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ON ψ-DIRECT SUMS OF BANACH SPACES AND CONVEXITY

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Dedicated to Maestro Ivry Gitlis on his 80th birthday with deep respect and affection

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Abstract

Let $X_1, X_2, \dots, X_N$ be Banach spaces and ψ a continuous convex function with some appropriate conditions on a certain convex set in $\mathbb{R}^{N-1}$. Let $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ be the direct sum of $X_1, X_2, \dots, X_N$ equipped with the norm associated with ψ. We characterize the strict, uniform, and locally uniform convexity of $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ by means of the convex function ψ. As an application these convexities are characterized for the $\ell_{p,q}$-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p,q}$ $(1 < q \leq p \leq \infty, q < \infty)$, which includes the well-known facts for the ℓ_p-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_p$ in the case $p = q$.

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1. Introduction and preliminaries

A norm $\|\cdot\|$ on $\mathbb{C}^N$ is called *absolute* if $\|(z_1, \dots, z_N)\| = \|(|z_1|, \dots, |z_N|)\|$ for all $(z_1, \dots, z_N) \in \mathbb{C}^N$, and *normalized* if $\|(1, 0, \dots, 0)\| = \cdots = \|(0, \dots, 0, 1)\| = 1$ (see for example [3, 2]). In case of $N = 2$, according to Bonsall and Duncan [3] (see also [12]), for every absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^2$ there corresponds a unique continuous convex function ψ on the unit interval $[0, 1]$ satisfying

$$\max\{1 - t, t\} \leq \psi(t) \leq 1$$

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under the equation $\psi(t) = \|(1-t, t)\|$. Recently in [11] Saito, Kato and Takahashi presented the N-dimensional version of this fact, which states that for every absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^N$ there corresponds a unique continuous convex function ψ satisfying some appropriate conditions on the convex set

$$\Delta_N = \left\{ t = (t_1, \dots, t_{N-1}) \in \mathbb{R}^{N-1} : \sum_{j=1}^{N-1} t_j \le 1, t_j \ge 0 \right\}$$

under the equation $\psi(t) = \left\| \left(1 - \sum_{j=1}^{N-1} t_j, t_1, \dots, t_{N-1}\right) \right\|$.

For an arbitrary finite number of Banach spaces $X_1, X_2, \dots, X_N$, we define the *ψ-direct sum* $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ to be their direct sum equipped with the norm

$$\|(x_1, x_2, \dots, x_N)\|_\psi = \|(\|x_1\|, \|x_2\|, \dots, \|x_N\|)\|_\psi \quad \text{for } x_j \in X_j,$$

where $\|\cdot\|_\psi$ term in the right-hand side is the absolute normalized norm on $\mathbb{C}^N$ with the corresponding convex function ψ. This extends the notion of ℓ_p-sum of Banach spaces. The aim of this paper is to characterize the strict, and uniform convexity of $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$. The locally uniform convexity is also included. For the case $N = 2$, the first two have been recently proved in Takahashi-Kato-Saito [13] and Saito-Kato [10], respectively. However the proof of the uniform convexity for the 2-dimensional case given in [10] seems difficult to be extended to the N-dimensional case, though it is of independent interest as it is of real analytic nature and maybe useful for estimating the modulus of convexity. Our proof for the N-dimensional case is essentially different from that in [10]. As an application we shall consider the $\ell_{p,q}$-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p,q}$ and show that $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p,q}$ is uniformly convex if and only if all X_j are uniformly convex, where $1 < q \le p \le \infty$, $q < \infty$. The same is true for the strict and locally uniform convexity. These results include the well-known facts for the ℓ_p-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_p$ as the case $p = q$.

Let us recall some definitions. A Banach space X or its norm $\|\cdot\|$ is called *strictly convex* if $\|x\| = \|y\| = 1$ $(x \ne y)$ implies $\|(x+y)/2\| < 1$. This is equivalent to the following statement: if $\|x + y\| = \|x\| + \|y\|$, $x \ne 0$, $y \ne 0$, then $x = \lambda y$ with some $\lambda > 0$ (see for example [9, page 432], [1]). X is called *uniformly convex* provided for any ϵ $(0 < \epsilon < 2)$ there exists $\delta > 0$ such that whenever $\|x - y\| \ge \epsilon$, $\|x\| = \|y\| = 1$, one has $\|(x+y)/2\| \le 1 - \delta$, or equivalently, provided for any ϵ $(0 < \epsilon < 2)$ one has $\delta_X(\epsilon) > 0$, where δ_X is the *modulus of convexity of* X, that is,

$$\delta_X(\epsilon) := \inf\{1 - \|(x+y)/2\|; \|x - y\| \ge \epsilon, \quad \|x\| = \|y\| = 1\} \quad (0 \le \epsilon \le 2).$$

We also have the following restatement: X is uniformly convex if and only if, whenever $\|x_n\| = \|y_n\| = 1$ and $\|(x_n + y_n)/2\| \to 1$, it follows that $\|x_n - y_n\| \to 0$. X is called *locally uniformly convex* (see for example [9, 4]) if for any $x \in X$ with $\|x\| = 1$ and for any ϵ ($0 < \epsilon < 2$) there exists $\delta > 0$ such that if $\|x - y\| \geq \epsilon$, $\|y\| = 1$, then $\|(x+y)/2\| \leq 1 - \delta$. Clearly the notion of locally uniform convexity is between those of uniform and strict convexities.

2. Absolute norms on $\mathbb{C}^N$ and ψ-direct sums $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$

Let AN_N denote the family of all absolute normalized norms on $\mathbb{C}^N$. Let

$$\Delta_N = \{(s_1, s_2, \ldots, s_{N-1}) \in \mathbb{R}^{N-1} : s_1 + s_2 + \cdots + s_{N-1} \leq 1, s_j \geq 0 (\forall j)\}.$$

For any $\|\cdot\| \in AN_N$ define the function ψ on Δ_N by

$$\text{(1)} \quad \psi(s) = \|(1 - s_1 - \cdots - s_{N-1}, s_1, \ldots, s_{N-1})\| \quad \text{for } s = (s_1, \ldots, s_{N-1}) \in \Delta_N.$$

Then ψ is continuous and convex on Δ_N, and satisfies the following conditions:

$$(A_0) \qquad \psi(0, \ldots, 0) = \psi(1, 0, \ldots, 0) = \cdots = \psi(0, \ldots, 0, 1) = 1,$$

$$(A_1) \qquad \psi(s_1, \ldots, s_{N-1}) \geq (s_1 + \cdots + s_{N-1})\psi\left(\frac{s_1}{\sum_{i=1}^{N-1} s_i}, \ldots, \frac{s_{N-1}}{\sum_{i=1}^{N-1} s_i}\right),$$

$$(A_2) \qquad \psi(s_1, \ldots, s_{N-1}) \geq (1 - s_1)\psi\left(0, \frac{s_2}{1 - s_1}, \ldots, \frac{s_{N-1}}{1 - s_1}\right),$$

....

$$(A_N) \qquad \psi(s_1, \ldots, s_{N-1}) \geq (1 - s_{N-1})\psi\left(\frac{s_1}{1 - s_{N-1}}, \ldots, \frac{s_{N-2}}{1 - s_{N-1}}, 0\right).$$

Note that from (A_0) it follows that $\psi(s_1, \ldots, s_{N-1}) \leq 1$ on Δ_N as ψ is convex. Denote Ψ_N be the family of all continuous convex functions ψ on Δ_N satisfying (A_0), $(A_1), \ldots, (A_N)$. Then the converse holds true: For any $\psi \in \Psi_N$ define

$$\text{(2)} \quad \|(z_1, \ldots, z_N)\|_\psi = \begin{cases} \left(\sum_{i=1}^N |z_i|\right) \psi\left(|z_2|/(\sum_{i=1}^N |z_i|), \ldots, |z_N|/(\sum_{i=1}^N |z_i|)\right) & \\ \qquad\qquad \text{if } (z_1, \ldots, z_N) \neq (0, \ldots, 0), \\ 0 \qquad\qquad \text{if } (z_1, \ldots, z_N) = (0, \ldots, 0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_N$ and $\|\cdot\|_\psi$ satisfies (1). Thus *the families AN_N and Ψ_N are in one-to-one correspondence under equation* (1) (Saito-Kato-Takahashi [11, Theorem 4.2]). The ℓ_p-norms

$$\|(z_1, \ldots, z_N)\|_p = \begin{cases} \{|z_1|^p + \cdots + |z_N|^p\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|z_1|, \ldots, |z_N|\} & \text{if } p = \infty \end{cases}$$

are typical examples of absolute normalized norms, and for any $\|\cdot\| \in AN_N$ we have

$$\|\cdot\|_\infty \le \|\cdot\| \le \|\cdot\|_1 \tag{3}$$

([11, Lemma 3.1], see also [3]). The functions corresponding to ℓ_p-norms on $\mathbb{C}^N$ are

$$\psi_p(s_1, \dots, s_{N-1}) = \begin{cases} \left\{\left(1 - \sum_{j=1}^{N-1} s_j\right)^p + s_1^p + \cdots + s_{N-1}^p\right\}^{1/p} & \text{if } 1 \le p < \infty, \\ \max\left\{1 - \sum_{j=1}^{N-1} s_j, s_1, \dots, s_{N-1}\right\} & \text{if } p = \infty \end{cases}$$

for $(s_1, \dots, s_{N-1}) \in \Delta_N$.

Let $X_1, X_2, \dots, X_N$ be Banach spaces. Let $\psi \in \Psi_N$ and let $\|\cdot\|_\psi$ be the corresponding norm in AN_N. Let $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ be the direct sum of $X_1, X_2, \dots, X_N$ equipped with the norm

$$\|(x_1, x_2, \dots, x_N)\|_\psi := \|(\|x_1\|, \|x_2\|, \dots, \|x_N\|)\|_\psi \quad \text{for } x_j \in X_j. \tag{4}$$

As is it immediately seen, $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ is a Banach space.

EXAMPLE. Let $1 \le q \le p \le \infty$, $q < \infty$. We consider the Lorentz $\ell_{p,q}$-norm $\|z\|_{p,q} = \left\{\sum_{j=1}^N j^{(q/p)-1} z_j^{*q}\right\}^{1/q}$ for $z = (z_1, \dots, z_N) \in \mathbb{C}^N$, where $\{z_j^*\}$ is the non-increasing rearrangement of $\{|z_j|\}$, that is, $z_1^* \ge z_2^* \ge \cdots \ge z_N^*$. (Note that in case of $1 \le p < q \le \infty$, $\|\cdot\|_{p,q}$ is not a norm but a quasi-norm (see [6, Proposition 1], [14, page 126])). Evidently $\|\cdot\|_{p,q} \in AN_N$ and the corresponding convex function $\psi_{p,q}$ is obtained by

$$\psi_{p,q}(s) = \|(1 - s_1 - \cdots - s_{N-1}, s_1, \dots, s_{N-1})\|_{p,q} \tag{5}$$

(for $s = (s_1, \dots, s_{N-1}) \in \Delta_N$), that is, $\|\cdot\|_{p,q} = \|\cdot\|_{\psi_{p,q}}$. Let $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p,q}$ be the direct sum of Banach spaces $X_1, X_2, \dots, X_N$ equipped with the norm

$$\|(x_1, \dots, x_N)\|_{p,q} := \|(\|x_1\|, \dots, \|x_N\|)\|_{p,q},$$

we call it the *$\ell_{p,q}$-sum* of $X_1, X_2, \dots, X_N$. If $p = q$ the $\ell_{p,p}$-sum is the usual ℓ_p-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_p$.

For some other examples of absolute norms on $\mathbb{C}^N$ we refer the reader to [11] (see also [12]).

3. Strict convexity of $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$

A function ψ on Δ_N is called *strictly convex* if for any $s, t \in \Delta_N$ $(s \ne t)$ one has $\psi((s+t)/2) < (\psi(s) + \psi(t))/2$. For absolute norms on $\mathbb{C}^N$, we have

LEMMA 3.1 (Saito-Kato-Takahashi [11, Theorem 4.2]). *Let $\psi \in \Psi_N$. Then $(\mathbb{C}^N, \|\cdot\|_\psi)$ is strictly convex if and only if ψ is strictly convex.*

The following lemma concerning the monotonicity property of the absolute norms on $\mathbb{C}^N$ is useful in the sequel.

LEMMA 3.2 (Saito-Kato-Takahashi [11, Lemma 4.1]). *Let $\psi \in \Psi_N$. Let $z = (z_1, \dots, z_N)$, $w = (w_1, \dots, w_N) \in \mathbb{C}^N$.*

(i) *If $|z_j| \le |w_j|$ for all j, then $\|z\|_\psi \le \|w\|_\psi$.*

(ii) *Let ψ be strictly convex. If $|z_j| \le |w_j|$ for all j and $|z_j| < |w_j|$ for some j, then $\|z\|_\psi < \|w\|_\psi$.*

THEOREM 3.3. *Let $X_1, X_2, \dots, X_N$ be Banach spaces and let $\psi \in \Psi_N$. Then $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ is strictly convex if and only if $X_1, X_2, \dots, X_N$ are strictly convex and ψ is strictly convex.*

PROOF. Let $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ be strictly convex. Then, each X_j and $(\mathbb{C}^N, \|\cdot\|_\psi)$ are strictly convex since they are isometrically imbedded into $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$. According to Lemma 3.1, ψ is strictly convex.

Conversely, let each X_j and ψ be strictly convex. Take arbitrary $x = (x_j)$, $y = (y_j)$, $x \ne y$, in $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ with $\|x\|_\psi = \|y\|_\psi = 1$. Let first $(\|x_1\|, \dots, \|x_N\|) = (\|y_1\|, \dots, \|y_N\|)$. Then, if $\|x + y\|_\psi = 2$,

$$\begin{aligned} 2 = \|x + y\|_\psi &= \|(\|x_1 + y_1\|, \dots, \|x_N + y_N\|)\|_\psi \\ &\le \|(\|x_1\| + \|y_1\|, \dots, \|x_N\| + \|y_N\|)\|_\psi \le \|x\|_\psi + \|y\|_\psi = 2, \end{aligned}$$

from which it follows that $\|x_j + y_j\| = \|x_j\| + \|y_j\|$ for all j by Lemma 3.2. As each X_j is strictly convex, $x_j = k_j y_j$ with $k_j > 0$. Since $\|x_j\| = \|y_j\|$, we have $k_j = 1$ and hence $x_j = y_j$ for all j, or $x = y$, which is a contradiction. Therefore we have $\|x + y\|_\psi < 2$. Let next $(\|x_1\|, \dots, \|x_N\|) \ne (\|y_1\|, \dots, \|y_N\|)$. Since ψ is strictly convex, $(\mathbb{C}^N, \|\cdot\|_\psi)$ is strictly convex by Lemma 3.1. Consequently we have

$$\begin{aligned} \|x + y\|_\psi &= \|(\|x_1 + y_1\|, \dots, \|x_N + y_N\|)\|_\psi \\ &\le \|(\|x_1\| + \|y_1\|, \dots, \|x_N\| + \|y_N\|)\|_\psi \\ &= \|(\|x_1\|, \dots, \|x_N\|) + (\|y_1\|, \dots, \|y_N\|)\|_\psi < 2, \end{aligned}$$

as is desired. □

Now we see that the function $\psi_{p,q}$ in the above example is strictly convex if $1 < q \le p \le \infty$, $q < \infty$. We need the next lemma.

LEMMA 3.4 ([5]). *Let* $\{\alpha_j\}, \{\beta_j\} \in \mathbb{R}^N$ *and* $\alpha_j \geq 0$, $\beta_j \geq 0$. *Let* $\{\alpha_j^*\}, \{\beta_j^*\}$ *be their non-increasing rearrangements, that is,* $\alpha_1^* \geq \alpha_2^* \geq \cdots \geq \alpha_N^*$ *and* $\beta_1^* \geq \beta_2^* \geq \cdots \geq \beta_N^*$. *Then* $\sum_{j=1}^N \alpha_j \beta_j \leq \sum_{j=1}^N \alpha_j^* \beta_j^*$.

PROPOSITION 3.5. *Let* $1 < q \leq p \leq \infty$, $q < \infty$. *Then the function* $\psi_{p,q}$ *given by* (5) *is strictly convex on* Δ_N.

PROOF. Let $s = (s_j)$, $t = (t_j) \in \Delta_N$, $s \neq t$. Without loss of generality we may assume that

$$2 - (s_1 + t_1) - \cdots - (s_{N-1} + t_{N-1}) \geq s_1 + t_1 \geq \cdots \geq s_{N-1} + t_{N-1} \geq 0.$$

Put

$$\begin{aligned} \sigma &= (1 - s_1 - \cdots - s_{N-1},\ 2^{1/p-1/q} s_1, \ldots, N^{1/p-1/q} s_{N-1}), \\ \tau &= (1 - t_1 - \cdots - t_{N-1},\ 2^{1/p-1/q} t_1, \ldots, N^{1/p-1/q} t_{N-1}). \end{aligned}$$

Then by Lemma 3.4 we have

$$\begin{aligned} \|\sigma\|_q &= \left\{(1 - s_1 - \cdots - s_{N-1})^q + 2^{q/p-1} s_1^q + \cdots + N^{q/p-1} s_{N-1}^q\right\}^{1/q} \\ &\leq \|(1 - s_1 - \cdots - s_{N-1}, s_1, \ldots, s_{N-1})\|_{p,q} = \psi_{p,q}(s) \end{aligned}$$

and $\|\tau\|_q \leq \psi_{p,q}(t)$. On the other hand,

$$\begin{aligned} \psi_{p,q}\left(\frac{s+t}{2}\right) &= \left\{\left(1 - \sum_{i=1}^{N-1} \frac{s_i + t_i}{2}\right)^q + \sum_{i=1}^{N-1} (i+1)^{q/p-1} \left(\frac{s_i + t_i}{2}\right)^q\right\}^{1/q} \\ &= \left[\left(\frac{1}{2}\left\{\left(1 - \sum_{i=1}^{N-1} s_i\right) + \left(1 - \sum_{i=1}^{N-1} t_i\right)\right\}\right)^q \right. \\ &\quad \left. + \sum_{i=1}^{N-1} \left(\frac{1}{2}\left\{(i+1)^{1/p-1/q} s_i + (i+1)^{1/p-1/q} t_i\right\}\right)^q\right]^{1/q} = \left\|\frac{\sigma + \tau}{2}\right\|_q. \end{aligned}$$

Since ℓ_q-norm $\|\cdot\|_q$ $(1 < q < \infty)$ is strictly convex and $s \neq t$, we have $\|\sigma + \tau\|_q < \|\sigma\|_q + \|\tau\|_q$. Indeed, if $\|\sigma + \tau\|_q = \|\sigma\|_q + \|\tau\|_q$, then $\sigma = k\tau$ with some $k > 0$ (note that $\sigma \neq 0$, $\tau \neq 0$). Hence $s_j = kt_j$ for all j, and $1 - \sum_{i=1}^{N-1} s_i = k\left(1 - \sum_{i=1}^{N-1} t_i\right)$. Therefore, $k = 1$ and we have $s = t$, which is a contradiction. Consequently,

$$\psi_{p,q}\left(\frac{s+t}{2}\right) = \left\|\frac{\sigma + \tau}{2}\right\|_q < \frac{\|\sigma\|_q + \|\tau\|_q}{2} \leq \frac{\psi_{p,q}(s) + \psi_{p,q}(t)}{2},$$

or $\psi_{p,q}$ is strictly convex. □

By Theorem 3.3 and Proposition 3.5 we have the following result for the $\ell_{p,q}$-sum of Banach spaces.

COROLLARY 3.6. *Let* $1 < q \le p \le \infty$, $q < \infty$. *Then,* $\ell_{p,q}$*-sum* $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p,q}$ *is strictly convex if and only if* $X_1, X_2, \ldots, X_N$ *are strictly convex.*

In particular, the ℓ_p*-sum* $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_p$, $1 < p < \infty$, *is strictly convex if and only if* $X_1, X_2, \ldots, X_N$ *are strictly convex.*

4. Uniform convexity of $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$

Let us characterize the uniform convexity of $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$.

THEOREM 4.1. *Let* $X_1, X_2, \ldots, X_N$ *be Banach spaces and let* $\psi \in \Psi_N$. *Then* $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ *is uniformly convex if and only if* $X_1, X_2, \ldots, X_N$ *are uniformly convex and* ψ *is strictly convex.*

PROOF. The necessity assertion is proved in the same way as the proof of Theorem 3.3. Assume that $X_1, X_2, \ldots, X_N$ are uniformly convex and ψ is strictly convex. Take an arbitrary $\epsilon > 0$ and put

$$\delta := 2\delta_X(\epsilon) = \inf\{2 - \|x + y\|_\psi : \|x - y\|_\psi \ge \epsilon,\ \|x\|_\psi = \|y\|_\psi = 1\}.$$

We show that $\delta > 0$. There exist sequences $\{x_n\}$ and $\{y_n\}$ in $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ so that

$$\|x_n - y_n\|_\psi \ge \epsilon, \qquad \|x_n\|_\psi = \|y_n\|_\psi = 1 \tag{6}$$

and

$$\lim_{n\to\infty} \|x_n + y_n\|_\psi = 2 - \delta. \tag{7}$$

Let $x_n = (x_1^{(n)}, \ldots, x_N^{(n)})$ and $y_n = (y_1^{(n)}, \ldots, y_N^{(n)})$. Since for each $1 \le j \le N$, $\|x_j^{(n)}\| = \|(0, \ldots, 0, x_j^{(n)}, 0, \ldots, 0)\|_\psi \le \|x_n\|_\psi = 1$ and $\|y_j^{(n)}\| \le \|y_n\|_\psi = 1$ for all n, the sequences $\{\|x_j^{(n)}\|\}_n$ and $\{\|y_j^{(n)}\|\}_n$ have a convergent subsequence respectively. So we may assume that $\|x_j^{(n)}\| \to a_j$, $\|y_j^{(n)}\| \to b_j$ as $n \to \infty$. Further, in the same way, we may assume that

$$\|x_j^{(n)} - y_j^{(n)}\| \to c_j \quad \text{as } n \to \infty \tag{8}$$

and

$$\|x_j^{(n)} + y_j^{(n)}\| \to d_j \quad \text{as } n \to \infty. \tag{9}$$

Put $K_n = \sum_{j=1}^N \|x_j^{(n)}\|$. Then $\|x_n\|_\psi = K_n\psi(\|x_2^{(n)}\|/K_n, \ldots, \|x_N^{(n)}\|/K_n) = 1$. Letting $n \to \infty$, as ψ is continuous, we have

$$\|(a_1, \ldots, a_N)\|_\psi = \left(\sum_{j=1}^N a_j\right) \psi\left(\frac{a_2}{\sum_{j=1}^N a_j}, \ldots, \frac{a_N}{\sum_{j=1}^N a_j}\right) = 1. \tag{10}$$

Also we have

$$\|(b_1, \cdots, b_N)\|_\psi = \left(\sum_{j=1}^N b_j\right) \psi\left(\frac{b_2}{\sum_{j=1}^N b_j}, \ldots, \frac{b_N}{\sum_{j=1}^N b_j}\right) = 1. \tag{11}$$

Next let $n \to \infty$ in (6), or in

$$\|x_n - y_n\|_\psi = \left(\sum_{j=1}^N \left\|x_j^{(n)} - y_j^{(n)}\right\|\right) \times \psi\left(\frac{\left\|x_2^{(n)} - y_2^{(n)}\right\|}{\sum_{j=1}^N \left\|x_j^{(n)} - y_j^{(n)}\right\|}, \ldots, \frac{\left\|x_N^{(n)} - y_N^{(n)}\right\|}{\sum_{j=1}^N \left\|x_j^{(n)} - y_j^{(n)}\right\|}\right) \geq \epsilon.$$

Then we have

$$\|(c_1, \ldots, c_N)\|_\psi = \left(\sum_{j=1}^N c_j\right) \psi\left(\frac{c_2}{\sum_{j=1}^N c_j}, \ldots, \frac{c_N}{\sum_{j=1}^N c_j}\right) \geq \epsilon \tag{12}$$

by (8). In the same way, according to (7) and (9), we have

$$\|(d_1, \ldots, d_N)\|_\psi = 2 - \delta. \tag{13}$$

Now, assume that $(a_1, \ldots, a_N) \neq (b_1, \ldots, b_N)$. Then, according to (10), (11) and the strict convexity of ψ we obtain that

$$2 - \delta = \|(d_1, \ldots, d_N)\|_\psi \leq \|(a_1 + b_1, \ldots, a_N + b_N)\|_\psi < 2,$$

which implies $\delta > 0$. Next, let $(a_1, \ldots, a_N) = (b_1, \ldots, b_N)$. Since $(c_1, \ldots, c_N) \neq (0, \ldots, 0)$ from (12), we may assume that $c_1 > 0$ without loss of generality. Then as

$$c_1 = \lim_{n\to\infty} \left\|x_1^{(n)} - y_1^{(n)}\right\| \leq \lim_{n\to\infty} \left(\left\|x_1^{(n)}\right\| + \left\|y_1^{(n)}\right\|\right) = a_1 + b_1 = 2a_1,$$

we have $a_1 > 0$ and

$$0 < \frac{c_1}{a_1} = \lim_{n\to\infty} \left\|\frac{x_1^{(n)}}{\|x_1^{(n)}\|} - \frac{y_1^{(n)}}{\|x_1^{(n)}\|}\right\| = \lim_{n\to\infty} \left\|\frac{x_1^{(n)}}{\|x_1^{(n)}\|} - \frac{y_1^{(n)}}{\|y_1^{(n)}\|}\right\|. \tag{14}$$

Indeed, we have the latter identity because

$$\left|\left\|\frac{x_1^{(n)}}{\|x_1^{(n)}\|} - \frac{y_1^{(n)}}{\|x_1^{(n)}\|}\right\| - \left\|\frac{x_1^{(n)}}{\|x_1^{(n)}\|} - \frac{y_1^{(n)}}{\|y_1^{(n)}\|}\right\|\right| \leq \|y_1^{(n)}\| \left|\frac{1}{\|x_1^{(n)}\|} - \frac{1}{\|y_1^{(n)}\|}\right| \to b_1 \left|\frac{1}{a_1} - \frac{1}{b_1}\right| = 0 \quad \text{as } n \to \infty.$$

Since X_1 is uniform convex, it follows from (14) that

$$\frac{d_1}{a_1} = \lim_{n\to\infty} \left\| \frac{x_1^{(n)}}{\|x_1^{(n)}\|} + \frac{y_1^{(n)}}{\|x_1^{(n)}\|} \right\| = \lim_{n\to\infty} \left\| \frac{x_1^{(n)}}{\|x_1^{(n)}\|} + \frac{y_1^{(n)}}{\|y_1^{(n)}\|} \right\| < 2,$$

whence $d_1 < 2a_1$. Accordingly, by (13) and Lemma 3.2 we obtain that

$$\begin{aligned} 2 - \delta &= \|(d_1, d_2, \ldots, d_N)\|_\psi \\ &< \|(2a_1, a_2 + b_2, \ldots, a_N + b_N)\|_\psi \\ &= \|(a_1 + b_1, a_2 + b_2, \ldots, a_N + b_N)\|_\psi \\ &\le \|(a_1, \ldots, a_N)\|_\psi + \|(b_1, \ldots, b_N)\|_\psi = 2, \end{aligned}$$

which implies $\delta > 0$. This completes the proof. □

The parallel argument works for the locally uniform convexity and we obtain the next result.

THEOREM 4.2. *Let $\psi \in \Psi_N$. Then $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ is locally uniformly convex if and only if $X_1, X_2, \ldots, X_N$ are locally uniformly convex and ψ is strictly convex.*

Indeed, for the sufficiency, take an arbitrary $x \in (X_1 \oplus X_2 \oplus \cdots \oplus X_N)_\psi$ with $\|x\|_\psi = 1$ and merely let $x_n = x$ in the above proof. By Theorem 4.1 and Theorem 4.2 combined with Proposition 3.5 we obtain the following corollary.

COROLLARY 4.3. *Let $1 < q \le p \le \infty$, $q < \infty$. Then, $\ell_{p,q}$-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_{p,q}$ is uniformly convex (locally uniformly convex) if and only if $X_1, X_2, \ldots, X_N$ are uniformly convex (locally uniformly convex).*

In particular, the ℓ_p-sum $(X_1 \oplus X_2 \oplus \cdots \oplus X_N)_p$, $1 < p < \infty$, is uniformly convex (locally uniformly convex) if and only if $X_1, X_2, \ldots, X_N$ are uniformly convex (locally uniformly convex).

References

[1] B. Beauzamy, *Introduction to Banach spaces and their geometry*, 2nd edition (North-Holland, Amsterdam, 1985).

[2] R. Bhatia, *Matrix analysis* (Springer, Berlin, 1997).

[3] F. F. Bonsall and J. Duncan, *Numerical ranges II*, London Math. Soc. Lecture Note Ser. 10 (Cambridge University Press, Cambridge, 1973).

[4] J. Diestel, *Geometry of Banach spaces*, Lecture Notes in Math. 485 (Springer, Berlin, 1975).

[5] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities* (Cambridge University Press, Cambridge, 1967).
[6] M. Kato, 'On Lorentz spaces $\ell_{p,q}\{E\}$', *Hiroshima Math. J.* **6** (1976), 73–93.
[7] D. Kutzarova and T. Landes, 'Nearly uniform convexity of infinite direct sums', *Indiana Univ. Math. J.* **41** (1992), 915–926.
[8] ———, 'NUC and related properties of finite direct sums', *Boll. Un. Math. Ital. (7)* **8A** (1994), 45–54.
[9] R. E. Megginson, *An introduction to Banach space theory* (Springer, New York, 1998).
[10] K.-S. Saito and M. Kato, 'Uniform convexity of ψ-direct sums of Banach spaces', *J. Math. Anal. Appl.* **277** (2003), 1–11.
[11] K.-S. Saito, M. Kato and Y. Takahashi, 'On absolute norms on $\mathbb{C}^n$', *J. Math. Anal. Appl.* **252** (2000), 879–905.
[12] ———, 'von Neumann-Jordan constant of absolute normalized norms on $\mathbb{C}^2$', *J. Math. Anal. Appl.* **244** (2000), 515–532.
[13] Y. Takahashi, M. Kato and K.-S. Saito, 'Strict convexity of absolute norms on $\mathbb{C}^2$ and direct sums of Banach spaces', *J. Inequal. Appl.* **7** (2002), 179–186.
[14] H. Triebel, *Interpolation theory, function spaces, differential operators* (North-Holland, Amsterdam, 1978).

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