilon} \right) + md(z, \Delta).$$

Let us apply Lemma 2.2 to $A_m(x, z) := g(x, z) + md(z, \Delta)$ with $g(x, z) := f(x) + \lambda b \circ \pi(x - z/\varepsilon)$. Then we obtain a point $(x_m, z_m) \in X \times X$ and a C^2 function ϕ_m such that $d(z_m, \Delta) < 1/m$; $\max(\|\phi_m\|_{\infty}; \|\phi'_m\|_{\infty}; \|\phi''_m\|_{\infty}) < 1/m$ and $A_m - \phi_m$ has a strong minimum at (x_m, z_m) .

Since $A_m - \phi_m$ has a strong minimum at (x_m, z_m) , if we first fix $z = z_m$ and then $x = x_m$, we obtain $(x_m^*, x_m^{\mathcal{B}}) \in D^{2^-} f(x_m)$ and $(z_m^*, z_m^{\mathcal{B}}) \in D^{2^-} md(\cdot, \Delta)(z_m)$ with

$$(2) \quad x_m^* := \left(\frac{\partial \phi_m}{\partial x} \right) (x_m, z_m) - \frac{\lambda}{\varepsilon} b' \left(\pi \left(\frac{x_m - z_m}{\varepsilon} \right) \right) \circ \pi,$$

$$(3) \quad x_m^{\mathcal{B}} := \left(\frac{\partial^2 \phi_m}{\partial x^2} \right) (x_m, z_m) - \frac{\lambda}{\varepsilon^2} b'' \left(\pi \left(\frac{x_m - z_m}{\varepsilon} \right) \right) \circ \pi^2,$$

$$(4) \quad z_m^* := \left(\frac{\partial \phi_m}{\partial z} \right) (x_m, z_m) + \frac{\lambda}{\varepsilon} b' \left(\pi \left(\frac{x_m - z_m}{\varepsilon} \right) \right) \circ \pi,$$

$$z_m^{\mathcal{B}} := \left(\frac{\partial^2 \phi_m}{\partial z^2} \right) (x_m, z_m) - \frac{\lambda}{\varepsilon^2} b'' \left(\pi \left(\frac{x_m - z_m}{\varepsilon} \right) \right) \circ \pi^2.$$

We prove that for a sufficiently large $m \in \mathbb{N}$, $x_0 := x_m$; $x_0^* := x_m^*$; $x_0^{\mathcal{B}} := x_m^{\mathcal{B}}$ (respectively $z_0 = z_m$; $z_0^* = z_m^*$ and $z_0^{\mathcal{B}} = z_m^{\mathcal{B}}$) satisfy our lemma.

From (2)–(3) and the fact that $\pi(y) = 0$ for all $y \in Y$, we have

$$\langle x_m^*, y \rangle = \left\langle \left(\frac{\partial \phi_m}{\partial x} \right) (x_m, z_m), y \right\rangle \quad \forall y \in Y$$

and

$$\langle x_m^{\mathcal{B}} y, y \rangle = \left\langle \left(\frac{\partial^2 \phi_m}{\partial x^2} \right) (x_m, z_m) y, y \right\rangle \quad \forall y \in Y.$$

Now since $\max(\|\phi_m\|_\infty; \|\phi'_m\|_\infty; \|\phi''_m\|_\infty) < 1/m$, it follows that $\|x_m^*\|_{Y^*} < 1/m$ and $\|x_m^{\mathcal{B}}\|_{\mathcal{B}(Y)} < 1/m$.

To complete the proof of (i), it suffices to show that

$$\|\pi(x_m - z_m)\|_{X/Y} := d(x_m - z_m, Y) < \varepsilon.$$

Indeed, since $A_m - \phi_m$ has a minimum at (x_m, z_m) , for all $x, z \in X$ we have

$$\begin{aligned} f(x_m) + md(z_m, \Delta) + \lambda b \circ \pi \left(\frac{x_m - z_m}{\varepsilon} \right) - \phi_m(x_m, z_m) \\ \leq f(x) + md(z, \Delta) + \lambda b \circ \pi \left(\frac{x - z}{\varepsilon} \right) - \phi_m(x, z). \end{aligned}$$

Now suppose that $\|\pi(x_m - z_m)\|_{X/Y} \geq \varepsilon$. Then, using the fact that $\text{supp}(b) \subset B_{X/Y}$, we get $b \circ \pi((x_m - z_m)/\varepsilon) = 0$. Now, taking $z = x \in \Delta$ in the above inequality and using the fact that $b(\pi(0)) = 1$, we obtain, for all $x \in \Delta$,

$$f(x_m) + md(z_m, \Delta) - \phi_m(x_m, z_m) \leq f(x) + \lambda - \phi_m(x, x).$$

From $\|\phi_m\| < 1/m$, we get

$$\lambda \geq f(x_m) - \inf_{\Delta} f(x) + md(z_m, \Delta) - \frac{2}{m} \geq \inf_X f - \inf_{\Delta} f(x) - \frac{2}{m}$$

Since $m > 2/\varepsilon^4$, it follows that $\lambda > -\beta - \varepsilon^4$, which is impossible and concludes the proof of (i).

From (2) and (4), we get: $\|x_m^* + z_m^*\| < 2/m$. This gives (ii).

Now, we prove (iii). Using Lemma 2.2 (iii), we obtain that for sufficiently large m (we can extract subsequences $(x_{m_k})_k$ and $(z_{m_k})_k$ of $(x_m)_m$ and $(z_m)_m$ respectively)

$$\begin{aligned} f(x_m) + \lambda b \circ \pi \left(\frac{x_m - z_m}{\varepsilon} \right) \\ < \liminf_{\eta \rightarrow 0} \left\{ f(x) + \lambda b \circ \pi \left(\frac{x - z}{\varepsilon} \right) : x \in X; d(z, \Delta) \leq \eta \right\} + \varepsilon \\ &\leq \liminf_{\eta \rightarrow 0} \{ f(x) + \lambda b \circ \pi(0) : x \in \Delta + \eta B_X \} + \varepsilon \\ &= \liminf_{\eta \rightarrow 0} \{ f(x) : x \in \Delta + \eta B_X \} + \lambda + \varepsilon. \end{aligned}$$

But $b \circ \pi((x_m - z_m)/\varepsilon) \leq 1$ and $\lambda < 0$, so $\lambda b \circ \pi((x_m - z_m)/\varepsilon) \geq \lambda$. Thus

$$f(x_m) < \liminf_{\eta \rightarrow 0} \left\{ f(x) : x \in \Delta + \eta B_X \right\} + \varepsilon.$$

Finally, (iv) comes from (2) and (3). Indeed, using (2), the fact that $m > 2/\varepsilon^4$, $0 < \varepsilon < \min\{1, 1/2a_{X/Y}\}$ and that $|\lambda| = \varepsilon^4 + (\inf_{\Delta} f - \inf_X f)$ we obtain

$$\begin{aligned} \|x_m^*\| &\leq \|\phi'_m\|_{\infty} + \frac{|\lambda|}{\varepsilon} \|b'\|_{\infty} \leq \frac{1}{m} + \varepsilon^3 a_{X/Y} + \left(\inf_{\Delta} f - \inf_X f \right) \frac{a_{X/Y}}{\varepsilon} \\ &\leq \varepsilon + \left(\inf_{\Delta} f - \inf_X f \right) \frac{a_{X/Y}}{\varepsilon}. \end{aligned}$$

In a similar way, using (3), we obtain that $\|x_m^{\otimes}\| \leq \varepsilon^2 + (\inf_{\Delta} f - \inf_X f) a_{X/Y} / \varepsilon^2$. \square

LEMMA 2.4. *Let X be a Banach space, Y a closed subspace of X , C a closed subset of X and $\hat{x} \in X$. Set $H := Y + C$ and $\Delta := [\hat{x}, H]$. Then, for every $(x_0, z_0) \in X \times X$, we have*

- (i) $d(x_0, \Delta) \leq d(x_0 - z_0, Y) + d(z_0, \Delta)$;
- (ii) $d(x_0, H) \leq d(x_0 - z_0, Y) + d(z_0, H)$.

PROOF. Note that for every $y \in Y$ and every $\lambda \in [0, 1]$, $H = H - y/(1 - \lambda)$. This is due to the linearity of the subspace $Y \subseteq X$. Now let us fix $(x_0, z_0) \in X \times X$. For all $y \in Y$, we have

$$\begin{aligned} d(x_0, \Delta) &= \inf_{v \in \Delta} \|x_0 - v\| = \inf_{\substack{\lambda \in [0, 1], \\ h \in H}} \|x_0 - (\lambda \hat{x} + (1 - \lambda)h)\| \\ &= \inf_{\substack{\lambda \in [0, 1], \\ h \in H}} \|x_0 - (\lambda \hat{x} + (1 - \lambda)h)\| \\ &\leq \|x_0 - (z_0 + y)\| + \inf_{\substack{\lambda \in [0, 1], \\ h \in H}} \|(z_0 + y) - (\lambda \hat{x} + (1 - \lambda)h)\| \\ &= \|x_0 - (z_0 + y)\| + \inf_{\substack{\lambda \in [0, 1], \\ h \in H}} \left\| z_0 - \left(\lambda \hat{x} + (1 - \lambda) \left(h - \frac{y}{1 - \lambda} \right) \right) \right\| \\ &= \|x_0 - (z_0 + y)\| + \inf_{\substack{\lambda \in [0, 1], \\ h \in H}} \|z_0 - (\lambda \hat{x} + (1 - \lambda)h)\| \\ &= \|(x_0 - z_0) - y\| + d(z_0, \Delta). \end{aligned}$$

Taking the infimum over $y \in Y$, we conclude the proof of (i).

In a similar way we prove (ii). \square

PROOF OF THEOREM 1.3. We first prove the theorem when $r = 0$ and then we deduce the general case.

Case 1: $r = 0$. Let $\varepsilon > 0$ and let us fix $\bar{h} \in]0, h/2[$ such that

$$\inf_{y \in H + 2\bar{h}B_X} f(y) > f(\hat{x}).$$

We assume without loss of generality that

$$(5) \quad \varepsilon < \min \left\{ \inf_{y \in H + 2\bar{h}B_X} f(y) - f(\hat{x}), \bar{h} \right\}.$$

Let us denote by $S := \overline{\Delta + \bar{h}B_X}$ the closure of $\Delta + \bar{h}B_X$ in X . The function f_1 defined by $f_1(x) := f(x) + \delta_S(x)$ for all $x \in X$ is lsc bounded below on X . Let us apply Lemma 2.3 to the function f_1 , the subspace Y and the set Δ . So, there exists $x_0, z_0 \in X$, $(x_0^*, x_0^{\otimes}) \in D^{2-} f_1(x_0)$ and $(z_0^*, z_0^{\otimes}) \in D^{2-} m_0 d(\cdot, \Delta)(z_0)$ such that

- (a) $\|x_0^*\|_{Y^*} < \varepsilon/2$, $\|x_0^{\otimes}\|_{\mathcal{B}(Y)} < \varepsilon/2$, $d(x_0 - z_0, Y) < \varepsilon/2$ and $d(z_0, \Delta) < \varepsilon/2$;
- (b) $\|x_0^* + z_0^*\| < \varepsilon/2$;
- (c) $f_1(x_0) < \lim_{\eta \rightarrow 0} \inf_{y \in \Delta + \eta B_X} f_1(y) + \varepsilon/2$;
- (d) $\|x_0^*\| < \varepsilon/2 + 2a_{X/Y}/\varepsilon(\inf_{\Delta} f_1 - \inf_X f_1)$; $\|x_0^{\otimes}\| < \varepsilon^2/4 + 4a_{X/Y}/\varepsilon^2(\inf_{\Delta} f_1 - \inf_X f_1)$

First, we show that $d(x_0, \Delta) < \varepsilon$, which implies that x_0 belongs to the interior of $\Delta + \bar{h}B_X$, and so $(x_0^*, x_0^{\otimes}) \in D^{2-} f(x_0)$. Indeed, thanks to Lemma 2.4 (i), we have the following inequality: $d(x_0, \Delta) \leq d(x_0 - z_0, Y) + d(z_0, \Delta)$. It follows from (a) that $d(x_0, \Delta) < \varepsilon$. This completes the proof of (i).

Proof of (ii). Since $x_0^* \in D^{-} m_0 d(\cdot, \Delta)(z_0) = \partial m_0 d(\cdot, \Delta)(z_0)$, it follows that

$$(6) \quad \langle z_0^*, z - z_0 \rangle \leq m_0 d(z, \Delta) - m_0 d(z_0, \Delta), \quad \forall z \in X.$$

Choose a bounded sequence $(u_n)_n \in \Delta$ such that $\|z_0 - u_n\| < d(z_0, \Delta) + 1/n$. For every $w \in \Delta$ we have $d(w - u_n + z_0, \Delta) \leq \|z_0 - u_n\| < d(z_0, \Delta) + 1/n$ and it follows from (6) that

$$(7) \quad \langle z_0^*, w - u_n \rangle \leq m_0 d(w - u_n + z_0, \Delta) - m_0 d(z_0, \Delta) \leq m_0/n.$$

By (b) we deduce $\langle x_0^*, w - u_n \rangle + \varepsilon \|w - u_n\| > -\langle z_0^*, w - u_n \rangle$. Using (7), for all $w \in \Delta$, we obtain

$$(8) \quad \langle x_0^*, w - u_n \rangle + \varepsilon \|w - u_n\| > -m_0/n.$$

To complete the proof of (ii), we need the following claim.

CLAIM. *There exists $\bar{h} > 0$ such that for all $n > 2/\varepsilon$, $d(u_n, H) \geq \bar{h}$.*

PROOF OF THE CLAIM. First we need to show that $d(x_0, H) > 2\bar{h}$. Suppose that the contrary holds. Using (5), it follows that

$$f(x_0) \geq \inf_{y \in H+2\bar{h}B_X} f(y) > f(\hat{x}) + \varepsilon.$$

On the other hand, by (c) and the fact that $\hat{x} \in \Delta$, we get $f(x_0) < f(\hat{x}) + \varepsilon$. This leads to a contradiction. Consequently,

$$(9) \quad d(x_0, H) > 2\bar{h}.$$

Now, from (9), the fact that $d(x_0 - z_0, Y) < \varepsilon/2 < \bar{h}/2$ and Lemma 2.4 (ii) we deduce that $d(z_0, H) \geq \bar{h}$. On the other hand,

$$d(z_0, H) \leq \|z_0 - u_n\| + d(u_n, H) < d(z_0, \Delta) + 1/n + d(u_n, H).$$

Thus $d(u_n, H) > d(z_0, H) - d(z_0, \Delta) - 1/n$ for all $n \in \mathbb{N}$. It follows that, for $n > 2/\varepsilon$,

$$d(u_n, H) > d(z_0, H) - d(z_0, \Delta) - \frac{\varepsilon}{2} \geq \bar{h} - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = \bar{h} - \varepsilon > 0$$

and the claim is proved with $\tilde{h} := \bar{h} - \varepsilon$. □

Now we complete the proof of (ii). Since $u_n \in \Delta$, there exists $t_n \in [0, 1]$ and $\bar{y}_n \in H$ such that

$$(10) \quad u_n = \hat{x} + t_n(\bar{y}_n - \hat{x}).$$

Let $(t_{n_k})_k$ be a subsequence of $(t_n)_n$ that converges to some point $t_0 \in [0, 1]$. We claim that $t_0 \neq 1$. Suppose the contrary holds, that is, $t_0 = 1$. Using the Claim for sufficiently large k , we get

$$\tilde{h} \leq d(u_{n_k}, H) \leq \|u_{n_k} - \bar{y}_{n_k}\| = (1 - t_{n_k})\|\hat{x} - \bar{y}_{n_k}\|.$$

Since $(u_{n_k})_k$ is bounded and $t_{n_k} \rightarrow t_0 = 1$, it follows from (10) that the sequence \bar{y}_{n_k} is also bounded and it follows from the above inequality that

$$\tilde{h} \leq \liminf_{k \rightarrow +\infty} ((1 - t_{n_k})\|\hat{x} - \bar{y}_{n_k}\|) = (1 - t_0) \liminf_{k \rightarrow +\infty} \|\hat{x} - \bar{y}_{n_k}\| = 0,$$

which is impossible since $\tilde{h} > 0$. Hence $t_0 \neq 1$.

Now, for each $y \in H$, we set $h_n(y) := y + t_n(\bar{y}_n - y)$ ($h_n(y) \in H$, by convexity of H). Using (10) we get $h_{n_k}(y) - u_{n_k} = (1 - t_{n_k})(y - \hat{x})$ for all $y \in H$. Taking $w = h_{n_k}(y) \in H \subset \Delta$ in (8), we get

$$\langle x_0^*, y - \hat{x} \rangle + \varepsilon \|y - \hat{x}\| > -\frac{m_0}{n_k} \frac{1}{1 - t_{n_k}}, \quad \forall y \in H.$$

Letting k tend to infinity, we obtain $\langle x_0^*, y - \hat{x} \rangle + \varepsilon \|y - \hat{x}\| \geq 0$, for all $y \in H$. This completes the proof of (ii). The proofs of (iii) and (iv) are given directly by (c) and (d).

Now we deduce the general case.

General case: On $X \times \mathbb{R}$, we consider the norm defined as follows:

$$\|(x, t)\| := \|x\| + |t| \quad \text{for all } (x, t) \in X \times \mathbb{R}.$$

Let $0 < \varepsilon < 1$ and choose ε' such that $\varepsilon' \in]0, \varepsilon/4[$, $\varepsilon'|r| < \varepsilon$ and

$$\lim_{\eta \rightarrow 0} \inf_{y \in H + \eta B_X} f(y) > f(\hat{x}) + r + \varepsilon'.$$

Let us define the function F on $X \times \mathbb{R}$ as follows: $F(x, t) := f(x) - (r + \varepsilon')t$. It is clear that F is lsc on $X \times \mathbb{R}$ and is bounded below on $[(\hat{x}, 0), H \times \{1\}] + hB_{X \times \mathbb{R}}$. On the other hand,

$$\lim_{\eta \rightarrow 0} \inf_{H \times \{1\} + \eta B_{X \times \mathbb{R}}} F = \left(\lim_{\eta \rightarrow 0} \inf_{H + \eta B_X} f \right) - (r + \varepsilon') > f(\hat{x}) = F(\hat{x}, 0).$$

Now we apply Case 1 with the function F , the set $H' = H \times \{1\} = C \times \{1\} + Y \times \{0\}$ and the point $(\hat{x}, 0)$. There exists $(x_0, t_0) \in [(\hat{x}, 0), H \times \{1\}] + \varepsilon' B_{X \times \mathbb{R}}$ (which implies, in particular, that $d(t_0, [0, 1]) < \varepsilon'$ and $d(x_0, \Delta) < \varepsilon'$) and $(x_0^*, x_0^{\otimes}) \in D^{2-} f(x_0)$ satisfying $\|x_0^*\|_Y < \varepsilon'$, $\|x_0^{\otimes}\|_Y < \varepsilon'$ and

$$F(x_0, t_0) = f(x_0) - (r + \varepsilon')t_0 < \lim_{\eta \rightarrow 0} \inf_{[(\hat{x}, 0), H \times \{1\}] + \eta B_{X \times \mathbb{R}}} (f(x) - (r + \varepsilon')t) + \varepsilon'.$$

It follows from the above inequality that

$$\begin{aligned} f(x_0) &< \lim_{\eta \rightarrow 0} \inf_{[(\hat{x}, 0), H \times \{1\}] + \eta B_{X \times \mathbb{R}}} (f(x) - (r + \varepsilon')(t - t_0)) + \varepsilon' \\ &< \lim_{\eta \rightarrow 0} \inf_{[\hat{x}, H] + \eta B_X} f + |r + \varepsilon'|(1 + \varepsilon') + \varepsilon' \\ &< \lim_{\eta \rightarrow 0} \inf_{[\hat{x}, H] + \eta B_X} f + |r| + \varepsilon. \end{aligned}$$

Using Case 1, we also deduce the following inequality:

$$0 \leq \langle x_0^*, y - \hat{x} \rangle - (r + \varepsilon') + \varepsilon'(\|y - \hat{x}\| + 1), \quad \forall y \in H.$$

This inequality implies that $r < \langle x_0^*, y - \hat{x} \rangle + \varepsilon \|y - \hat{x}\|$ for all $y \in H$. The proof is completed. \square

The following corollary will permit us to prove uniqueness of second order viscosity solution of Hamilton-Jacobi equations.

COROLLARY 2.5. *Let Y be a Banach space satisfying (H). Let $\tilde{x} \in Y$, $T \in \mathbb{R}$ and $r \in \mathbb{R}$. Suppose that $f : \mathbb{R} \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ is lsc function bounded below on $[0, T+h] \times Y$ for some $h > 0$. Assume further that $f(0, y) \geq f(T, \tilde{x}) + r$, for all $y \in Y$. Then, for all $\varepsilon > 0$, there exists $(t_0, x_0) \in [0, T+\varepsilon] \times Y$, $(a, p) \in \mathbb{R} \times Y^*$ and $(\frac{Q_1}{Q_2}, \frac{Q_2}{Q_3}) \in \mathcal{B}(\mathbb{R} \times Y)$ with $((a, p), (\frac{Q_1}{Q_2}, \frac{Q_2}{Q_3})) \in D^{2^-} f(t_0, x_0)$, such that*

- (i) $\|p\| < \varepsilon$; $\|Q_3\| < \varepsilon$;
- (ii) $a < -r/T + \varepsilon$;
- (iii) $f(x_0, t_0) < \lim_{\eta \rightarrow 0} \inf_{(t,x) \in [(T,\tilde{x}), \{0\} \times Y] + \eta B_{\mathbb{R} \times Y}} f(t, x) + |r| + \varepsilon$.

PROOF. Set $H = \{0\} \times Y$, $\Delta = [(T, \tilde{x}), \{0\} \times Y]$ and $F := f + \delta_{[0, T+h] \times Y}$. The function F is lsc bounded below on $[-h, T+h] \times Y$. Let us remark that $\Delta + hB_{\mathbb{R} \times Y} \subset [-h, T+h] \times Y$. Then we have that F is bounded below on $\Delta + hB_{\mathbb{R} \times Y}$. Now let us observe that

$$\lim_{\eta \rightarrow 0} \inf_{x \in H + \eta B_{\mathbb{R} \times Y}} F(x) = \inf_{y \in Y} f(0, y) > f(T, \tilde{x}) + r - \varepsilon T = F(T, \tilde{x}) + r - \varepsilon T.$$

Consider $X = \mathbb{R} \times Y$ and $\hat{x} = (T, \tilde{x})$. Then we apply Theorem 1.3 to the set H , the space X , the point $\hat{x} \in X$, the real number $r - \varepsilon T \in \mathbb{R}$, and the function F . We get a point $(t_0, x_0) \in (\Delta + \varepsilon B_X) \cap \text{dom } F$ (this implies that $(t_0, x_0) \in [0, T+\varepsilon] \times Y$ and $((a, p), (\frac{Q_1}{Q_2}, \frac{Q_2}{Q_3})) \in D^{2^-} f(t_0, x_0)$ satisfying the corollary. \square

REMARK 2.6. If we suppose in Corollary 2.5 that $f(0, y) \geq 0$ for all $y \in H$, then we can set $r = -f(T, \tilde{x})$, and we obtain in (ii) that $a < f(T, \tilde{x})/T + \varepsilon$.

3. Application to Hamilton-Jacobi equations

The purpose of this section is to recover, by a simple proof, the uniqueness of viscosity solution of second order. Note that the formula for the second order subdifferential of the sum of two lower semicontinuous functions is not available in infinite dimensions. A counterexamples in infinite dimensional Hilbert spaces are given in [8].

LEMMA 3.1. *Let u_1, u_2 be two lsc functions defined on a finite dimensional Banach space X . Let $x_0 \in X$ and $(p, Q) \in D^{2^-}(u_1 + u_2)(x_0)$. Then for all $\varepsilon > 0$, there exists $x_1, x_2 \in X$, $(p_1, Q_1) \in D^{2^-}(u_1)(x_1)$ and $(p_2, Q_2) \in D^{2^-}(u_2)(x_2)$ such that*

- (i) $\|x_1 - x_0\| < \varepsilon$ and $\|x_2 - x_0\| < \varepsilon$;
- (ii) $\|u_1(x_1) - u_1(x_0)\| < \varepsilon$ and $\|u_2(x_2) - u_2(x_0)\| < \varepsilon$;
- (iii) $\|p_1 + p_2 - p\| < \varepsilon$ and $\|Q_1 + Q_2 - Q\| < \varepsilon$.

Let X be a Banach space and $H : \mathbb{R} \times X \times X^* \times \mathcal{B}(X) \rightarrow \mathbb{R}$ be a uniformly continuous function. We consider the associated evolution equation:

$$(11) \quad \begin{cases} u_t + H(t, x, Du, D^2u) = 0 \\ u(0, x) = u_0(x), \end{cases}$$

where $u_0 : X \rightarrow \mathbb{R}$ is the initial condition, u is defined on $\mathbb{R} \times X$, u_t denotes the partial derivative with respect to the real variable, and Du and D^2u denote the first and second partial derivatives with respect to the x -variable.

Here we focus our attention here on the uniqueness of a continuous viscosity solution $u : \mathbb{R}^+ \times X \rightarrow \mathbb{R}$ of (11).

As in Definition 1.2, we define the viscosity superdifferential of second order of f at x by $D^{2+} f(x) := \{(\phi'(x), \phi''(x)); \phi : X \rightarrow \mathbb{R} \text{ is } C^2 \text{ and } f - \phi \text{ has a local maximum at } x\}$.

DEFINITION 3.2. A function $u : \mathbb{R}^+ \times X \rightarrow \mathbb{R}$ is a *viscosity subsolution* of (11) if u is upper semi continuous (usc) and, for every $(t, x) \in \mathbb{R}^+ \times X$ and every $((a, p), \begin{pmatrix} A & C \\ C & D \end{pmatrix}) \in D^{2+} f(t, x)$, we have

$$\begin{cases} a + H(t, x, p, D) \leq 0, \\ u(0, x) \leq u_0(x). \end{cases}$$

The function u is a viscosity supersolution of (11) if u is lower semi continuous (lsc) and, for every $(t, x) \in \mathbb{R}^+ \times X$ and every $((a, p), \begin{pmatrix} A & C \\ C & D \end{pmatrix}) \in D^{2-} f(t, x)$, we have

$$\begin{cases} a + H(t, x, p, D) \geq 0, \\ u(0, x) \geq u_0(x). \end{cases}$$

Finally, u is a viscosity solution of (11) if u is both a viscosity subsolution and a viscosity supersolution of (11).

PROPOSITION 3.3. Let X be a finite dimensional Banach space and let u, v be two real valued functions defined on $\mathbb{R}^+ \times X$ such that u is usc bounded above and v is lsc bounded below. If u is a viscosity subsolution of (11) and if v is a viscosity supersolution of (11), then $u \leq v$.

PROOF. Let us fix $T \in]0, +\infty[$ and, in order to get a contradiction, let us assume that $\inf_{[0, T] \times X} (v - u) < 0$. The function $v - u$ is lsc bounded below on $[0, T] \times X$. Thus, for $\varepsilon > 0$ sufficiently small, there exists $(t_0, x_0) \in]0, T] \times X$ such that

$$(v - u)(t_0, x_0) < \inf_{[0, T] \times X} (v - u) + \varepsilon T < 0.$$

According to the initial condition, we have $(v - u)(0, x) \geq 0$ for every $x \in X$. Now, let us apply Corollary 2.5 and Remark 2.6 to the function $v - u$. Thus, there exists $(t, x) \in [0, t_0 + \varepsilon] \times X$ and $((a, p), \begin{pmatrix} A & C \\ C & D \end{pmatrix}) \in D^{2^-}(v - u)(t, x)$ such that $a < (v - u)(t_0, x_0)/t_0 + \varepsilon$, $\|p\| < \varepsilon$ and $\|D\| < \varepsilon$. Let us apply Lemma 3.1 to the functions $u_1 = v$ and $u_2 = -u$. There exist $(t_1, x_1), (t_2, x_2) \in \mathbb{R}^+ \times X$,

$$\left((a_1, p_1), \begin{pmatrix} A_1 & C_1 \\ C_1 & D_1 \end{pmatrix} \right) \in D^{2^-}v(t_1, x_1)$$

and

$$\left((a_2, p_2), \begin{pmatrix} A_2 & C_2 \\ C_2 & D_2 \end{pmatrix} \right) \in D^{2^+}u(t_2, x_2)$$

satisfying:

- (i) $\|x_1 - x\| < \varepsilon$, $\|x_2 - x\| < \varepsilon$, $|t_1 - t| < \varepsilon$, $|t_2 - t| < \varepsilon$;
- (ii) $\|D_1 - D_2 - D\| < \varepsilon$, $\|p_1 - p_2 - p\| < \varepsilon$ and $|a_1 - a_2 - a| < \varepsilon$.

The function u is a viscosity subsolution of (11), so $a_2 + H(t_2, x_2, p_2, D_2) \leq 0$. On the other hand, the function v is a viscosity supersolution of (11), so

$$a_1 + H(t_1, x_1, p_1, D_1) \geq 0.$$

Consequently,

$$\begin{aligned} \frac{\inf_{[0,T] \times X}(v - u)}{T} &> \frac{(v - u)(t_0, x_0)}{T} - \varepsilon > \frac{(v - u)(t_0, x_0)}{t_0} - \varepsilon \\ &> a - 2\varepsilon > a_1 - a_2 - 3\varepsilon \\ &\geq H(t_2, x_2, p_2, D_2) - H(t_1, x_1, p_1, D_1) - 3\varepsilon. \end{aligned}$$

Moreover, $\|x_1 - x_2\| \leq \|x_1 - x_0\| + \|x_0 - x_2\| < 2\varepsilon$, $\|t_1 - t_2\| \leq \|t_1 - t_0\| + \|t_0 - t_2\| < 2\varepsilon$, $\|p_1 - p_2\| \leq \|p_1 - p_2 - p\| + \|p\| < 2\varepsilon$ and $\|D_1 - D_2\| \leq \|D_1 - D_2 - D\| + \|D\| < 2\varepsilon$. Using the uniform continuity of H and sending ε to zero, we get

$$\frac{\inf_{[0,T] \times X}(v - u)}{T} \geq 0,$$

which is a contradiction. □

REMARK 3.4. Proposition 3.3 clearly implies the uniqueness of viscosity solution for (11). The existence of viscosity solutions for (11) was established in [4].

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