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OPERATOR-VALUED MULTIPLIER THEOREMS CHARACTERIZING HILBERT SPACES

WOLFGANG ARENDT and SHANGQUAN BU

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Abstract

We show that the operator-valued Marcinkiewicz and Mikhlin Fourier multiplier theorem are valid if and only if the underlying Banach space is isomorphic to a Hilbert space.

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1. Introduction

Mikhlin's multiplier theorem is of great importance in analysis. It says that a bounded function $m \in C^1(\mathbb{R} \setminus \{0\})$ such that $tm'(t)$ is bounded, defines an $L^p(\mathbb{R})$-multiplier for $1 < p < \infty$. In the context of partial differential equations vector-valued spaces $L^p(\mathbb{R}; X)$ occur in a natural way, where X is a Banach space. Thus the function m should take its values in $\mathscr{L}(X)$. Our aim is to show that Mikhlin's multiplier theorem does hold for such operator-valued functions if and only if X is isomorphic to a Hilbert space.

The phenomenon that operator-valued versions of certain classical multiplier theorems are only valid in Hilbert spaces was first observed by Pisier (unpublished) as a consequence of Kwapien's deep characterization of Hilbert spaces. More recently, new versions of operator-valued multiplier theorems turned out to be most useful in the theory of evolution equations (see the references and comments below) and it

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seems to us that it is important to elaborate in some detail why the classical result merely holds on Hilbert spaces.

In another context, it helps to impose a Mikhlin's condition of order k

$$m \in C^k(\mathbb{R} \setminus \{0\}; \mathcal{L}(X)), \qquad \sup_{t \in \mathbb{R} \setminus \{0\}, 0 \le l \le k} \|t^l m^{(l)}(t)\| < \infty. \tag{1}$$

In fact, Amann [1] discovered that if m satisfies (1) with $k = 2$, then m is a multiplier for Besov spaces and in particular for the space $C^\theta(\mathbb{R}; X)$, $0 < \theta < 1$ (see also [2] and [11]). We show here that imposing higher order Mikhlin's conditions does not help in the context of operator-valued L^p-multipliers.

We also consider the groups $\mathbb{T}$ and $\mathbb{Z}$ instead of $\mathbb{R}$. In fact, the case $\mathbb{T}$ corresponds to Marcinkiewicz's classical theorem and its operator-valued version is already treated in [3] for the order-1-case.

Now we would like to comment on the new vector-valued multiplier theorems which were found recently. It were Berkson-Gillespie [4] who introduced the notion of R-boundedness (after implicit use of Bourgain [6]). They use R as an abbreviation for Riesz, but in many subsequent papers people seem think rather of Rademacher or 'Randomized' because the definition involves Rademacher functions. A multiplier theorem of Marcinkiewicz type was established by Clément-de Pagter-Sukochev-Witvliet [8] for multipliers of the form $m(t)I$ (I is the identity operator) clarifying the role of R-boundedness. Then Weis [18] established Mikhlin's theorem for operator-valued functions (without restriction) replacing boundedness by the stronger condition of R-boundedness. Then in [3] the corresponding periodic theorem (that is, Marcinkiewicz's theorem) was proved on the basis of results in [8]. Štrkalj and Weis [17] gave an R-version of the variational version of the Marcinkiewicz theorem. Further important contributions were given by Clément-Prüss [9], Denk-Hieber-Prüss [10], and Girardi-Weis [11].

2. Periodic multipliers

Let us first recall some notions. Let X be a Banach space. Denote by r_j the j-th Rademacher function on $[0, 1]$. For $x \in X$, we denote by $r_j \otimes x$ the vector-valued function $t \mapsto r_j(t)x$. Let Y be another Banach space. We denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X to Y. If $X = Y$ we will denote $\mathcal{L}(X, Y)$ simply by $\mathcal{L}(X)$. A family $\mathbf{T} \subset \mathcal{L}(X, Y)$ is called *R-bounded* if for some $q \in [1, \infty)$ there exists a constant $c_q \ge 0$ such that

$$\left\| \sum_{j=1}^{n} r_j \otimes T_j x_j \right\|_{L^q(0,1;Y)} \le c_q \left\| \sum_{j=1}^{n} r_j \otimes x_j \right\|_{L^q(0,1;X)} \tag{2}$$

for all $T_1, \dots, T_n \in \mathbf{T}$, $x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$. By Kahane's inequality [14, Theorem 1.e.13] if such constant c_q exists for some $q \in [1, \infty)$, then it also exists for each $q \in [1, \infty)$.

It is known that R-boundedness is strictly stronger than boundedness in operator norm unless the underlying Banach space is isomorphic to a Hilbert space. More precisely, each bounded subset in $\mathscr{L}(X, Y)$ is R-bounded if and only if X is of cotype 2 and Y is of type 2 (see [3, Proposition 1.13]). In particular, by a result of Kwapien [14, pages 73–74], each bounded subset in $\mathscr{L}(X)$ is R-bounded if and only if X is isomorphic to a Hilbert space.

For $1 \le p < \infty$, consider the Banach space $L^p(0, 2\pi; X)$ with norm $\|f\|_p := \left(\int_0^{2\pi} \|f(t)\|^p \, dt\right)^{1/p}$. For $f \in L^p(0, 2\pi; X)$ we denote by

$$\hat{f}(k) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) \, dt,$$

the k-th *Fourier coefficient* of f, where $k \in \mathbb{Z}$. For $k \in \mathbb{Z}, x \in X$ we let $e_k(t) = e^{ikt}$ and $(e_k \otimes x)(t) = e_k(t)x \quad (t \in \mathbb{R})$. A function $f \in L^p(0, 2\pi; X)$ is called a *trigonometric polynomial* if f is given by $f = \sum_{k \in \mathbb{Z}} e_k \otimes x_k$, where $x_k \in X$ is 0 for all but finitely many $k \in \mathbb{Z}$.

Let $M = (M_k)_{k \in \mathbb{Z}} \subset \mathscr{L}(X, Y)$ be a sequence and let $1 \le p, q < \infty$. We say that $(M_k)_{k \in \mathbb{Z}}$ is a *periodic L^p-L^q-Fourier multiplier* if there exists a constant $C > 0$ such that

$$\left\| \sum_{k \in \mathbb{Z}} e_k \otimes M_k x_k \right\|_{L^q(0, 2\pi; Y)} \le C \left\| \sum_{k \in \mathbb{Z}} e_k \otimes x_k \right\|_{L^p(0, 2\pi; X)}$$

for all X-valued trigonometric polynomials $\sum_{k \in \mathbb{Z}} e_k \otimes x_k$. In this case, there exists a unique operator $M \in \mathscr{L}(L^p(0, 2\pi; X), L^q(0, 2\pi; Y))$ such that $(Mf)\hat{}(k) = M_k \hat{f}(k)$ for $k \in \mathbb{Z}$ [3]. When $p = q$, we say simply that $(M_k)_{k \in \mathbb{Z}}$ is a *periodic L^p-Fourier multiplier*. For $k \in \mathbb{Z}$, we let $(\Delta^1 M)(k) = M_{k+1} - M_k$ and $(\Delta^m M)(k) = (\Delta^1(\Delta^{m-1} M))(k)$ for $m \ge 2$. Notice that $\Delta^m M$ is a discrete analogue of the m-th derivative of M.

The classical Marcinkiewicz Fourier multiplier theorem has been extended to the operator-valued case in the following way: let X and Y be UMD spaces and let $(M_k)_{k \in \mathbb{Z}} \subset \mathscr{L}(X, Y)$; if both $\{M_k : k \in \mathbb{Z}\}$ and $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ are R-bounded, then $(M_k)_{k \in \mathbb{Z}}$ defines a periodic L^p-Fourier multiplier for each $1 < p < \infty$ [3]. Indeed, $(M_k)_{k \in \mathbb{Z}}$ is a periodic L^p-L^q-Fourier multiplier whenever $1 \le q \le p < \infty$.

We will need the following inequality of Pisier [15]. Let $1 \le p < \infty$ and let $\Lambda = \{n_k : k \in N\} \subset \mathbb{Z}$ be a Sidon subset [16, page 120]. Then there exists $C > 0$ such that for any Banach space X and for any finite sequence $(y_k)_{1 \le k \le N}$ of X, we have

(3)
$$C^{-1} \left\| \sum_k r_k \otimes y_k \right\|_2 \le \left\| \sum_k e_{n_k} \otimes y_k \right\|_p \le C \left\| \sum_k r_k \otimes y_k \right\|_2 .$$

Note that if $\lambda > 1$, then any subset $\{n_k : k \in \mathbb{N}\}$ satisfying $n_{k+1}/n_k \geq \lambda$ $(k \in \mathbb{N})$ is a Sidon subset of $\mathbb{Z}$ [16, page 127].

The following result shows that one cannot replace R-boundedness in the operator-valued Marcinkiewicz theorem above by boundedness in operator norm unless the underlying Banach space is isomorphic to a Hilbert space.

THEOREM 1. *Let X be a Banach space. Then the following assertions are equivalent:*

(i) *X is isomorphic to a Hilbert space.*

(ii) *For some $1 \leq q < p < \infty$, each sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X)$ satisfying*

(a) $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$,

(b) $\sup_{k\in\mathbb{Z}} \|k^l(\Delta^l M)(k)\| < \infty$ *for* $l \in \mathbb{N}$,

(c) $M_k = 0$ *for* $k \leq 0$,

is a periodic L^p-L^q-Fourier multiplier.

(iii) *For all $1 < p < \infty$, each sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X)$ satisfying*

(a) $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$,

(b) $\sup_{k\in\mathbb{Z}} \|k(\Delta^1 M)(k)\| < \infty$,

is a periodic L^p-Fourier multiplier.

REMARK 2. For $l = 1$, the condition formulated in (iii) is the classical condition considered by Marcinkiewicz in the scalar case. For arbitrary $l \in \mathbb{N}$ we therefore speak of the *Marcinkiewicz condition of order l*. For $p = q$ and $l = 1$, Theorem 1 has been proved in [3, Proposition 1.17.]. However, a more refined choice of test functions is needed here for the general case. The motivation to consider $l > 1$ stems from the results on Fourier multipliers for spaces of Hölder continuous functions where, indeed, the Marcinkiewicz condition of order 2 suffices (see [2] and also the Concluding Remarks at the end of this article). Theorem 1 shows that this is not the case in the L^p-context even if we consider weaker multipliers by allowing $q < p$. This has also been done by Kalton-Lancien in the context of maximal regularity for Cauchy problems [13] (see also the Concluding Remarks 5 (b) below).

PROOF. (i) $\Rightarrow$ (iii). Assume that X is isomorphic to a Hilbert space, then considering an orthonormal basis one easily verifies that each bounded subset in $\mathscr{L}(X)$ is actually R-bounded, so the result follows from the operator-valued Marcinkiewicz Fourier multiplier theorem in [3].

(iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i). Assume that for some $1 \leq q < p < \infty$, each sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X)$ satisfying $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^l(\Delta^l M)(k)\| < \infty$ for $l \in \mathbb{N}$ and $M_k = 0$ for $k \leq 0$, is a periodic L^p-L^q-Fourier multiplier. Let $N = (N_k)_{k\in\mathbb{N}} \subset \mathscr{L}(X)$ be a bounded sequence.

Let $\mathscr{S}(\mathbb{R})$ be the Schwartz space. Let $\phi_1 \in \mathscr{S}(\mathbb{R})$ be such that $\operatorname{supp}(\phi_1) \subset [2,4]$ and $\phi_1(3) = 1$. For $n \geq 1$, we let $h_n = 2^{2n-2}$. Define $\phi_n = \phi_1(\cdot/h_n)$. Then $\operatorname{supp}(\phi_n) \subset [2h_n, 4h_n]$ and $\phi_n(3h_n) = 1$. Let $\phi : \mathbb{R} \to \mathscr{L}(X)$ be defined by

$$\phi(t) = \begin{cases} \phi_n(t)N_n & \text{if } 2h_n \leq t \leq 4h_n \text{ for some } n \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Let $M = (\phi(k))_{k\in\mathbb{Z}}$. We claim that

$$\sup_{k\in\mathbb{Z}} \|\phi(k)\| < \infty, \tag{4}$$

$$\sup_{k\in\mathbb{Z}} \|k^l(\Delta^l M)(k)\| < \infty, \tag{5}$$

for $l \in \mathbb{N}$. Indeed (4) is clearly true. We will only give the proof for (5) when $l = 2$, the proof for the general case is similar.

First notice that when $4h_n \leq k \leq 8h_n - 2$ for some $n \in \mathbb{N}$, or $k \leq 0$, then $(\Delta^2 M)(k) = 0$. While when $2h_n - 2 < k < 4h_n$ for some $n \in \mathbb{N}$

$$\begin{aligned}(\Delta^2 M)(k) &= (\phi_n(k+2) - 2\phi_n(k+1) + \phi_n(k))N_n \\ &= \left(\phi_1\left(\frac{k+1}{h_n} + \frac{1}{h_n}\right) - 2\phi_1\left(\frac{k+1}{h_n}\right) + \phi_1\left(\frac{k+1}{h_n} - \frac{1}{h_n}\right)\right) N_n \\ &= \frac{1}{2h_n^2}(\phi_1''(\eta_1) + \phi_1''(\eta_2))N_n\end{aligned}$$

for some $\eta_1, \eta_2 \in \mathbb{R}$. We deduce that

$$\sup_{k\in\mathbb{Z}} \|k^2(\Delta^2 M)(k)\| \leq \sup_{n\in\mathbb{N}} \frac{16h_n^2}{4h_n^2}\|N_n\| \sup_{x\in\mathbb{R}} |\phi_1''(x)| \leq 4 \sup_{n\in\mathbb{N}} \|N_n\| \sup_{x\in\mathbb{R}} |\phi_1''(x)|.$$

Thus $M = (\phi(k))_{k\in\mathbb{Z}}$ is a periodic L^p-L^q-Fourier multiplier by assumption. Hence there exists $C > 0$ such that for $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$, we have

$$\left\|\sum_k e_k \otimes \phi(k)x_k\right\|_q \leq C\left\|\sum_k e_k \otimes x_k\right\|_p,$$

and, in particular,

$$\left\|\sum_{n\geq 1} e_{3h_n} \otimes M_k x_{3h_n}\right\|_q \leq C\left\|\sum_{n\geq 1} e_{3h_n} \otimes x_{3h_n}\right\|_p.$$

By (3), this implies that the sequence $(M_k)_{k\geq 1}$ is R-bounded. It is easy to check that if each countable subset of T is R-bounded then so is T. We deduce from this that each bounded subset in $\mathscr{L}(X)$ is actually R-bounded. By [3, Proposition 1.13], this implies that X is isomorphic to a Hilbert space. $\square$

3. Multipliers on the line

Let X be a Banach space and consider the Banach space $L^p(\mathbb{R};X)$ for $1 < p < \infty$. We denote by $\mathscr{D}(\mathbb{R};X)$ the space of all X-valued C^∞-functions with compact support. $\mathscr{S}(\mathbb{R};X)$ will be the X-valued Schwartz space and we let $\mathscr{S}'(\mathbb{R};X) := \mathscr{L}(\mathscr{S}(\mathbb{R});X)$, where $\mathscr{S}(\mathbb{R})$ denotes the $\mathbb{C}$-valued Schwartz space. Let Y be another Banach space. Then given $M \in L^1_{\mathrm{loc}}(\mathbb{R};\mathscr{L}(X,Y))$, we may define an operator $T : \mathscr{F}^{-1}\mathscr{D}(\mathbb{R};X) \to \mathscr{S}'(\mathbb{R};Y)$ by means of

$$T\phi := \mathscr{F}^{-1}M\mathscr{F}\phi \quad \text{for all } \mathscr{F}\phi \in \mathscr{D}(\mathbb{R};X),$$

where $\mathscr{F}$ denotes the Fourier transform. Since $\mathscr{F}^{-1}\mathscr{D}(\mathbb{R};X)$ is dense in $L^p(\mathbb{R};X)$, we see that T is well defined on a dense subset of $L^p(\mathbb{R};X)$. We say that M is an *L^p-Fourier multiplier* on $L^p(\mathbb{R};X)$ if T can be extended to a bounded linear operator from $L^p(\mathbb{R};X)$ to $L^p(\mathbb{R};Y)$.

The classical Mikhlin Fourier multiplier theorem has been extended to the operator-valued case by Weis. Let X and Y be UMD spaces, $1 < p < \infty$ and let $M \in C^1(\mathbb{R}\setminus\{0\},\mathscr{L}(X,Y))$. If both $\{M(x) : x \neq 0\}$ and $\{xM'(x) : x \neq 0\}$ are R-bounded, then M defines a L^p-Fourier multiplier on $L^p(\mathbb{R};X)$ [18].

The following result shows that one cannot replace R-boundedness in the operator-valued Mikhlin theorem above by boundedness in operator norm unless the underlying Banach space is isomorphic to a Hilbert space.

THEOREM 3. *Let X be a Banach space. Then the following assertions are equivalent*:

(i) *X is isomorphic to a Hilbert space.*

(ii) *For some $1 < p < \infty$, each function $M \in C^\infty(\mathbb{R};\mathscr{L}(X))$ satisfying*

(a) *$M(x) = 0$ for $x \le 0$,*

(b) *$\sup_{x\in\mathbb{R}} \|M(x)\| < \infty$,*

(c) *$\sup_{x\in\mathbb{R}}(1+|x|)^l\|M^{(l)}(x)\| < \infty$ for $l \in \mathbb{N}$,*

defines an L^p-Fourier multiplier on $L^p(\mathbb{R};X)$.

(iii) *For all $1 < p < \infty$, each function $M \in C^1(\mathbb{R}\setminus\{0\};\mathscr{L}(X))$ satisfying the conditions*

(a) *$\sup_{x\neq0}\|M(x)\| < \infty$,*

(b) *$\sup_{x\neq0}\|xM'(x)\| < \infty$,*

defines an L^p-Fourier multiplier on $L^p(\mathbb{R};X)$.

PROOF. (i) $\Rightarrow$ (iii). Assume that X is isomorphic to a Hilbert space. Then considering an orthonormal basis one easily verifies that each bounded subset in $\mathscr{L}(X)$ is actually R-bounded, so the result follows from the operator-valued Mikhlin Fourier multiplier theorem of Weis [18].

(iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i). Assume (ii) holds. Let $(M_k)_{k\geq 0} \subset \mathscr{L}(X)$ be a bounded sequence and let $\phi \in \mathscr{D}(\mathbb{R})$ satisfying $\operatorname{supp}(\phi) \subset [1,2]$, $\sup_{x\in\mathbb{R}} |\phi(x)| = 1$ and $\phi(3/2) = 1$. Define $M \in C^\infty(\mathbb{R}; \mathscr{L}(X))$ by

$$M(x) = \begin{cases} 0 & \text{if } x \leq 1; \\ \phi(2^{-k}x)M_k & \text{if } 2^k \leq x < 2^{k+1} \text{ for some } k \geq 0. \end{cases}$$

Then $\sup_{x\in\mathbb{R}} \|M(x)\| = \sup_{k\geq 0} \|M_k\| < \infty$ and for $l \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}} (1+|x|)^l \|M^{(l)}(x)\| \leq 2^m \left(\sup_{x\in\mathbb{R}} |x^l \phi^{(l)}(x)| \sup_{k\geq 0} \|M_k\| + \sup_{x\in\mathbb{R}} |\phi^{(l)}(x)| \sup_{k\geq 0} 2^{-lk} \|M_k\| \right) < \infty.$$

So M is an L^p-Fourier multiplier on $L^p(\mathbb{R}; X)$ by assumption. By [9, Proposition 1] this implies that the set $\{M(x) : x \in \mathbb{R}\}$ is R-bounded. In particular, the sequence $(M_k)_{k\geq 0}$ is R-bounded. We deduce from this that each bounded subset in $\mathscr{L}(X)$ is R-bounded, by [3, Proposition 1.13] X is isomorphic to a Hilbert space. $\square$

4. Multipliers on $\mathbb{Z}$

Let X, Y be Banach spaces and consider the Banach space $\ell^p(\mathbb{Z}; X)$ for $1 < p < \infty$. Let $\mathbb{T} = \{e^{it} : 0 \leq t < 2\pi\}$ be the torus. We consider the dense subspace P of $\ell^p(\mathbb{Z}; X)$ consisting of all elements having a finite support. Then for $f = (f_n)_{n\in\mathbb{Z}} \in P$, the Fourier transform of f is a function on $[-\pi, \pi]$ defined by $(\mathscr{F}f)(t) = \sum_{n\in\mathbb{Z}} f_n e^{int}$. Let $M \in L^\infty(-\pi, \pi; \mathscr{L}(X, Y))$. Then the function $M\mathscr{F}f$ is in $L^\infty(-\pi, \pi; Y)$, where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. We deduce that that $Tf := \mathscr{F}^{-1}(M\mathscr{F}f) \in c_0(\mathbb{Z}; Y)$ makes sense. We say that M is an *L^p-Fourier multiplier* on $\ell^p(\mathbb{Z}; X)$ if the mapping T can be extended to a bounded linear operator from $\ell^p(\mathbb{Z}; X)$ to $\ell^p(\mathbb{Z}; Y)$.

The classical Mikhlin Fourier multiplier theorem on $\ell^p(\mathbb{Z})$ has been extended to the operator-valued case by Blunck. Let $1 < p < \infty$, X be a UMD space, let $M \in C^1((-\pi, 0)\cup(0, \pi); \mathscr{L}(X))$ be such that both $\{M(t) : t \in (-\pi, 0)\cup(0, \pi)\}$ and $\{(e^{it} - 1)(e^{it}+1)M'(t) : t \in (-\pi, 0) \cup (0, \pi)\}$ are R-bounded. Then M is an L^p-Fourier multiplier on $\ell^p(\mathbb{Z}; X)$ [5]. In particular, each $M \in C^1([-\pi, 0) \cup (0, \pi]; \mathscr{L}(X))$ such that both $\{M(t) : t \neq 0\}$ and $\{tM'(t) : t \neq 0\}$ are R-bounded, defines an L^p-Fourier multiplier on $\ell^p(\mathbb{Z}; X)$. Blunck has also established the R-boundedness of

L^p-Fourier multipliers on $\ell^p(\mathbb{Z}; X)$: when M is an L^p-Fourier multiplier on $\ell^p(\mathbb{Z}; X)$, then $\{M(t) : t \text{ is a Lebesgue point of } M\}$ is R-bounded.

The following result shows that one cannot replace the R-boundedness in Blunck's result by the boundedness in operator norm unless the underlying Banach space is isomorphic to a Hilbert space. As the proof is similar to that of Theorem 3, we omit it.

THEOREM 4. *Let X be a Banach space. Then the following assertions are equivalent*:

(i) *X is isomorphic to a Hilbert space.*

(ii) *For some $1 < p < \infty$, each function $M \in C^\infty([-\pi, \pi]; \mathscr{L}(X))$ satisfying*

(a) $\sup_{x\in[-\pi,\pi]} \|M(x)\| < \infty$,

(b) $\sup_{x\in[-\pi,\pi]} |x|^l \|M^{(l)}(x)\| < \infty$ *for* $l \in \mathbb{N}$,

(c) $M(x) = 0$ *for* $x \leq 0$,

defines an L^p-Fourier multiplier on $\ell^p(\mathbb{Z}; X)$.

(iii) *For all $1 < p < \infty$, each function $M \in C^1([-\pi, 0) \cup (0, \pi]; \mathscr{L}(X))$ satisfying*

(a) $\sup_{x\neq0} \|M(x)\| < \infty$,

(b) $\sup_{x\neq0} \|xM'(x)\| < \infty$,

defines an L^p-Fourier multiplier on $\ell^p(\mathbb{Z}; X)$.

5. Concluding remarks

(a) One can actually show by using [3, Theorem 1.3] and the same argument as in the proof of Theorem 1, that when X and Y are UMD-spaces, then the assertions (ii) and (iii) in Theorem 1 are still equivalent for sequences in $\mathscr{L}(X, Y)$. Similarly, using [18, Theorem 3.4] (respectively, [5, Theorem 1.3]) one can show that when X and Y are UMD-spaces, the assertions (ii) and (iii) in Theorem 3 (respectively, Theorem 4) are still equivalent for functions with values in $\mathscr{L}(X, Y)$. Furthermore, these assertions are equivalent to X having cotype 2 and Y having type 2. This contains our Theorem 1, Theorem 3 and Theorem 4 by a result of Kwapien [14, pages 73–74], saying that a Banach space X is isomorphic to a Hilbert space if and only if X is of cotype 2 and of type 2.

(b) A restricted version of our results follows from the recent work of Kalton and Lancien on the maximal regularity problem [12]. In particular, the counterexample constructed in [12] can be used to show that the equivalences in Theorem 1 and Theorem 3 are true within the class of UMD Banach spaces which have an unconditional basis.

(c) In contrast to the L^p-spaces case, the situation for Hölder continuous function spaces is quite different. It has been shown that the operator-valued Marcinkiewicz (respectively, Mikhlin) Fourier multiplier theorem holds true on $C^{\alpha}_{\mathrm{per}}([0, 2\pi]; X)$ (respectively, $C^{\alpha}(\mathbb{R}; X)$) for every Banach space X and $0 < \alpha < 1$ and for each sequence $M = (M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X)$ satisfying a second order condition:

$$\sup_{k\in\mathbb{Z}} \|M_k\| + \sup_{k\in\mathbb{Z}} \|k(\Delta^1 M)(k)\| + \sup_{k\in\mathbb{Z}} \|k^2(\Delta^2 M)(k)\| < \infty$$

(respectively, each function $M \in C^2(\mathbb{R} \setminus \{0\}; \mathscr{L}(X))$ satisfying a second order condition: $\sup_{x\neq 0} \|M(x)\| + \sup_{x\neq 0} \|xM'(x)\| + \sup_{x\neq 0} \|x^2 M''(x)\| < \infty$) (see Amann [1] and [2]). Here $C^{\alpha}_{\mathrm{per}}([0, 2\pi]; X)$ denotes the space of all functions in $C^{\alpha}(\mathbb{R}, X)$ which are 2π-periodic. If the Banach space has a non-trivial type, then even the Marcinkiewicz condition of order 1 suffices (see [2] and [11]).

(d) Periodic L^p-Fourier multipliers (respectively, L^p-Fourier multipliers on $L^p(\mathbb{R}; X)$) of the form $M = (m_k I)_{k\in\mathbb{Z}}$, where $m_k \in \mathbb{C}$ for $k \in \mathbb{Z}$ (respectively, $M = f\, I$, where $f \in C^1(\mathbb{R} \setminus \{0\})$) on $L^p(0, 2\pi; X)$ (respectively, on $L^p(\mathbb{R}; X)$) have been studied by Zimmermann [19], where I denotes the identity operator on X. Actually Zimmermann's results follow from the operator-valued Marcinkiewicz (respectively, Mikhlin) Fourier multiplier theorem established in [3] (respectively, in [18]) as each subset $M \subset \mathscr{L}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R-bounded whenever $\Omega \subset \mathbb{C}$ is bounded. Zimmermann's results together with a result of Burkholder [7] show that the scalar-valued Marcinkiewicz (respectively, Mikhlin) Fourier multiplier theorem holds true for $L^p(0, 2\pi; X)$ (respectively, $L^p(\mathbb{R}; X)$) for some $1 < p < \infty$ if and only if X is a UMD space. A similar result characterizing UMD spaces via a scalar-valued Fourier multiplier theorem on $\ell^p(\mathbb{Z}; X)$ can be established based on results in [4].

(e) It is remarkable that in all three cases we consider here (Theorem 1, Theorem 3 and Theorem 4), the sequence $(M_k)_{k\in\mathbb{Z}}$ (or the function M) satisfying the Marcinkiewicz condition (of order l) without being a Fourier multiplier consists of operators of rank 1 (see [3, Proposition 1.13.]).

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Abteilung Angewandte Analysis
Universität Ulm
89069 Ulm
Germany
e-mail: arendt@mathematik.uni-ulm.de

Department of Mathematical Science
University of Tsinghua
Beijing 100084
China
e-mail: sbu@math.tsinghua.edu.cn