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HIGHER DIMENSIONAL COHOMOLOGY OF WEIGHTED SEQUENCE ALGEBRAS

A. POURABBAS

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Abstract

It is well known that $c_0(\mathbb{Z})$ is amenable and so its global dimension is zero. In this paper we will investigate the cyclic and Hochschild cohomology of Banach algebra $c_0(\mathbb{Z}, \omega^{-1})$ and its unitisation with coefficients in its dual space, where ω is a weight on $\mathbb{Z}$ which satisfies $\inf\{\omega(i)\} = 0$. Moreover we show that the weak homological bi-dimension of $c_0(\mathbb{Z}, \omega^{-1})$ is infinity.

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1. Introduction

The Banach algebra $\mathscr{A}$ is amenable if $\mathscr{H}^1(\mathscr{A}, \mathscr{X}') = 0$ for every Banach $\mathscr{A}$-bimodule $\mathscr{X}$. This definition was introduced by Johnson in (1972) [8]. The Banach algebra $\mathscr{A}$ is weakly amenable if $\mathscr{H}^1(\mathscr{A}, \mathscr{A}') = 0$. This definition generalizes the one which was introduced by Bade, Curtis and Dales in [1], where it was noted that a commutative Banach algebra $\mathscr{A}$ is weakly amenable if and only if $\mathscr{H}^1(\mathscr{A}, \mathscr{X}) = 0$ for every symmetric Banach $\mathscr{A}$-bimodule $\mathscr{X}$.

Johnson in [8] proved that for an amenable Banach algebra $\mathscr{A}$, the cohomology groups $\mathscr{H}^n(\mathscr{A}, \mathscr{X}')$ vanish for every Banach $\mathscr{A}$-bimodule $\mathscr{X}$ and all $n \geq 1$. The question was raised whether in general $\mathscr{H}^n(\mathscr{A}, \mathscr{A}') = 0$ for a weakly amenable Banach algebra $\mathscr{A}$ and all $n \geq 1$. The question was answered in the negative in [14] by showing that $\mathscr{H}^2(\ell^1(\mathbb{F}_2), \ell^\infty(\mathbb{F}_2)) \neq 0$. In fact Johnson [8] showed that $\mathscr{H}^2(\ell^1(\mathbb{F}_2), \mathbb{C}) \neq 0$ and in [14] Sinclair and Smith showed that the non-trivial cohomology group $\mathscr{H}^2(\ell^1(\mathbb{F}_2), \mathbb{C})$ is naturally embedded as a direct summand of

$\mathscr{H}^2(\ell^1(\mathbb{F}), \ell^\infty(\mathbb{F}))$. In this paper we will give an example of a weakly amenable Banach algebra, such that the n^{th} cohomology groups with coefficients in the dual space do not vanish for all $n > 1$.

It is a question of general interest whether or not the n^{th} cohomology group is necessarily zero. This, and closely related questions have stimulated much of the recent development of the theory of cohomology groups.

Bade, Curtis and Dales in [1] showed that $\mathscr{H}^1(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+)') \neq 0$. This may lead one to believe that $\mathscr{H}^n(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+)')$ for all $n \geq 2$ are also non-zero. However, Johnson showed in [10] that the alternating cohomology of $\ell^1(\mathbb{Z}_+)$ vanishes in all dimensions strictly greater than one. Then Dales and Duncan [2, Theorem 3.2] showed that $\mathscr{H}^2(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+)') = 0$. Gourdeau and White in [4] with a complicated proof showed that $\mathscr{H}^3(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+)') = 0$. This leads to the conjecture that all the cohomology groups of $\ell^1(\mathbb{Z}_+)$ with coefficients in $\ell^1(\mathbb{Z}_+)'$ vanish for $n > 3$.

In this paper for the weakly amenable Banach algebra $\mathscr{A}^\#$, the unitisation of $\mathscr{A} = c_0(\mathbb{Z}, \omega^{-1})$, we show that the cyclic cohomology group $\mathscr{H}\mathscr{C}^n(\mathscr{A}^\#)$ and the Hochschild cohomology group $\mathscr{H}^n(\mathscr{A}^\#, (\mathscr{A}^\#)')$ are non-trivial for every $n \geq 2$.

Let ω be a weight sequence on $\mathbb{Z}$, that is, ω is a non-zero, positive valued function on $\mathbb{Z}$ such that $\omega(n) \leq 1$ for every $n \in \mathbb{Z}$. Set

$$c_0(\mathbb{Z}, \omega^{-1}) = \left\{ a = \{a_n\} : n \in \mathbb{Z},\ \lim_{|n| \to \infty} \frac{|a_n|}{\omega(n)} = 0 \right\},$$

where $c_0(\mathbb{Z}, \omega^{-1})$ is a closed subalgebra of

$$\ell^\infty(\mathbb{Z}, \omega^{-1}) = \left\{ a = \{a_n\} : n \in \mathbb{Z},\ \|a\|_{\omega^{-1}} = \sup \left\{ \frac{|a_n|}{\omega(n)} : n \in \mathbb{Z} \right\} < \infty \right\}$$

and $c_0(\mathbb{Z}, \omega^{-1})'$ (the dual space of $c_0(\mathbb{Z}, \omega^{-1})$) is equal to

$$\ell^1(\mathbb{Z}, \omega) = \left\{ a = \{a_n\} : n \in \mathbb{Z},\ \sum_{n=-\infty}^{\infty} |a_n| \omega(n) < \infty \right\}.$$

The element $e_i = \{\delta_{ij}\}_{j \in \mathbb{Z}}$, $i \in \mathbb{Z}$ is an idempotent, where δ_{ij} denotes the Kronecker delta. We denote the linear span of such elements by E, which is a dense subset of $c_0(\mathbb{Z}, \omega^{-1})$; since if $a \in c_0(\mathbb{Z}, \omega^{-1})$, then we define

$$a^n = \sum_{i=-n}^{n} a_i e_i = \{\dots, 0, a_{-n}, \dots, a_n, 0, \dots\}$$

and

$$\|a - a^n\|_{\omega^{-1}} = \sup_{|i| > |n|} \frac{|a_i|}{\omega(i)} \to 0 \quad \text{as} \quad |n| \to \infty.$$

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Since a commutative Banach algebra which is the closed linear span of its idempotents is weakly amenable [9], then $c_0(\mathbb{Z}, \omega^{-1})$ is weakly amenable, and by [3, Proposition 1.4] $\mathscr{A}^{\#}$, the unitisation of $\mathscr{A} = c_0(\mathbb{Z}, \omega^{-1})$ is also weakly amenable.

NOTE. In this paper every weight ω on $\mathbb{Z}$ which we consider must satisfy the condition $\inf\{\omega(i)\} = 0$, because if $\inf\{\omega(i)\} \neq 0$, then ω^{-1} is a bounded weight and so $c_0(\mathbb{Z}, \omega^{-1}) \cong c_0(\mathbb{Z})$ which is amenable.

Throughout $\mathscr{A}^{\#}$ means the unitisation of $\mathscr{A} = c_0(\mathbb{Z}, \omega^{-1})$. Let $\mathbf{1}$ be the unit element of $\mathscr{A}^{\#}$. Suppose E_N is the closed linear span of $\{e_i\}_{i=1}^{N}$. Then E_N is a closed subalgebra of $\mathscr{A}^{\#}$. If $a \in \mathscr{A}^{\#}$, then $a = a' + \alpha\mathbf{1}$, where $a' = \{a'_n\}_{n\in\mathbb{Z}}$ is in $c_0(\mathbb{Z}, \omega^{-1})$ and $\alpha \in \mathbb{C}$. The norm on $\mathscr{A}^{\#}$ is defined by $\|a\|_{\omega^{-1}} = \|a'\|_{\omega^{-1}} + |\alpha|$. Also for every $a = a' + \alpha\mathbf{1}$ and $b = b' + \beta\mathbf{1}$ in $\mathscr{A}^{\#}$ we define $ab = a'b' + \alpha b' + \beta a' + \alpha\beta\mathbf{1}$. Clearly $E_N \cong \mathbb{C}^N$ and since a direct sum of amenable algebras is amenable, then E_N is an amenable closed subalgebra of $\mathscr{A}^{\#}$.

Note that for every $\phi \in \mathscr{Z}^n(\mathscr{A}^{\#}, (\mathscr{A}^{\#})')$, the space of all bounded n-cocycles, by [11] there exists ψ_N in $\mathscr{C}^{n-1}(\mathscr{A}^{\#}, (\mathscr{A}^{\#})')$ such that

$$(\phi - \delta\psi_N)(a_1, \dots, a_n) = 0 \quad \text{if any one of } a_1, \dots, a_n \text{ lies in } E_N.$$

But we will show that this is not true for the whole of $\mathscr{A}^{\#}$, in fact for every $n \geq 2$ we will find a (cyclic) cocycle $\phi \in \mathscr{Z}^n(\mathscr{A}^{\#}, (\mathscr{A}^{\#})')$ which does not co-bound.

The weak homological bi-dimension of a Banach algebra $\mathscr{A}$, denoted by wdb $\mathscr{A}$, is the smallest integer n such that $\mathscr{H}^m(\mathscr{A}, X') = 0$ for all Banach $\mathscr{A}$-bimodules X and all $m > n$, or wdb $\mathscr{A} = \infty$ if there is no such n. If $\mathscr{A}$ is an amenable Banach algebra, then wdb $\mathscr{A} = 0$ [7, Section 2.5]. The weak homological bi-dimension of a Banach algebra is a number that measures how much this algebra is homologically worse than amenable. The homological bi-dimension of a Banach algebra $\mathscr{A}$, denoted by db $\mathscr{A}$, is the smallest integer n such that $\mathscr{H}^m(\mathscr{A}, X) = 0$ for all Banach $\mathscr{A}$-bimodules X and all $m > n$, or wdb $\mathscr{A} = \infty$ if there is no such n. For every Banach algebra $\mathscr{A}$, we have wdb $\mathscr{A} \leq$ db $\mathscr{A}$ (see [7, VII, Section 3.4] and [13]).

A consequence of the main results of this paper (Theorem 2.2 and Theorem 3.4) is that the weak homological bi-dimension of $c_0(\mathbb{Z}, \omega^{-1})$ is infinity, that is,

$$\operatorname{wdb} c_0(\mathbb{Z}, \omega^{-1}) = \infty.$$

The paper is organized as follows. In Section 2 we calculate the even dimensional cyclic and Hochschild cohomology groups of $\mathscr{A}^{\#}$ with coefficients in $(\mathscr{A}^{\#})'$, the dual space of $\mathscr{A}^{\#}$. In Section 3 we will continue our argument for the odd dimensional case.

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2. Even dimensional cohomology groups of weighted sequence algebras

In this section we prove that $\mathscr{H}^{2n}(\mathscr{A}^\#, (\mathscr{A}^\#)') \neq 0$ and $\mathscr{H}\mathscr{C}^{2n}(\mathscr{A}^\#) \neq 0$ for every $n \in \mathbb{N}$.

LEMMA 2.1. *Let $\sum_{i=-\infty}^{\infty} \alpha_i$ be an absolutely convergent series of real numbers, and let*

$$\phi : \overbrace{\mathscr{A}^\# \times \mathscr{A}^\# \times \cdots \times \mathscr{A}^\#}^{2n \text{ times}} \to (\mathscr{A}^\#)'$$

be the function defined by

$$\phi(a_1, \ldots, a_{2n})(a_{2n+1}) = \sum_{i=-\infty}^{\infty} \frac{a'_{1i} \cdots a'_{(2n+1)i}}{\omega(i)^{(2n+1)}} \alpha_i,$$

where $a_k = a'_k + \beta_k \mathbf{1}$ and $a'_k = \{a'_{ki}\}_{i\in\mathbb{Z}}$ $(k = 1, 2, \ldots, 2n+1)$. Then ϕ is a bounded cyclic $2n$-cocycle for every $n \in \mathbb{N}$.

PROOF. It is easy to see that ϕ is a $2n$-linear map. Also

$$\begin{aligned} |\phi(a_1, \ldots, a_{2n})(a_{2n+1})| &\le \sum_{i=-\infty}^{\infty} \frac{|a'_{1i} \cdots a'_{(2n+1)i}|}{\omega(i)^{2n+1}} |\alpha_i| \\ &\le \sup_i \left\{ \frac{|a'_{1i}|}{\omega(i)} \right\} \cdots \sup_i \left\{ \frac{|a'_{(2n+1)i}|}{\omega(i)} \right\} \left(\sum_{i=-\infty}^{\infty} |\alpha_i| \right) \\ &\le \|a_1\|_{\omega^{-1}} \cdots \|a_{2n+1}\|_{\omega^{-1}} \left(\sum_{i=-\infty}^{\infty} |\alpha_i| \right). \end{aligned}$$

Thus ϕ is bounded and $\|\phi\| \le \sum_{i=-\infty}^{\infty} |\alpha_i|$. Now we want to show that ϕ is a $2n$-cocycle, that is,

$$\begin{aligned} \delta\phi(a_1, \ldots, a_{2n+1})(a_{2n+2}) &= a_1\phi(a_2, \ldots, a_{2n+1})(a_{2n+2}) \\ &\quad + \sum_{i=1}^{2n} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_{2n+1})(a_{2n+2}) \\ &\quad + (-1)^{2n+1} \big(\phi(a_1, \ldots, a_{2n}) a_{2n+1}\big)(a_{2n+2}) = 0. \end{aligned}$$

Now we calculate all terms on the right-hand side of the above equation and we obtain the following $(2n+2)$ terms respectively;

$$\sum_{i=-\infty}^{\infty} \frac{\alpha_i}{\omega(i)^{2n+1}} \big\{ a'_{1i} \cdots a'_{(2n+2)i} + \beta_1 a'_{2i} \cdots a'_{(2n+2)i} + \beta_{(2n+2)} a'_{1i} \cdots a'_{(2n+1)i}$$

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$$
\begin{aligned}
&- \left(a'_{1i}\cdots a'_{(2n+2)i} + \beta_1 a'_{2i}\cdots a'_{(2n+2)i} + \beta_2 a'_{1i}a'_{3i}\cdots a'_{(2n+2)i}\right)\\
&\pm \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots\\
&+ \left(a'_{1i}\cdots a'_{(2n+2)i} + \beta_{2n} a'_{1i}\cdots a'_{(2n-1)i}a'_{(2n+1)i}a'_{(2n+2)i}\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \beta_{(2n+1)} a'_{1i}\cdots a'_{(2n)i}a'_{(2n+2)i}\right)\\
&- \left(a'_{1i}\cdots a'_{(2n+2)i} + \beta_{(2n+1)} a'_{1i}\cdots a'_{(2n)i}a'_{(2n+2)i} + \beta_{(2n+2)} a'_{1i}\cdots a'_{(2n+1)i}\right)\Big\}.
\end{aligned}
$$

So all terms in the above equation cancel in pairs. Thus ϕ is a $2n$-cocycle, and obviously it is cyclic, that is,

$$\phi(a_1, \dots, a_{2n})(a_{2n+1}) = (-1)^{2n}\phi(a_2, \dots, a_{2n}, a_{2n+1})(a_1). \qquad \square$$

THEOREM 2.2. *Let ω be a weight on $\mathbb{Z}$ such that* $\inf\{\omega(i)\} = 0$. *Then*

$$\mathscr{H}^{2n}(\mathscr{A}^{\#}, (\mathscr{A}^{\#})') \neq 0 \quad \textit{and also} \quad \mathscr{H}\mathscr{C}^{2n}(\mathscr{A}^{\#}) \neq 0$$

for every $n \in \mathbb{N}$.

PROOF. Let ϕ be the bounded cyclic $2n$-cocycle which was introduced in Lemma 2.1 and let α_i be defined as below. Since $\inf\{\omega(i)\} = 0$, then there exist numbers m_k, $(k = 1, 2, \dots)$ such that $m_i \neq m_j$ whenever $i \neq j$ and $\omega(m_k) \leq 1/2^k$. Now we define

$$\alpha_i = \begin{cases} 1/k^2 & \text{if } i = m_k \quad (k = 1, 2, \dots); \\ 0 & \text{otherwise} \end{cases}$$

and so $\sum_{i=-\infty}^{\infty} \alpha_i = \sum_{k=1}^{\infty} 1/k^2$ which converges. Thus by Lemma 2.1

$$\phi(a_1, \dots, a_{2n})(a_{2n+1}) = \sum_{k=1}^{\infty} \frac{a'_{1m_k}\cdots a'_{(2n+1)m_k}}{\omega(m_k)^{2n+1}k^2}$$

is a bounded cyclic $2n$-cocycle for every $n \in \mathbb{N}$. Now if there exists a ψ in $\mathscr{C}^{2n-1}(\mathscr{A}^{\#}, (\mathscr{A}^{\#})')$ such that

$$
\begin{aligned}
\phi(a_1, \dots, a_{2n})(a_{2n+1}) &= \delta\psi(a_1, \dots, a_{2n})(a_{2n+1})\\
&= a_1\psi(a_2, \dots, a_{2n})(a_{2n+1})\\
&\quad + \sum_{i=1}^{2n-1}(-1)^i\psi(a_1, \dots, a_i a_{i+1}, \dots, a_{2n})(a_{2n+1})\\
&\quad + (-1)^{2n}(\psi(a_1, \dots, a_{2n-1})a_{2n})(a_{2n+1}),
\end{aligned}
$$

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where $a_k \in \mathscr{A}^\#$ $(k = 1, 2, \dots, 2n+1)$, in particular, if $a_1 = \cdots = a_{2n+1} = e_{m_j}$, $(j = 1, 2, \dots)$, then

$$\phi(\overbrace{e_{m_j}, \dots, e_{m_j}}^{2n \text{ times}})(e_{m_j}) = \psi(\overbrace{e_{m_j}, \dots, e_{m_j}}^{2n-1 \text{ times}})(e_{m_j}) = \frac{1}{\omega(m_j)^{2n+1} j^2}.$$

So since $\omega(j) \leq 1/2^j$

$$\begin{aligned}\|\psi\| &\geq \sup_j \left\{ \left| \psi(\overbrace{\omega(m_j)e_{m_j}, \dots, \omega(m_j)e_{m_j}}^{2n \text{ times}})(\omega(m_j)e_{m_j}) \right| \right\} \\ &= \sup_j \left\{ \frac{\omega(m_j)^{2n}}{\omega(m_j)^{2n+1} j^2} \right\} = \sup_j \left\{ \frac{1}{\omega(m_j) j^2} \right\} \geq \sup_j \left\{ \frac{2^j}{j^2} \right\} = \infty\end{aligned}$$

which is a contradiction. So $\mathscr{H}^{2n}(\mathscr{A}^\#, (\mathscr{A}^\#)') \neq 0$ and also $\mathscr{H}\mathscr{C}^{2n}(\mathscr{A}^\#) \neq 0$. $\square$

3. Odd dimensional cohomology groups of weighted sequence algebras

In this section we will show that $\mathscr{H}^{2n+1}(\mathscr{A}^\#, (\mathscr{A}^\#)') \neq 0$ and also $\mathscr{H}\mathscr{C}^{2n+1}(\mathscr{A}^\#) \neq 0$ for every $n \geq 1$. Note that the structure of the function ϕ which is a base for Theorem 3.4, for the three dimensional case is different from the structure of the corresponding functions in the other cases.

LEMMA 3.1. *Let $\sum_{i=-\infty}^{\infty} \alpha_i$ be an absolutely convergent series of real numbers, and let $\phi : \mathscr{A}^\# \times \mathscr{A}^\# \times \mathscr{A}^\# \to (\mathscr{A}^\#)'$ be the function defined by*

$$\phi(a, b, c)(d) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \frac{a'_j b'_i c'_i d'_j - a'_i b'_i c'_j d'_j}{\omega(i)^2 \omega(j)^2} \alpha_i \alpha_j,$$

where $a = a' + \alpha \mathbf{1}$, $b = b' + \beta \mathbf{1}$, $c = c' + \gamma \mathbf{1}$ and $d = d' + \lambda \mathbf{1}$. Then ϕ is a bounded cyclic 3-cocycle.

PROOF. It is easy to see that ϕ is a trilinear map and also

$$\begin{aligned}|\phi(a, b, c)(d)| &\leq \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \frac{|a'_j b'_i c'_i d'_j| + |a'_i b'_i c'_j d'_j|}{\omega(i)^2 \omega(j)^2} |\alpha_i| |\alpha_j| \\ &\leq 2\|a\|_{\omega^{-1}} \|b\|_{\omega^{-1}} \|c\|_{\omega^{-1}} \|d\|_{\omega^{-1}} \left(\sum_{i=-\infty}^{\infty} |\alpha_i| \right)^2.\end{aligned}$$

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Thus ϕ is bounded and $\|\phi\| \le 2\{\sum_{i=-\infty}^{\infty}|\alpha_i|\}^2$. Now we want to show that ϕ satisfies

$$(1)\qquad a\phi(b,c,d)(h) - \phi(ab,c,d)(h) + \phi(a,bc,d)(h) - \phi(a,b,cd)(h) + (\phi(a,b,c)d)(h) = 0,$$

where $a = a' + \alpha\mathbf{1}$, $b = b' + \beta\mathbf{1}$, $c = c' + \gamma\mathbf{1}$, $d = d' + \lambda\mathbf{1}$ and $h = h' + \theta\mathbf{1}$. By definition of ϕ and (1)

$$\begin{aligned}
\sum_i\sum_j \frac{\alpha_i\alpha_j}{\omega(i)^2\omega(j)^2}\Big(&\{(b_j'c_i'd_i'h_j'a_j' + \alpha b_j'c_i'd_i'h_j' + \theta b_j'c_i'd_i'a_j') \\
&\quad - (b_i'c_i'd_j'h_j'a_j' + \alpha b_i'c_i'd_j'h_j' + \theta b_i'c_i'd_j'a_j')\} \\
&- \{(a_j'b_j'c_i'd_i'h_j' + \alpha b_j'c_i'd_i'h_j' + \beta a_j'c_i'd_i'h_j') \\
&\quad - (a_i'b_i'c_i'd_j'h_j' + \alpha b_i'c_i'd_j'h_j' + \beta a_i'c_i'd_j'h_j')\} \\
&+ \{(a_j'b_i'c_i'd_i'h_j' + \beta a_j'c_i'd_i'h_j' + \gamma a_j'b_i'd_i'h_j') \\
&\quad - (a_i'b_i'c_i'd_j'h_j' + \beta a_i'c_i'd_j'h_j' + \gamma a_i'b_i'd_j'h_j')\} \\
&- \{(a_j'b_i'c_i'd_i'h_j' + \gamma a_j'b_i'd_i'h_j' + \lambda a_j'b_i'c_i'h_j') \\
&\quad - (a_i'b_i'c_j'd_j'h_j' + \gamma a_i'b_i'd_j'h_j' + \lambda a_i'b_i'c_j'h_j')\} \\
&+ \{(a_j'b_i'c_i'd_j'h_j' + \lambda a_j'b_i'c_i'h_j' + \theta a_j'b_i'c_i'd_j') \\
&\quad - (a_i'b_i'c_j'd_j'h_j' + \lambda a_i'b_i'c_j'h_j' + \theta a_i'b_i'c_j'd_j')\}\Big) \\
= \sum_i\sum_j \frac{\theta b_j'c_i'd_i'a_j'}{\omega(i)^2\omega(j)^2}\alpha_i\alpha_j &- \sum_i\sum_j \frac{\theta a_i'b_i'c_j'd_j'}{\omega(i)^2\omega(j)^2}\alpha_i\alpha_j = 0
\end{aligned}$$

and so ϕ is a 3-cocycle. Also ϕ is cyclic, since

$$\begin{aligned}
\phi(d,a,b,c) &= \sum_i\sum_j \frac{d_j'a_i'b_i'c_j' - d_i'a_i'b_j'c_j'}{\omega(i)^2\omega(j)^2}\alpha_i\alpha_j \\
&= -\sum_i\sum_j \frac{a_i'b_j'c_j'd_i'}{\omega(i)^2\omega(j)^2}\alpha_i\alpha_j + \sum_i\sum_j \frac{a_i'b_i'c_j'd_j'}{\omega(i)^2\omega(j)^2}\alpha_i\alpha_j \\
&= -\phi(a,b,c,d) = (-1)^3\phi(a,b,c,d).
\end{aligned}$$

□

Now we are going to construct the $2n+1$-cocycle ϕ for higher dimensions.

LEMMA 3.2. *Let* $\psi_{ij} : \overbrace{\mathscr{A}^\# \times \mathscr{A}^\# \times \cdots \times \mathscr{A}^\#}^{2n\text{ times}} \to (\mathscr{A}^\#)'$ *be a* $2n$*-linear function defined by*

$$\psi_{ij}(a_1,\dots,a_{2n})(a_{2n+1}) = \sum_{k=1}^{2n+1} a_{1i}'\cdots a_{kj}'\cdots a_{(2n+1)i}',$$

where $a_k = a'_k + \beta_k \mathbf{1}$ *and* $a'_k = \{a'_{ki}\}_{i\in\mathbb{Z}}$ $(k = 1, \dots, 2n+1)$. *Then*

$$\delta\psi_{ij}(a_1, \dots, a_{2n+1})(a_{2n+2}) = a'_{1j}a'_{2i}\cdots a'_{(2n+1)i}a'_{(2n+2)j} + \sum_{k=1}^{2n+1}(-1)^k a'_{1i}\cdots a'_{kj}a'_{(k+1)j}\cdots a'_{(2n+2)i}.$$

PROOF. By the coboundary formula we have

$$(2)\quad \delta\psi_{ij}(a_1, \dots, a_{2n+1})(a_{2n+2}) = \psi_{ij}(a_2, \dots, a_{2n+1})(a_{2n+2}a_1) + \sum_{k=1}^{2n}(-1)^k\psi_{ij}(a_1, \dots, a_k a_{k+1}, \dots, a_{2n+1})(a_{2n+2}) - \psi_{ij}(a_1, \dots, a_{2n})(a_{2n+1}a_{2n+2}).$$

Using the definition of ψ_{ij} we obtain the value of all terms on the right-hand side of the above equation as follows

$$\begin{aligned}\psi_{ij}&(a_2, \dots, a_{2n+1})(a_{2n+2}a_1)\\ &= a'_{2i}\cdots a'_{(2n+1)i}a'_{(2n+2)j}a'_{1j} + \sum_{k=2}^{2n+1} a'_{1i}\cdots a'_{kj}\cdots a'_{(2n+2)i}\\ &\quad+ \sum_{k=2}^{2n+2}\beta_1 a'_{2i}\cdots a'_{kj}\cdots a'_{(2n+2)i} + \sum_{k=1}^{2n+1}\beta_{2n+2}a'_{1i}\cdots a'_{kj}\cdots a'_{(2n+1)i}.\end{aligned}$$

For $l = 1, \dots, 2n$,

$$\begin{aligned}\psi_{ij}&(a_1, \dots, a_l a_{l+1}, \dots, a_{2n+1})(a_{2n+2})\\ &= a'_{1i}\cdots a'_{lj}a'_{(l+1)j}\cdots a'_{(2n+2)i} + \sum_{\substack{k=1\\ k\neq l,l+1}}^{2n+2} a'_{1i}\cdots a'_{kj}\cdots a'_{(2n+2)i}\\ &\quad+ \sum_{\substack{k=1\\ k\neq l}}^{2n+2}\beta_l a'_{2i}\cdots a'_{kj}\cdots \widehat{a'}_l\cdots a'_{(2n+2)i} + \sum_{\substack{k=1\\ k\neq l+1}}^{2n+2}\beta_{l+1}a'_{1i}\cdots a'_{kj}\cdots \widehat{a'}_{l+1}\cdots a'_{(2n+2)i},\end{aligned}$$

where symbol $\widehat{\cdot}$ shows the element in that position is removed.

$$\begin{aligned}\psi_{ij}&(a_1, \dots, a_{2n})(a_{2n+1}a_{2n+2})\\ &= a'_{1i}\cdots a'_{(2n)i}a'_{(2n+1)j}a'_{(2n+2)j} + \sum_{k=1}^{2n} a'_{1i}\cdots a'_{kj}\cdots a'_{(2n+2)i}\end{aligned}$$

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$$+\sum_{\substack{k=1\\k\neq 2n+1}}^{2n+2}\beta_{2n+1}a'_{1i}\cdots a'_{kj}\cdots a'_{(2n)i}a'_{(2n+2)i}+\sum_{k=1}^{2n+1}\beta_{2n+2}a'_{1i}\cdots a'_{kj}\cdots a'_{(2n+1)i}.$$

Substitute the values for ψ_{ij} obtained above in (2). Then all summations with β_k ($k = i, \dots, 2n+2$) coefficients cancel in pairs, and we obtain

$$\begin{aligned}\delta\psi_{ij}(a_1,\dots,a_{2n+1})(a_{2n+2})\\ &= a'_{1j}a'_{2i}\cdots a'_{(2n+1)i}a'_{(2n+2)j}+\sum_{k=1}^{2n+1}(-1)^k a'_{1i}\cdots a'_{kj}a'_{(k+1)j}\cdots a'_{(2n+2)i}\\ &\quad+\sum_{k=2}^{2n+1}a'_{1i}\cdots a'_{kj}\cdots a'_{(2n+2)i}+\sum_{l=1}^{2n+1}(-1)^l\sum_{\substack{k=1\\k\neq l,l+1}}^{2n+2}a'_{1i}\cdots a'_{kj}\cdots a'_{(2n+2)i}\end{aligned}$$

and the sum of the last two terms is zero because, they contain $2n$ terms like $a'_{1i}\cdots a'_{kj}\cdots a'_{(2n+2)i}$ for every $k = 1,\dots,2n+2$, half with a positive sign and the other half with a negative sign which cancel in pairs. So this finishes the proof. $\square$

LEMMA 3.3. *Let $\sum_i \alpha_i$ be an absolutely convergent series of real numbers, and let $\phi : \underbrace{\mathscr{A}^\# \times \mathscr{A}^\# \times \cdots \times \mathscr{A}^\#}_{2n+1 \text{ times}} \to (\mathscr{A}^\#)'$ be the function defined by*

$$\phi(a_1,\dots,a_{2n+1})(a_{2n+2})=\sum_i\sum_j\frac{\alpha_i\alpha_j}{\omega(i)^{2n}\omega(j)^2}\delta\psi_{ij}(a_1,\dots,a_{2n+1})(a_{2n+2}),$$

where ψ_{ij} is defined as in Lemma 3.2. Then ϕ is a bounded cyclic $(2n+1)$-cocycle for every $n > 1$.

PROOF. It is easy to see that ϕ is a $2n+1$-linear map and also

$$|\phi(a_1,\dots,a_{2n+1})(a_{2n+2})|\le(2n+2)\|a_1\|_{\omega^{-1}}\cdots\|a_{2n+2}\|_{\omega^{-1}}\left(\sum_i|\alpha_i|\right)^2.$$

Thus ϕ is bounded and $\|\phi\|\le(2n+2)\left(\sum_{i=-\infty}^{\infty}|\alpha_i|\right)^2$. Also ϕ is a $(2n+1)$-cocycle, that is,

$$\delta\phi=\sum_i\sum_j\frac{\alpha_i\alpha_j}{\omega(i)^{2n}\omega(j)^2}\delta\delta\psi_{ij}=0$$

because $\delta\delta\psi_{ij}=0$. Furthermore we show that ϕ is cyclic, that is, it satisfies

$$\phi(a_1,\dots,a_{2n+1})(a_{2n+2})=(-1)^{2n+1}\phi(a_2,\dots,a_{2n+2})(a_1).$$

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For this we have to calculate the right-hand side of the above equation. We have the following:

$$
\begin{aligned}
&\phi(a_2, \ldots, a_{2n+2})(a_1) \\
&= \sum_i \sum_j \left\{ \frac{\alpha_i \alpha_j}{\omega(i)^{2n}\omega(j)^2} \left(a'_{2j} a'_{3i} \cdots a'_{(2n+2)i} a'_{1j} - a'_{2j} a'_{3j} \cdots a'_{(2n+2)i} a'_{1i} \right.\right. \\
&\qquad \left.\left. + a'_{2i} a'_{3j} a'_{4j} \cdots a'_{(2n+2)i} a'_{1i} \mp \cdots - a'_{2i} \cdots a'_{(2n+1)i} a'_{(2n+2)j} a'_{1j} \right) \right\} \\
&= \sum_i \sum_j \left\{ \frac{\alpha_i \alpha_j}{\omega(i)^{2n}\omega(j)^2} \left(- a'_{1j} a'_{2i} \cdots a'_{(2n+1)i} a'_{(2n+2)j} + a'_{1j} a'_{2j} a'_{3i} \cdots a'_{(2n+2)i} \right.\right. \\
&\qquad \left.\left. \mp \cdots - a'_{1i} \cdots a'_{(2n)j} a'_{(2n+1)j} a'_{(2n+2)i} + a'_{1i} \cdots a'_{(2n)i} a'_{(2n+1)j} a'_{(2n+2)j} \right) \right\} \\
&= -\phi(a_1, \ldots, a_{2n+1})(a_{2n+2}).
\end{aligned}
$$

Therefore ϕ is a cyclic $(2n+1)$-cocycle. $\square$

THEOREM 3.4. *Let ω be a weight on $\mathbb{Z}$ such that* $\inf\{\omega(i)\} = 0$. *Then*

$$\mathscr{H}^{2n+1}(\mathscr{A}^{\#}, (\mathscr{A}^{\#})') \neq 0$$

and also $\mathscr{HC}^{2n+1}(\mathscr{A}^{\#}) \neq 0$ for every $n \in \mathbb{N}$.

PROOF. Let ϕ be the bounded $2n+1$-cocycle which was introduced in Lemma 3.1 for $n = 1$ and in Lemma 3.3 for $n > 1$. Consider the sequence α_i which was defined in the proof of Theorem 2.2. Note that $m_i \neq m_j$ whenever $i \neq j$ and $\omega(m_k) \leq 1/2^k$. Also if $i < j$, since $1/2^j < 1/2^i$, then $\max\{\omega(m_i), \omega(m_j)\} \leq 1/2^i$.

Now if $\psi \in \mathscr{C}^{2n}(\mathscr{A}^{\#}, (\mathscr{A}^{\#})')$ such that $\phi = \delta\psi$, then by the definition of ϕ and the coboundary formula we have

$$
\begin{aligned}
\phi(e_{m_j}, \overbrace{e_{m_i}, \ldots, e_{m_i}}^{2n \text{ times}})(e_{m_j}) &= \psi(\overbrace{e_{m_i}, \ldots, e_{m_i}}^{2n \text{ times}})(e_{m_j}) \\
&\quad + \overbrace{\psi(e_{m_j}, \underbrace{e_{m_i}, \ldots, e_{m_i}}_{2n-1 \text{ times}})(e_{m_j}) \pm \cdots + \psi(e_{m_j}, \underbrace{e_{m_i}, \ldots, e_{m_i}}_{2n-1 \text{ times}})(e_{m_j})}^{2n-1 \text{ times}} \\
&= \psi(e_{m_i}, \ldots, e_{m_i})(e_{m_j}) + \psi(e_{m_j}, e_{m_i}, \ldots, e_{m_i})(e_{m_j}) \\
&= \psi(e_{m_i} + e_{m_j}, \overbrace{e_{m_i}, \ldots, e_{m_i}}^{2n-1 \text{ times}})(e_{m_j}).
\end{aligned}
$$

Therefore by the definition of ϕ

$$\psi(e_{m_i} + e_{m_j}, \overbrace{e_{m_i}, \ldots, e_{m_i}}^{2n-1 \text{ times}})(e_{m_j}) = \frac{\alpha_{m_i}\alpha_{m_j}}{\omega(m_i)^{2n}\omega(m_j)^2}.$$

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Suppose $\min\{\omega(m_i), \omega(m_j)\} = C_{ij}$, then

$$\|C_{ij}(e_{m_i} + e_{m_j})\|_{\omega^{-1}} = 1 \quad \text{and} \quad \|\omega(m_i)e_{m_i}\|_{\omega^{-1}} = 1.$$

If we let $i < j$, then

$$\begin{aligned}\|\psi\| &\geq \sup_{i,j}\left\{|\psi\left(C_{ij}(e_{m_i} + e_{m_j}), \omega(m_i)e_{m_i}, \dots, \omega(m_i)e_{m_i}\right)(\omega(m_j)e_{m_j})|\right\} \\ &= \sup_{i,j}\left\{\frac{\min\{\omega(m_i), \omega(m_j)\}\alpha_{m_i}\alpha_{m_j}}{\omega(m_i)\omega(m_j)}\right\} \\ &= \sup_{i,j}\left\{\frac{1}{\max\{\omega(m_i), \omega(m_j)\}i^2j^2}\right\} \geq \sup_{i,j}\left\{\frac{2^i}{j^4}\right\}.\end{aligned}$$

In particular, for $j = i + 1$, we have $\|\psi\| \geq \sup_i 2^i/(i+1)^4 = \infty$ which contradicts $\psi \in \mathscr{C}^{2n}(\mathscr{A}^\#, (\mathscr{A}^\#)')$. So $\mathscr{H}^{2n+1}(\mathscr{A}^\#, (\mathscr{A}^\#)') \neq 0$ and $\mathscr{H}\mathscr{C}^{2n+1}(\mathscr{A}^\#) \neq 0$. $\square$

REMARK. Consider the short exact sequence $0 \to \mathscr{A} \to \mathscr{A}^\# \to \mathbb{C} \to 0$. The dual of this short exact sequence, is the short exact sequence,

$$0 \to \mathbb{C} \to (\mathscr{A}^\#)' \to \mathscr{A}' \to 0.$$

This gives the long exact sequence of cohomology (see [6, III. Corollary 4.11])

$$\cdots \to \mathscr{H}^n(\mathscr{A}^\#, \mathbb{C}) \to \mathscr{H}^n(\mathscr{A}^\#, (\mathscr{A}^\#)') \to \mathscr{H}^n(\mathscr{A}^\#, \mathscr{A}') \to \cdots.$$

From this, one can show that $\mathscr{H}^n(\mathscr{A}^\#, \mathbb{C}) \neq 0$ for every $n \geq 2$.

As we noticed in Section 1, E_N is an amenable closed subalgebra of $\mathscr{A}^\#$. So $\mathscr{A}^\#$ satisfies the conditions of [12, Theorem 2.6 and Theorem 5.1]. We can therefore apply Theorem 2.2 and Theorem 3.4 to conclude that for each $n \geq 2$, the E_N-relative (cyclic) cohomology of $\mathscr{A}^\#$ does not vanish.

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References

[1] W. G. Bade, P. C. Curtis, Jr. and H. G. Dales, 'Amenability and weak amenability for Beurling and Lipschitz algebras', *Proc. London Math. Soc.* **55** (1987), 359–377.

[2] H. G. Dales and J. Duncan, 'Second order cohomology groups of some semigroup algebras', in: *Banach Algebra'97 (Blaubeuren)* (Walter de Gruyter, Berlin, 1998) pp. 101–117.

[3] H. G. Dales, F. Ghahramani and N. Grønbæk, 'Derivation into iterated duals of Banach algebras', *Studia Math.* **128** (1998), 19–54.

[4] F. Gourdeau and M. C. White, 'Vanishing of the third simplicial cohomology group of $\ell^1(Z_+)$', *Trans. Amer. Math. Soc.* **353** (2001), 2003–2017.

[5] N. Grønbæk, 'Weak and cyclic amenability for non-commutative Banach algebras', *Proc. Edinburgh Math. Soc.* **35** (1992), 315–328.

[6] A. Ya. Helemskii, *The homology of Banach and topological algebras* (Kluwer Academic Publishers, Dordrecht, 1986).

[7] ———, *Banach and locally convex algebras* (Oxford University Press, Oxford, 1993).

[8] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. 127 (Amer. Math. Soc. Providence, 1972).

[9] ———, *Derivations from $L^1(G)$ into $L^1(G)$ and $L^\infty(G)$*, Lecture Notes in Math. 1359 (Springer, Berlin, 1988) pp. 191–198.

[10] ———, 'Alternating cohomology', preprint.

[11] B. E. Johnsona, R. V. Kadison and J. R. Ringrose, 'Cohomology of operator algebra III. Reduction to normal cohomology', *Bull. Soc. Math. France* **100** (1972), 73–96.

[12] Z. A. Lykova, 'Relative cohomology of Banach algebras', *J. Operator Theory* **41** (1999), 23–53.

[13] Yu. V. Selivanov, 'Weak homological bi-dimension and its values in the class of biflat Banach algebras', *Extracta Math.* **11** (1996), 348–365.

[14] A. M. Sinclair and R. R. Smith, *Hochschild cohomology of von Neumann algebras*, London Math. Soc. Lecture Note Ser. 204 (Cambridge Univ. Press, Cambridge, 1995) pp. 196.

Faculty of Mathematics and Computer Science
Amir Kabir University
424 Hafez Avenue
Tehran 15914
Iran
e-mail: arpabbas@aut.ac.ir