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THE EXPONENTIAL REPRESENTATION OF HOLOMORPHIC FUNCTIONS OF UNIFORMLY BOUNDED TYPE

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Abstract

It is shown that if E, F are Fréchet spaces, $E \in (H_{ub})$, $F \in (DN)$ then $H(E, F) = H_{ub}(E, F)$ holds. Using this result we prove that a Fréchet space E is nuclear and has the property (H_{ub}) if and only if every entire function on E with values in a Fréchet space $F \in (DN)$ can be represented in the exponential form. Moreover, it is also shown that if $H(F^*)$ has a LAERS and $E \in (H_{ub})$ then $H(E \times F^*)$ has a LAERS, where E, F are nuclear Fréchet spaces, F^* has an absolute basis, and conversely, if $H(E \times F^*)$ has a LAERS and $F \in (DN)$ then $E \in (H_{ub})$.

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1. Introduction

Let E and F be locally convex spaces. A holomorphic function f from E to F is said to be *of uniformly bounded type* if there exists a neighbourhood U of $0 \in E$ such that $f(rU)$ is bounded for all $r > 0$. By $H_{ub}(E, F)$ we denote the linear subspace of the space of holomorphic functions from E to F, consisting of all functions of uniformly bounded type. We write $H_{ub}(E)$ rather than $H_{ub}(E, \mathbb{C})$. We say that a locally convex space E has the *property* (H_{ub}) (and write $E \in (H_{ub})$) if the identity $H(E) = H_{ub}(E)$ holds.

Recently Le Mau Hai and Thai Thuan Quang [3] have shown that $H_b(E, F) = H_{ub}(E, F)$ for Fréchet spaces E, F and $E \in (H_{ub})$, $F \in (\overline{DN})$.

In Section 3, we extend the above result to the more general case. Namely, the property $(\overline{DN})$ of the space F is replaced by the property (DN) (Theorem 3.1).

By using Theorem 3.1, in Section 4 we prove that a Fréchet space E is nuclear and $E \in (H_{ub})$ if and only if every entire function on E with values in a Fréchet space $F \in (DN)$ can be represented in the exponential form (Theorem 4.1). The final portion of this section will deal with the exponential representation of holomorphic functions on $E \times F^*$, where E is Fréchet, $E \in (H_{ub})$ and F is nuclear Fréchet such that F^* has an absolute basis and $H(F^*)$ has a linearly absolutely exponential representation system (Theorem 4.2). The proof is based on Theorems 3.1, 4.1 and Proposition 4.3 which say that if F is a Fréchet space such that F^* has an absolute basis then $F \in (DN)$ if and only if $H(F^*) \in (DN)$.

2. Preliminaries

We may frequently use the standard notation of the theory of locally convex spaces as presented in the book of Pietsch [9]. A locally convex space always is a complex vector space with a locally convex Hausdorff topology.

For a Fréchet space E, we always assume that its locally convex structure is generated by an increasing system $\{\|\cdot\|_k\}$ of semi-norms. Then we denote by E_k the completion of the canonically normed space $E/\ker\|\cdot\|_k$ and $\omega_k : E \to E_k$ denotes the canonical map and U_k denotes the set $\{x \in E : \|x\|_k < 1\}$.

2.1. Holomorphic function Let E and F be locally convex spaces and let $D \subset E$ be open, $D \neq \emptyset$. A function $f : D \to F$ is called *holomorphic* if f is continuous and Gâteaux-analytic. By $H(D, F)$ we denote the vector space of all holomorphic functions on D with values in F. We use $H_b(E, F)$ to denote the space of holomorphic functions from E to F which are bounded on every bounded set in E. The space $H_b(E, F)$ is equipped with the topology τ_b of uniform convergence on all bounded sets.

2.2. The property (DN) Let E be a Fréchet space with a fundamental system of semi-norms $\{\|\cdot\|_k\}$. We say that E has the *property* (DN) if

$$\exists p\ \exists d\ \forall q\ \exists k, C > 0 \quad \text{such that} \quad \|\cdot\|_q^{1+d} \le C\|\cdot\|_k\|\cdot\|_p^d.$$

2.3. Separately holomorphic function Let E, F be locally convex spaces. For a function $f : E \times F \to \mathbb{C}$, we put

$$f_x(y) = f(x, y) \quad \text{for} \quad y \in F;$$
$$f_y(x) = f(x, y) \quad \text{for} \quad x \in E.$$

The function f is called *separately holomorphic* if $f_x : F \to \mathbb{C}$ and $f_y : E \to \mathbb{C}$ are holomorphic for all $x \in E$ and $y \in F$ respectively.

2.4. Linearly absolutely exponential representation system Let E be a locally convex space and $\{x_k\}$ be a sequence in E. We say that $\{x_k\}$ is a *linearly absolute representation system* (abbreviated LARS) if every element $x \in E$ can be written in the form $x = \sum_{k\geq 1} \xi_k(x)x_k$, where the series is absolutely convergent.

A LARS in $H(D)$ of the form $\{\exp u_k\}$, where u_k are continuous linear functionals on E and D is an open set in E, is said to be a linearly absolutely exponential representation system of $H(D)$. It is denoted by LAERS.

3. Holomorphic functions of uniformly bounded type

In this section we prove the following theorem which was proved in [3] by Le Mau Hai and Thai Thuan Quang in the case when $F \in (\overline{DN})$.

THEOREM 3.1. *Let E be a Fréchet space. Then $E \in (H_{ub})$ if and only if*

$$H(E, F) = H_{ub}(E, F)$$

for all Fréchet spaces $F \in (DN)$.

PROOF. 1. *Necessary*. Since $F \in (DN)$, by Vogt [13] F can be considered as a subspace of the space $B\widehat{\otimes}_\pi s$ for some Banach space, where s denotes the space of rapidly decreasing sequences.

On the other hand, $B\widehat{\otimes}_\pi s$ is a subspace of $H_b((B\widehat{\otimes}_\pi s)'_b)$ and $H_b((B\widehat{\otimes}_\pi s)'_b) \cong H_b(B'\widehat{\otimes}_\pi s')$.

Given $f \in H(E, F) \subset H(E, H_b(B'\widehat{\otimes}_\pi s'))$, we write the Taylor expansion of $f(z)$ at $0 \in B'\widehat{\otimes}_\pi s'$ for $z \in E$

$$f(z)(t) = \sum_{n\geq 0} \sum_{\substack{j_1,\dots,j_n\geq 1 \\ k_1,\dots,k_n\geq 1}} \widehat{P_n f(z)}(u_{k_1} \otimes e^*_{j_1}, \dots, u_{k_n} \otimes e^*_{j_n}),$$

where

$$t = \sum_{k\geq 1} u_k \otimes v_k \in B'\widehat{\otimes}_\pi s', \quad P_n f(z)(t) = \frac{1}{2\pi i} \int_{|\varrho|=r>0} \frac{f(z)(\varrho t)}{\varrho^{n+1}} d\varrho,$$

and $\{e_j\}$ is the canonical basis of s with the dual basis $\{e^*_j\}$, $\widehat{P_n f}$ is the symmetric n-linear form associated to $P_n f$.

Since $\{e_j\}_{j\geq 1}$ is an absolute basis, for $p \geq 1$, choose $q \geq p$ such that

$$\sum_{j\geq 1} \|e^*_j\|^*_q \|e_j\|_p < \frac{1}{(p+1)e^2},$$

where $\|e_j^*\|_q^* = \sup\{|e_j^*(x)|, \|x\|_q \le 1\} = 1/j^q$.

For each $p \ge 1$, consider a family $\mathscr{F}_p = \{f_{p,n,u_1,\ldots,u_n}\}_{n\ge 0} \subset H_b(E)$ given by

$$f_{p,n,u_1,\ldots,u_n}(z) = \sum_{j_1,\ldots,j_n\ge 1} p^n \widehat{P_n f(z)}(u_1 \otimes e_{j_1}^*, \ldots, u_n \otimes e_{j_n}^*)\|e_{j_1}\|_p \cdots \|e_{j_n}\|_p$$

where $u_1, \ldots, u_n \in W$, the unit ball of B'.

Then for each $p \ge 1$, the family $\mathscr{F}_p$ is bounded in $H_b(E)$. Indeed, for every bounded set K in E, we have

$$\begin{aligned}
&\sup_{z\in K}\sup_{\substack{u_1,\ldots,u_n\in W\\ n\ge 0}} \left\{p^n \sum_{j_1,\ldots,j_n\ge 1} \left|\widehat{P_n f(z)}(u_1 \otimes e_{j_1}^*, \ldots, u_n \otimes e_{j_n}^*)\right| \|e_{j_1}\|_p \cdots \|e_{j_n}\|_p\right\} \\
&\quad = \sup_{z\in K}\sup_{\substack{n\ge 0\\ u_1,\ldots,u_n\in W}} \left\{p^n \sum_{j_1,\ldots,j_n\ge 1} \left|\widehat{P_n f(z)}\left(u_1 \otimes \frac{e_{j_1}^*}{\|e_{j_1}^*\|_q^*}, \ldots, u_n \otimes \frac{e_{j_n}^*}{\|e_{j_n}^*\|_q^*}\right)\right|\right. \\
&\qquad \left. \times \|e_{j_1}\|_p \|e_{j_1}^*\|_q^* \cdots \|e_{j_n}\|_p \|e_{j_n}^*\|_q^*\right\} \\
&\quad \le \sup_{z\in K}\sup_{t\in \mathrm{conv}(W\otimes U_q^o)} |f(z)(t)| \left\{\sup_{n\ge 0} \frac{n^n p^n}{n!} \sum_{j_1,\ldots,j_n\ge 1} \|e_{j_1}\|_p \|e_{j_1}^*\|_q^* \cdots \|e_{j_n}\|_p \|e_{j_n}^*\|_q^*\right\} \\
&\quad \le \|f\|_{K\times \mathrm{conv}(W\otimes U_q^o)} \sup_{n\ge 0}\left\{\left(\frac{np}{(p+1)e^2}\right)^n \frac{1}{n!}\right\} \\
&\quad \le C_p \|f\|_{K\times \mathrm{conv}(W\otimes U_q^o)},
\end{aligned}$$

where

$$U_q = \{x \in s : \|x\|_q < 1\} \quad \text{with the polar } U_q^o,$$
$$C_p = \sup_{n\ge 0}\left\{\left(\frac{np}{(p+1)e^2}\right)^n \frac{1}{n!}\right\}.$$

Since $E \in (H_{ub})$, by Meise and Vogt [5], there exists $\alpha \ge 1$ such that the family $\mathscr{F}_p$ is bounded in $H_b(E_\alpha)$.

However, for every bounded set $K \subset E_\alpha$, $p \ge 1$, we have the following estimate

$$\begin{aligned}
(3.1)\quad &\sup_{z\in K}\sup_{t\in \mathrm{conv}(W\otimes U_q^o)} \sum_{n\ge 0} |P_n f(z)(t)| \\
&\quad \le \sup_{z\in K}\sup_{\substack{u_{k_1},\ldots,u_{k_n}\in W\\ \sum_{k\ge 1}|\lambda_k|\le 1}} \left\{\sum_{n\ge 0} \frac{1}{p^n} \sum_{k_1,\ldots,k_n\ge 1} |\lambda_{k_1}| \cdots |\lambda_{k_n}|\right. \\
&\qquad \left. \times \sum_{j_1,\ldots,j_n\ge 1} p^n \left|\widehat{P_n f(z)}(u_{k_1} \otimes e_{j_1}^*, \ldots, u_{k_n} \otimes e_{j_n}^*)\right| \|e_{j_1}\|_p \cdots \|e_{j_n}\|_p\right\}
\end{aligned}$$

$$\leq \sup_{z\in K} \sup_{\substack{u_1,\dots,u_n\in W\\ n\geq 0}} \left\{ p^n \sum_{j_1,\dots,j_n\geq 1} \left| \widehat{P_n f\,(z)}(u_1 \otimes e^*_{j_1}, \dots, u_n \otimes e^*_{j_n}) \right| \right.$$

$$\left. \times \|e_{j_1}\|_p \cdots \|e_{j_n}\|_p \right\} \sup_{\sum_{k\geq 1}|\lambda_k|\leq 1} \left\{ \sum_{n\geq 0} \frac{1}{p^n} \sum_{k_1,\dots,k_n\geq 1} |\lambda_{k_1}| \cdots |\lambda_{k_n}| \right\}$$

$$\leq C_K \sum_{n\geq 0} \frac{1}{p^n}$$

where

$$C_K = \sup_{z\in K} \sup_{\substack{u_1,\dots,u_n\in W\\ n\geq 0}} \left\{ p^n \sum_{j_1,\dots,j_n\geq 1} \left| \widehat{P_n f\,(z)}(u_{k_1} \otimes e^*_{j_1}, \dots, u_{k_n} \otimes e^*_{j_n}) \right| \right.$$

$$\left. \times \|e_{j_1}\|_p \cdots \|e_{j_n}\|_p \right\} < \infty.$$

Put

$$g(z, t) = \sum_{n\geq 0} P_n f\,(z)(t) \tag{3.2}$$

with $z \in E_\alpha$, $t \in (B\widehat{\otimes}_\pi s)'$.

From (3.1), it follows that the right-hand side of (3.2) converges and defines a seperately holomorphic function on $E_\alpha \times (B\widehat{\otimes}_\pi s)'$.

It is easy to see that g is bounded on every bounded set on $(B\widehat{\otimes}_\pi s)'$. By Galindo, Garcia, Maestre [1] the holomorphic function of bounded type

$$\widehat{g} : (B\widehat{\otimes}_\pi s)' \to H_b(E_\alpha)$$

which is induced by g, can be factorized through a Banach space by an entire function of bounded type. Because every ball in a Banach space is bounded we infer $f \in H_{ub}(E, F)$.

2. *Sufficient*. In the case $F = \mathbb{C}$, by hypothesis we obtain $E \in (H_{ub})$.

The theorem is proved. □

4. The exponential representation

First we recall that a locally convex space E has the property (H_u) and write $E \in (H_u)$ if every holomorphic function f on E is of uniform type. This means that there exists a continuous semi-norm ϱ on E such that f can be factorized holomorphically through the canonical map $\omega_\varrho : E \to E_\varrho$, where E_ϱ denote the space associated to ϱ.

In [7] Nguyen Minh Ha and Nguyen Van Khue proved that a Fréchet space E is nuclear and $E \in (H_u)$ if and only if every holomorphic function on E with values in a Banach space B can be written in the form $f(x) = \sum_{k\geq 1} \xi_k \exp u_k(x)$, where the series is absolutely convergent in the space $H(E, B)$ of holomorphic functions on E with values in B equipped with the compact-open topology.

In this section we shall consider the above result in another situation with the note that if $E \in (H_{ub})$ then $E \in (H_u)$ and if F is Banach then $F \in (DN)$. Namely, we are going to prove the following:

THEOREM 4.1. *Let E be a Fréchet space. Then E is nuclear and $E \in (H_{ub})$ if and only if every holomorphic function on E with values in a Fréchet space $F \in (DN)$ can be written in the form $f(x) = \sum_{k\geq 1} \xi_k \exp u_k(x)$, where the sequences $(\xi_k) \subset F$, $(u_k) \subset E^*$, the dual space of E, and the series is absolutely convergent in the space $H_b(E, F)$.*

PROOF. First we prove sufficiency of the theorem.

Let $\{p_\alpha\}$ be a fundamental system of semi-norms on E. To prove the nuclearity of E, for every continuous semi-norm ϱ on E write the canonical map $\omega_\varrho : E \to E_\varrho$ in the form $\omega_\varrho(x) = \sum_{k\geq 1} \xi_k \exp u_k(x)$ in which $\sum_{k\geq 1} \|\xi_k\| \exp \|u_k\|_B^* < \infty$ for every bounded set B in E. This follows from the hypothesis and the property (DN) of the space Banach E_ϱ. Then

$$\omega_\varrho(x) = d\omega_\varrho(0)(x) = \sum_{k\geq 1} \xi_k u_k(x)$$

for $x \in E$ and $\sum_{k\geq 1} \|\xi_k\| \|u_k\|_B^* < \infty$ for every bounded set B in E.

Now we prove that there exists a continuous semi-norm $\beta > \varrho$ in E such that

$$\sum_{k\geq 1} \|\xi_k\| \exp \|u_k\|_\beta^* < \infty.$$

Indeed, if this does not hold, for every α we have $\sum_{k\geq 1} \|\xi_k\| \exp \|u_k\|_\alpha^* = \infty$. Hence for every α there exists k_α such that $\sum_{k\leq k_\alpha} \|\xi_k\| \exp \|u_k\|_\alpha^* > \alpha$. This inequality implies that for each $k \leq k_\alpha$ there exists x_k^α with $\|x_k^\alpha\|_\alpha \leq 1$ such that

$$\sum_{k\leq k_\alpha} \|\xi_k\| \exp |u_k(x_k^\alpha)| > \alpha.$$

Put $B = \{x_1^1, \dots, x_{k_1}^1, \dots, x_1^\alpha, \dots, x_{k_\alpha}^\alpha, \dots\} \cup \{0\}$. Then B is bounded in E and

$$\sum_{k\geq 1} \|\xi_k\| \exp \|u_k\|_B^* > \alpha \quad \text{for every} \quad \alpha \geq 1.$$

This is impossible, because $\sum_{k\geq 1} \|\xi_k\| \exp \|u_k\|_B^* < \infty$.

By the same argument as above, there exists a continuous semi-norm $\beta > \varrho$ in E such that $\sum_{k\geq 1} \|\xi_k\| \|u_k\|_\beta^* < \infty$. This means that the canonical map $\omega_{\beta\varrho} : E_\beta \to E_\varrho$ is nuclear. Hence E is nuclear.

Now, since E is nuclear, to prove $E \in (H_{ub})$ by [4] it suffices to show that if E is a topological subspace of a locally convex space G with a fundamental system of continuous semi-norm induced by semi-inner products then every $f \in H(E)$ has an extension $g \in H(G)$.

Given $f \in H(E, \mathbb{C}) = H(E)$, by the hypothesis, we can write

$$f(x) = \sum_{k\geq 1} \xi_k \exp u_k(x)$$

such that $\sum_{k\geq 1} |\xi_k| \exp |u_k(x)| < \infty$. Applying the Hahn-Banach theorem to $u_k \in E^*$, $k \geq 1$, there exist $\hat{u}_k \in G^*$ such that $\hat{u}_k|_E = u_k$ and $\|\hat{u}_k\|^* = \|u_k\|^*$, for all $k \geq 1$. Then the function $g(x) = \sum_{k\geq 1} \xi_k \exp \hat{u}_k(x)$ defines a holomorphic function on G and $g|_E = f$.

Now, assume that E is nuclear and $E \in (H_{ub})$. By Theorem 3.1, we have $H(E, F) = H_{ub}(E, F)$. Then every $f \in H(E, F)$ is of uniform type. It implies that there exists a continuous semi-norm ϱ on E and a holomorphic function g on E_ϱ such that $f = g\omega_\varrho$. Take a continuous semi-norm $\beta > \varrho$ on E such that $T = \omega_{\beta\varrho}$ is nuclear. Write

$$T(x) = \sum_{j\geq 1} t_j u_j(x) e_j$$

with $a = \sum_{j\geq 1} |t_j| < \infty$ and $\|u_j^*\| < 1$, $\|e_j\| < 1$, $e_j \in E_\varrho$, $u_j \in E_\beta^*$ for $j \geq 1$. Consider the Taylor expansion of g at $0 \in E$

$$g(x) = \sum_{n\geq 0} P_n g(x),$$

where

$$P_n g(x) = \frac{1}{2\pi i} \int_{|t|=r} \frac{g(tx)}{t^{n+1}}\, dt.$$

Choose two sequences $\{\xi_k\}$ and $\{\alpha_k\}$ in $\mathbb{C}$ such that

$$z = \sum_{k\geq 1} \xi_k \exp(\alpha_k z) \tag{4.1}$$

for $z \in \mathbb{C}$ and

$$C_r = \sum_{k\geq 1} |\xi_k| \exp(r|\alpha_k|) < \infty \tag{4.2}$$

for all $r > 0$. Such sequences exist by [2]. Formally, we have

$$\begin{aligned}
(gT)(x) = g(Tx) &= \sum_{n \geq 0} P_n g(Tx) = \sum_{n \geq 0} P_n g\left(\sum_{j \geq 1} t_j u_j(x) e_j\right) \\
&= \sum_{n \geq 0} \sum_{j_1, \dots, j_n \geq 1} t_{j_1} \cdots t_{j_n} u_{j_1}(x) \cdots u_{j_n}(x) \widehat{P_n g}(e_{j_1}, \dots, e_{j_n}) \\
&= \sum_{n \geq 0} \sum_{j_1, \dots, j_n \geq 1} t_{j_1} \cdots t_{j_n} \widehat{P_n g}(e_{j_1}, \dots, e_{j_n}) \\
&\quad \times \left(\sum_{k \geq 1} \xi_k \exp \alpha_k u_{j_1}(x)\right) \cdots \left(\sum_{k \geq 1} \xi_k \exp \alpha_k u_{j_n}(x)\right) \\
&= \sum_{n \geq 0} \sum_{\substack{j_1, \dots, j_n \geq 1 \\ k_1, \dots, k_n \geq 1}} t_{j_1} \cdots t_{j_n} \xi_{k_1} \cdots \xi_{k_n} \widehat{P_n g}(e_{j_1}, \dots, e_{j_n}) \\
&\quad \times \exp[\alpha_{k_1} u_{j_1}(x) + \cdots + \alpha_{k_n} u_{j_n}(x)]
\end{aligned}$$

where $\widehat{P_n g}$ is the symmetric n-linear form associated to $P_n g$.

It remains to check that the right-hand side is absolutely convergent in $H(E, F)$. For each $r > 0$, take $s > C_r a e$. Since

$$\| P_n g(e_{j_1}, \dots, e_{j_n}) \|_q \leq \left(\frac{n^n}{n! s^n}\right) \|g\|_{s,q}$$

where

$$\|g\|_{s,q} = \sup\{\|g(x)\|_q : \|x\| < s\}$$

and without loss generality by the nuclearity of E, we may assume that g is bounded on every bounded set in E_ϱ. We have

$$\begin{aligned}
\|g(Tx)\|_q &\leq \sum_{n \geq 0} \sum_{\substack{j_1, \dots, j_n \geq 1 \\ k_1, \dots, k_n \geq 1}} |t_{j_1}| \cdots |t_{j_n}| |\xi_{k_1}| \cdots |\xi_{k_n}| \\
&\quad \times \left\| \widehat{P_n g}(e_{j_1}, \dots, e_{j_n}) \right\|_q \exp[r(|\alpha_{k_1}| + \cdots + |\alpha_{k_n}|)] \\
&\leq \|g\|_{s,q} \sum_{n \geq 0} \frac{C_r^n a^n n^n}{n! s^n} < \infty
\end{aligned}$$

for $\|x\| < r$.

The theorem is proved. □

In order to complete this section we will prove the following:

THEOREM 4.2. *Let E and F be nuclear Fréchet spaces such that F^* has an absolute basis. Then*

(i) *$H(E \times F^*)$ has a LAERS if $H(F^*)$ has a LAERS and $E \in (H_{ub})$;*
(ii) *conversely, if $H(E \times F^*)$ has a LAERS and $F \in (DN)$ then $E \in (H_{ub})$.*

The proof is based on Theorem 3.1 and the following:

PROPOSITION 4.3. *Let F be a Fréchet space such that F^* has an absolute basis. Then $F \in (DN)$ if and only if $H(F^*) \in (DN)$.*

PROOF. Sufficiency is obvious because F can be considered as a subspace of $H(F^*)$.

The proof of necessary condition is based on the results of Ryan [10] which introduces a convenient system of semi-norms defining the topology of $H(F^*)$.

Assume that $F \in (DN)$ such that F^* has an absolute basis $\{e_j^*\}$. By the open mapping theorem, the topology of F can be defined by the system of semi-norms

$$|||x|||_k = \sum_{j\geq 1} |e_j^*(x)| \|e_j\|_k,$$

where $\{e_j\}_j$ is the sequence of coefficient functionals associated to $\{e_j^*\}$. Choose $p \geq 1$ such that (DN) holds. Since $\|\cdot\|_p$ is a norm, we have $\|e_j\|_p \neq 0$, for $j \geq 1$. Hence

$$\|e_j^*\|_q^* \leq \|e_j^*\|_p^* \leq \frac{1}{\|e_j\|_p} < \infty, \quad \forall j \geq 1, \forall q \geq p.$$

Moreover, by the definition of (DN) and by the equality

$$|||e_j^*|||_q^* = \frac{1}{|||e_j|||_q}, \quad \forall j \geq 1,$$

there exists d sucth that for every q there exist k, $C > 0$ such that

$$|||e_j^*|||_q^{*1+d} \geq C|||e_j^*|||_k^* |||e_j^*|||_p^{*d} \quad \forall j \geq 1. \tag{4.3}$$

For each $k \geq 1$, put

$$F^*(k) = \left\{ x^* \in F^* : |||x^*|||_k^* = \sum_{j\geq 1} |e_j^*(x)| |||e_j^*|||_k^* < \infty \right\}.$$

It is easy to check that $F^*(k)$ is a Banach space and $\{e_j^*\}$ is also an absolute basis for $F^*(k)$. On the other hand, since $F^* = \bigcup_{k=1}^{\infty} F^*(k)$ and every bounded set in F^* is contained and bounded in some $F^*(k)$, we conclude that the topology of $H(F^*)$ can be defined by the system of semi-norms $\{\|\cdot\|_{k,r} : k \geq 1, r > 0\}$, where

$$\|f\|_{k,r} = \sup\{\|f(x^*)\| : \|x^*\|_k \leq r\}.$$

By applying results of Ryan [10], we obtain, for every $f \in H(F^*)$, the representation

$$f(x^*) = f\left(\sum_{j\geq 1} t_j e_j^*\right) = \sum_M b_m(f) t^m$$

which converges absolutely and uniformly on every bounded set in F^*, where

$$M = \{(m_1, m_2, \dots, m_n, 0, \dots),\ m_i \in \mathbb{N},\ i = 1, 2, \dots\},$$
$$b_m(f) = \left(\frac{1}{2\pi i}\right)^n \int_{|t_1|=\varrho_1} \cdots \int_{|t_n|=\varrho_n} \frac{f(t_1 e_1^* + \cdots + t_n e_n^*)}{t_1^{m_1+1} \cdots t_n^{m_n+1}} dt$$

with $dt = dt_1 \cdots dt_n$.

Since for every $k \geq p$ the map

$$\varphi_k : \ell^1 \to F^*(k)$$
$$(\xi_j) \mapsto \sum_{j\geq 1} \xi_j \frac{e_j^*}{\|e_j^*\|_k^*}$$

is an isomorphism, again by Ryan [10], the system $\{\|\cdot\|_{k,r} : k \geq p, r > 0\}$ is equivalent to the system of semi-norms $\{|||\cdot|||_{k,r} : k \geq p, r > 0\}$ where

$$|||f|||_{k,r} = \sup\left\{\frac{r^{|m|}|b_m(f)|m^m}{a^m_{\cdot,k}|m|^{|m|}},\ m \in M\right\}$$

and $|m| = m_1 + \cdots + m_n$; $a^m_{\cdot,k} = |||e_1^*|||_k^{*m_1} \cdots |||e_n^*|||_k^{*m_n}$, $m^m = m_1^{m_1} \cdots m_n^{m_n}$.

From (4.3) we get

$$\begin{aligned}
|||\cdot|||_{q,r}^{1+d} &\leq \sup\left\{\left[\frac{r^{|m|}|b_m(\cdot)|m^m}{a^m_{\cdot,q}|m|^{|m|}}\right]^{1+d},\ m \in M\right\} \\
&\leq \sup\left\{\left(\frac{r^{1+d}}{C}\right)^{|m|}\frac{|b_m(\cdot)|m^m}{a^m_{\cdot,k}|m|^{|m|}},\ m \in M\right\} \sup\left\{\left[\frac{|b_m(\cdot)|m^m}{a^m_{\cdot,p}|m|^{|m|}}\right]^d,\ m \in M\right\} \\
&= |||\cdot|||_{k,r^{1+d}/C}|||\cdot|||_{p,1}^d
\end{aligned}$$

Hence $H(F^*) \in (DN)$. The proposition is proved. $\square$

Now we prove Theorem 4.2.

(i) First note that F is reflexive, that is, $F^{**} = F$, because it is a Fréchet nuclear space. Since F is nuclear and $H(F^*)$ has a LAERS, by [7], $F \in (DN)$. According to Proposition 4.3, we have $H(F^*) \in (DN)$. Because E is nuclear and $H(E \times F^*) \cong H(E, H(F^*))$, it follows from Theorem 4.1 that every $f \in H(E, H(F^*))$ can be written in the form $f(x) = \sum_{k\geq 1} \xi_k \exp u_k(x)$, where $(\xi_k) \subset H(F^*)$ and the series is absolutely convergent in the $H(E, H(F^*))$.

Moreover, since $H(F^*)$ has a LAERS, every $\xi_k \in H(F^*)$ can be also written in the form

$$\xi_k(y^*) = \sum_{j\geq 1} \eta_{j,k} \exp v_j(y^*), \quad \forall k \geq 1,$$

where $(\eta_{j,k}) \subset \mathbb{C}$ and $v_j \in F^{**} = F, j \geq 1$ and the series is absolutely convergent in the $H(F^*)$. Thus

$$\begin{aligned} f(x, y^*) &= \sum_{k,j\geq 1} \eta_{j,k} \exp v_j(y^*) \exp u_k(x) \\ &= \sum_{k,j\geq 1} \eta_{j,k} \exp\left[\langle u_k, x\rangle + \langle v_j, y^*\rangle\right] \end{aligned}$$

and $\sum_{k,j\geq 1} |\eta_{j,k}| \exp\left[\|u_k\|_K^* + \|v_j\|_L^*\right] < \infty$ for every compact set $K \subset E$, $L \subset F^*$.

(ii) It is an immediate consequence of Theorem 4.1.

This completes the proof of the theorem. □

REMARK 4.4. Recently, Phan Thien Danh and Duong Luong Son [8] have proved that every separately holomorphic function on an open subset $U \times V$ of $E \times F^*$, where $E \in (\widetilde{\Omega})$ is a Fréchet nuclear space having a basis, has a local Dirichlet representation if and only if $F \in (DN)$ for every Fréchet nuclear space F.

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