

## ADDITIVE RESULTS FOR THE GENERALIZED DRAZIN INVERSE

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### Abstract

Additive perturbation results for the generalized Drazin inverse of Banach space operators are presented. Precisely, if  $A^d$  denotes the generalized Drazin inverse of a bounded linear operator  $A$  on an arbitrary complex Banach space, then in some special cases  $(A + B)^d$  is computed in terms of  $A^d$  and  $B^d$ . Thus, recent results of Hartwig, Wang and Wei (*Linear Algebra Appl.* **322** (2001), 207–217) are extended to infinite dimensional settings with simplified proofs.

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### 1. Introduction

Let  $X$  denote an arbitrary complex Banach space and  $\mathcal{L}(X)$  denote the Banach algebra of all bounded operators on  $X$ . If  $A \in \mathcal{L}(X)$ , then  $\mathcal{R}(A)$ ,  $\mathcal{N}(A)$  and  $\sigma(A)$ , respectively, denote the range, kernel and spectrum of  $A$ .

If  $S$  is a subset of the complex plane, then  $\text{acc } S$  and  $\text{iso } S$ , respectively, denote the set of all points of accumulation and the set of all isolated points of  $S$ .

If  $0 \notin \text{acc } \sigma(A)$ , then the function  $z \mapsto f(z)$  can be defined as  $f(z) = 0$  in a neighbourhood of 0 and  $f(z) = 1/z$  in a neighbourhood of  $\sigma(A) \setminus \{0\}$ . Then  $z \mapsto f(z)$  is regular in a neighbourhood of  $\sigma(A)$  and the generalized Drazin inverse

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of  $A$  is defined using the functional calculus as  $A^d = f(A)$  (see the well-known Koliha's paper [14]). Notice that  $A^d A = A A^d$ ,  $A^d A A^d = A^d$  and that  $A A^d A - A$  is quasinilpotent. We say that  $A \in \mathcal{L}(X)$  is *GD-invertible* if  $0 \notin \text{acc } \sigma(A)$ . If  $A$  is GD-invertible, then the spectral idempotent  $P$  of  $A$  corresponding to  $\{0\}$  is given by  $P = I - A A^d$ . The matrix form of  $A$  with respect to the decomposition  $X = \mathcal{N}(P) \oplus \mathcal{R}(P)$  is given by  $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ , where  $A_1$  is invertible and  $A_2$  is quasinilpotent. We can also write

$$(1) \quad A = C_A + Q_A, \quad C_A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_A = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix},$$

and  $A = C_A + Q_A$  is known as the *core-quasinilpotent decomposition* of  $A$ .

If  $A$  is GD-invertible, then the resolvent function  $z \mapsto (zI - A)^{-1}$  is defined in a punctured neighbourhood of  $\{0\}$ . If  $z = 0$  is a pole of the resolvent function of  $A$ , then the order of this pole is known as the *Drazin index* of  $A$ , denoted by  $\text{ind}(A)$ . In this case we say that  $A$  is *D-invertible*. If  $\text{ind}(A) = k$ , then  $A^d$  reduces to the ordinary Drazin inverse of  $A$ , denoted by  $A^D$ . Thus,  $A^D$  is the unique operator satisfying conditions

$$(2) \quad A^{m+1} A^D = A^m, \quad A^D A A^D = A^D, \quad A A^D = A^D A,$$

and  $m$  is the least integer  $\text{ind}(A)$  such that (2) is satisfied. Now the core-quasinilpotent decomposition reduces to the core-nilpotent decomposition. Precisely,  $\text{ind}(A) = k$  holds if and only if  $k$  is the least integer (if it exists) such that  $A_2^k = 0$  (recall notations from (1)).

If  $\text{ind}(A) \leq 1$ , then  $A^D$  is known as the *group inverse* of  $A$ , denoted by  $A^\#$ . Also,  $\text{ind}(A) = 0$  if and only if  $A$  is invertible and in this case  $A^D = A^{-1}$ . Notice that  $\text{ind}(C_A) \leq 1$  always holds and  $C_A^\# = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$  (see (1)).

The Drazin inverse in semigroups and associative rings was first introduced in [7]. The Drazin inverse of complex square matrices was investigated, among other papers and books, in [1] and [2]. A detailed treatment of the Drazin inverse in infinite dimensional spaces is given in [3] and [10]. The generalized Drazin inverse in Banach algebras was introduced in [14]. We mention that an alternative definition of a generalized Drazin inverse in a ring is also given in [9, 10, 11]. These two concepts of a generalized Drazin inverse are equivalent in the case when the ring is actually a complex Banach algebra with a unit.

In this paper we investigate the generalized Drazin inverse  $(A + B)^d$  in terms of  $A^d$  and  $B^d$ . Hartwig, Wang and Wei investigated this problem in a finite dimensional case (see [13]). In this paper we generalize their results to infinite dimensional setting, using an alternative approach based on matrix representation of operators relative to a space decomposition. This enables us to simplify the proofs.

## 2. Results

First we state the following auxiliary result.

**LEMMA 2.1.** *If  $P, Q \in \mathcal{L}(X)$  are quasinilpotent and  $PQ = 0$  or  $PQ = QP$ , then  $P + Q$  is also quasinilpotent. Hence,  $(P + Q)^d = 0$ .*

**PROOF.** Case I. Let  $PQ = 0$  and  $\lambda \neq 0$ . Then  $\lambda - P$  and  $\lambda - Q$  are invertible and consequently  $\lambda - P - Q = \lambda^{-1}(\lambda - P)(\lambda - Q)$  is invertible.

Case II. Let  $PQ = QP$  and let  $\sigma(P, Q)$  denote the Harte joint spectrum of  $(P, Q)$ . It follows that  $\sigma(P, Q) = \{(0, 0)\}$ . An application of the spectral mapping theorem for this spectrum leads to  $\sigma(P + Q) = \{0\}$  (see [10] for details). Thus, the proof is completed.  $\square$

The following result is a generalization of [13, Lemma 1.1].

**THEOREM 2.1.** *Let  $A, B, A + B \in \mathcal{L}(X)$  be GD-invertible and  $AB = BA$ . Then  $(A + B)^d = (C_A + C_B)^d[I + (C_A + C_B)^d(Q_A + Q_B)]^{-1}$ .*

**PROOF.** Let  $P$  be the spectral idempotent of  $A$  corresponding to  $\{0\}$ . Then  $A$  has the matrix form  $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$  with respect to the decomposition  $X = \mathcal{N}(P) \oplus \mathcal{R}(P)$ , where  $A_1$  is invertible and  $A_2$  is quasinilpotent. Now we have  $C_A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $Q_A = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}$ . Since  $B$  commutes with  $P$ , we conclude that  $B$  has the form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix}.$$

From  $\sigma(B) = \sigma(B_1) \cup \sigma(B_2)$  we get that  $B_1$  and  $B_2$  are GD-invertible. From the definition of the generalized Drazin inverse and properties of the functional calculus we get that  $B^d = \begin{bmatrix} B_1^d & 0 \\ 0 & B_2^d \end{bmatrix}$ , and  $C_B = \begin{bmatrix} C_{B_1} & 0 \\ 0 & C_{B_2} \end{bmatrix}$ ,  $Q_B = \begin{bmatrix} Q_{B_1} & 0 \\ 0 & Q_{B_2} \end{bmatrix}$ . From  $AB = BA$  we conclude that  $A_i B_i = B_i A_i$  for  $i = 1, 2$ . Let  $P_1$  be the spectral idempotent of  $B_1$  in the Banach algebra  $\mathcal{L}(\mathcal{N}(P))$  corresponding to  $\{0\}$  and let  $P_2$  be the spectral idempotent of  $B_2$  in the algebra  $\mathcal{L}(\mathcal{R}(P))$  corresponding to  $\{0\}$ . We have the matrix form of  $B_1 = \begin{bmatrix} B_3 & 0 \\ 0 & B_4 \end{bmatrix}$  with respect to the decomposition  $\mathcal{N}(P) = \mathcal{N}(P_1) \oplus \mathcal{R}(P_1)$ , where  $B_3$  is invertible and  $B_4$  is quasinilpotent. The matrix form of  $B_2$  is given as  $B_2 = \begin{bmatrix} B_5 & 0 \\ 0 & B_6 \end{bmatrix}$ , with respect to the decomposition  $\mathcal{R}(P) = \mathcal{N}(P_2) \oplus \mathcal{R}(P_2)$ , where  $B_5$  is invertible and  $B_6$  is quasinilpotent. Since  $A_i$  commutes with  $P_i$  for  $i = 1, 2$ , we conclude that  $A_1$  and  $A_2$ , respectively, have the following matrix forms:  $A_1 = \begin{bmatrix} A_3 & 0 \\ 0 & A_4 \end{bmatrix}$ , where  $A_1$  and  $A_3$  are invertible, and  $A_2 = \begin{bmatrix} A_5 & 0 \\ 0 & A_6 \end{bmatrix}$ , where  $A_5$  and  $A_6$  are quasinilpotent. The last statement follows from the fact  $\sigma(A_2) = \sigma(A_5) \cup \sigma(A_6)$ .

Notice that

$$A + B = \begin{bmatrix} A_3 + B_3 & 0 & 0 & 0 \\ 0 & A_4 + B_4 & 0 & 0 \\ 0 & 0 & A_5 + B_5 & 0 \\ 0 & 0 & 0 & A_6 + B_6 \end{bmatrix}.$$

Since  $0 \notin \text{acc } \sigma(A + B)$  and  $\sigma(A + B) = \bigcup_{i=3}^6 \sigma(A_i + B_i)$ , we conclude that  $A_i + B_i$  is GD-invertible for all  $i = 3, 4, 5, 6$ . Since  $A_4$  is invertible and  $B_4$  is quasinilpotent, we get  $A_4 + B_4 = A_4(I + A_4^{-1}B_4)$ . Also,  $A_4$  commutes with  $B_4$ . Hence,  $A_4^{-1}B_4$  is quasinilpotent and  $I + A_4^{-1}B_4$  is invertible. We get that  $A_4 + B_4$  is invertible. Analogously,  $A_5 + B_5$  is invertible. Finally,  $A_6 + B_6$  is a sum of mutually commuting quasinilpotent elements. From Lemma 2.1 it follows that  $A_6 + B_6$  is quasinilpotent.

We have  $C_{B_1} = \begin{bmatrix} B_3 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $C_{B_2} = \begin{bmatrix} B_5 & 0 \\ 0 & 0 \end{bmatrix}$  and

$$C_A + C_B = \begin{bmatrix} A_3 + B_3 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 \\ 0 & 0 & B_5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since  $A_3 + B_3$  is GD-invertible and  $A_4$  and  $B_5$  are invertible, we conclude that  $C_A + C_B$  is GD-invertible and

$$(C_A + C_B)^d = \begin{bmatrix} (A_3 + B_3)^d & 0 & 0 & 0 \\ 0 & A_4^{-1} & 0 & 0 \\ 0 & 0 & B_5^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We also have

$$Q_A + Q_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B_4 & 0 & 0 \\ 0 & 0 & A_5 & 0 \\ 0 & 0 & 0 & A_6 + B_6 \end{bmatrix}$$

and

$$(C_A + C_B)^d(Q_A + Q_B) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_4^{-1}B_4 & 0 & 0 \\ 0 & 0 & B_5^{-1}A_5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now it follows that

$$[I + (C_A + C_B)^d(Q_A + Q_B)]^{-1} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & (I + A_4^{-1}B_4)^{-1} & 0 & 0 \\ 0 & 0 & (I + A_5B_5^{-1})^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and we easily conclude that

$$\begin{aligned} & (C_A + C_B)^d[I + (C_A + C_B)^d(Q_A + Q_B)] \\ &= \begin{bmatrix} (A_3 + B_3)^d & 0 & 0 & 0 \\ 0 & (A_4 + B_4)^{-1} & 0 & 0 \\ 0 & 0 & (A_5 + B_5)^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = (A + B)^d. \end{aligned}$$

Thus, the proof is completed.  $\square$

Now we consider the non-commutative case. The following result is proved in [6] (see also [12] and [17] for a finite dimensional case).

**LEMMA 2.2.** *If  $A \in \mathcal{L}(X)$  and  $B \in \mathcal{L}(Y)$  are GD-invertible,  $C \in \mathcal{L}(Y, X)$  and  $D \in \mathcal{L}(X, Y)$ , then  $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$  and  $N = \begin{bmatrix} A & 0 \\ D & B \end{bmatrix}$  are also GD-invertible and  $M^d = \begin{bmatrix} A^d & S \\ 0 & B^d \end{bmatrix}$ ,  $N^d = \begin{bmatrix} A^d & 0 \\ R & B^d \end{bmatrix}$ , where*

$$\begin{aligned} S &= (A^d)^2 \left[ \sum_{n=0}^{\infty} (A^d)^n C B^n \right] (I - B B^d) + (I - A A^d) \left[ \sum_{n=0}^{\infty} A^n C (B^d)^n \right] (B^d)^2 - A^d C B^d \\ \text{and} \\ R &= (B^d)^2 \left[ \sum_{n=0}^{\infty} (B^d)^n D A^n \right] (I - A A^d) + (I - B B^d) \left[ \sum_{n=0}^{\infty} B^n D (A^d)^n \right] (A^d)^2 - B^d D A^d. \end{aligned}$$

We need one particular case of our main result.

**THEOREM 2.2.** *If  $P, Q \in \mathcal{L}(X)$  are GD-invertible,  $Q \in \mathcal{L}(X)$  is quasinilpotent and  $PQ = 0$ , then  $P + Q$  is GD-invertible and  $(P + Q)^d = \sum_{n=0}^{\infty} Q^n (P^d)^{n+1}$ .*

**PROOF.** Since  $P$  is GD-invertible, we conclude that  $P$  has the matrix form  $\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$ , where  $P_1$  is invertible and  $P_2$  is quasinilpotent. From  $PQ = 0$  we conclude that  $Q$  has the matrix form  $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$ , where  $P_2 Q_1 = 0$  and  $P_2 Q_2 = 0$ . Since  $\{0\} = \sigma(Q) = \{0\} \cup \sigma(Q_2)$ , we conclude that  $Q_2$  is quasinilpotent. Now we have

$$P + Q = \begin{bmatrix} P_1 & 0 \\ Q_1 & P_2 + Q_2 \end{bmatrix},$$

where  $P_1$  is invertible and  $P_2 + Q_2$  is quasinilpotent (see Lemma 2.1). From Lemma 2.2 we get that

$$(P + Q)^d = \begin{bmatrix} P_1^{-1} & 0 \\ \sum_{n=0}^{\infty} (P_2 + Q_2)^n Q_1 P_1^{-n-2} & 0 \end{bmatrix}.$$

By induction on  $n$  and using  $P_2 Q_1 = 0$  and  $P_2 Q_2 = 0$ , we prove that  $(P_2 + Q_2)^n Q_1 = Q_2^n Q_1$  holds for all  $n \geq 0$ . Hence,

$$\sum_{n=0}^{\infty} (P_2 + Q_2)^n Q_1 P_1^{-n-2} = \sum_{n=0}^{\infty} Q_2^n Q_1 P_1^{-n-2}.$$

On the other hand,

$$\begin{aligned} \sum_{n=0}^{\infty} Q^n (P^d)^{n+1} &= P^d + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ Q_2^{n-1} Q_1 & Q_2^n \end{bmatrix} \begin{bmatrix} P_1^{-n-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} P_1^{-1} & 0 \\ \sum_{n=0}^{\infty} Q_2^n Q_1 P_1^{-n-2} & 0 \end{bmatrix}. \end{aligned}$$

Thus, the proof is completed.  $\square$

The following result is a generalization of [13, Theorem 2.1].

**THEOREM 2.3.** *If  $P, Q \in \mathcal{L}(X)$  are GD-invertible and  $PQ = 0$ , then  $P + Q$  is GD-invertible and*

$$(P + Q)^d = (I - QQ^d) \left[ \sum_{n=0}^{\infty} Q^n (P^d)^n \right] P^d + Q^d \left[ \sum_{n=0}^{\infty} (Q^d)^n P^n \right] (I - PP^d).$$

**PROOF.** Since  $Q$  is GD-invertible, we can write  $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$ , where  $Q_1$  is invertible and  $Q_2$  is quasinilpotent. From  $PQ = 0$  we obtain that  $P = \begin{bmatrix} 0 & P_1 \\ 0 & P_2 \end{bmatrix}$ , where  $P_1 Q_2 = 0$  and  $P_2 Q_2 = 0$ . From  $\sigma(P) = \{0\} \cup \sigma(P_2)$  and  $0 \notin \text{acc } \sigma(P)$  we conclude that  $P_2$  is GD-invertible. Now we have

$$P + Q = \begin{bmatrix} Q_1 & P_1 \\ 0 & P_2 + Q_2 \end{bmatrix}.$$

From Theorem 2.2 we know that  $P_2 + Q_2$  is GD-invertible. Hence, using Lemma 2.2 we get that  $P + Q$  is GD-invertible and

$$(P + Q)^d = \begin{bmatrix} Q_1^{-1} & S \\ 0 & (P_2 + Q_2)^d \end{bmatrix},$$

where

$$S = \left[ \sum_{n=0}^{\infty} Q_1^{-n-2} P_1 (P_2 + Q_2)^n \right] \left[ I - (P_2 + Q_2)(P_2 + Q_2)^d \right] - Q_1^{-1} P_1 (P_2 + Q_2)^d.$$

Using  $P_1 Q_2 = 0$  and  $P_2 Q_2 = 0$  we prove  $P_1(P_2 + Q_2)^n = P_1 P_2^n$  for all  $n \geq 0$ . Now, using Lemma 2.2, Theorem 2.2, and facts  $P_1 Q_2 = 0$  and  $P_2 Q_2 = 0$ , we compute

$$\begin{aligned}
 & \left[ \sum_{n=0}^{\infty} Q_1^{-n-2} P_1 (P_2 + Q_2)^n \right] [I - (P_2 + Q_2)(P_2 + Q_2)^d] \\
 &= \left[ Q_1^{-2} P_1 + \sum_{n=1}^{\infty} Q_1^{-n-2} P_1 P_2^n \right] \left[ I - (P_2 + Q_2) \left( P_2^d + \sum_{n=1}^{\infty} Q_2^n (P_2^d)^{n+1} \right) \right] \\
 &= \left[ Q_1^{-2} P_1 + \sum_{n=1}^{\infty} Q_1^{-n-2} P_1 P_2^n \right] \left[ I - P_2 P_2^d - \sum_{n=1}^{\infty} Q_2^n (P_2^d)^n \right] \\
 &= Q_1^{-2} P_1 (I - P_2 P_2^d) - Q_1^{-2} P_1 \sum_{n=1}^{\infty} Q_2^n (P_2^d)^n + \sum_{n=1}^{\infty} Q_1^{-n-2} P_1 P_2^n (I - P_2 P_2^d) \\
 &\quad - \left[ \sum_{n=1}^{\infty} Q_1^{-n-2} P_1 P_2^n \right] \left[ \sum_{n=1}^{\infty} Q_2^n (P_2^d)^n \right] \\
 &= \sum_{n=0}^{\infty} Q_1^{-n-2} P_1 P_2^n (I - P_2 P_2^d).
 \end{aligned}$$

Also, using Theorem 2.2 we get

$$Q_1^{-1} P_1 (P_2 + Q_2)^d = Q_1^{-1} P_1 \sum_{n=0}^{\infty} Q_2^n (P_2^d)^{n+1} = Q_1^{-1} P_1 P_2^d.$$

On the other hand, notice that from Lemma 2.2 we get  $P^d = \begin{bmatrix} 0 & P_1 (P_2^d)^2 \\ 0 & P_2^d \end{bmatrix}$ . Hence we have

$$\begin{aligned}
 & (I - Q Q^d) \left[ P^d + \sum_{n=1}^{\infty} Q^n (P^d)^{n+1} \right] + \left[ Q^d + \sum_{n=1}^{\infty} (Q^d)^{n+1} P^n \right] (I - P P^d) \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & P_2^d \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_1^n & 0 \\ 0 & Q_2^n \end{bmatrix} \begin{bmatrix} 0 & P_1 (P_2^d)^{n+2} \\ 0 & (P_2^d)^{n+1} \end{bmatrix} \\
 &\quad + \begin{bmatrix} Q_1^{-1} & -Q_1^{-1} P_1 P_2^d \\ 0 & 0 \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & Q_1^{-n-1} P_1 P_2^{n-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & -P_1 P_2^d \\ 0 & I - P_2 P_2^d \end{bmatrix} \\
 &= \begin{bmatrix} Q_1^{-1} & \sum_{n=1}^{\infty} Q_1^{-n-1} P_1 P_2^{n-1} (I - P_2 P_2^d) - Q_1^{-1} P_1 P_2^d \\ 0 & \sum_{n=0}^{\infty} Q_2^n (P_2^d)^{n+1} \end{bmatrix}.
 \end{aligned}$$

Thus, the proof is completed. □

We can also prove the following result, generalizing [13, Corollary 2.2].

**THEOREM 2.4.** *Let  $A, H \in \mathcal{L}(X)$  and let  $A$  be a GD-invertible operator. Let  $F \in \mathcal{L}(X)$  be an idempotent commuting with  $A$  such that  $FH = H$ . If  $R = (A - H)F$  is GD-invertible, then  $A - H$  is GD-invertible and*

$$(A - H)^d = R^d - \sum_{n=0}^{\infty} (R^d)^{n+2} H(I - F)A^n(I - AA^d) \\ + (I + R^d H)(I - F)A^d - (I - RR^d) \sum_{n=0}^{\infty} (A - H)^n H(I - F)(A^d)^{n+2}.$$

**PROOF.** Since  $F^2 = F$ , we have  $X = \mathcal{R}(F) \oplus \mathcal{N}(F)$  and  $F = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$  with respect to this decomposition. Operators  $A$  and  $F$  mutually commute, hence  $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ . From  $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$  we know that  $A_1 \in \mathcal{L}(\mathcal{R}(F))$  and  $A_2 \in \mathcal{L}(\mathcal{N}(F))$  are GD-invertible and  $A^d = \begin{bmatrix} A_1^d & 0 \\ 0 & A_2^d \end{bmatrix}$ . Since  $FH = H$ , we conclude that the matrix form of  $H$  is given by  $H = \begin{bmatrix} H_1 & H_2 \\ 0 & 0 \end{bmatrix}$ . Now we have

$$A - H = \begin{bmatrix} A_1 - H_1 & -H_2 \\ 0 & A_2 \end{bmatrix}.$$

Notice that  $R = \begin{bmatrix} A_1 - H_1 & 0 \\ 0 & 0 \end{bmatrix}$  is GD-invertible, implying that  $A_1 - H_1 \in \mathcal{L}(\mathcal{R}(F))$  is GD-invertible. From Lemma 2.2 it follows that  $A - H$  is GD-invertible and

$$(A - H)^d = \begin{bmatrix} (A_1 - H_1)^d & 0 \\ 0 & A_2^d \end{bmatrix} + \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix},$$

where

$$S = (A_1 - H_1)^d H_2 A_2^d - \sum_{n=0}^{\infty} [(A_1 - H_1)^d]^{n+2} H_2 A_2^n (I - A_2 A_2^d) \\ - [I - (A_1 - H_1)(A_1 - H_1)^d] \sum_{n=0}^{\infty} (A_1 - H_1)^n H_2 (A_2^d)^{n+2}.$$

Notice that  $R^d = \begin{bmatrix} (A_1 - H_1)^d & 0 \\ 0 & 0 \end{bmatrix}$ . Now a straightforward computation shows:

$$(I - RR^d)(A - H)^n H(I - F)(A^d)^{n+2} \\ = \begin{bmatrix} 0 & [I - (A_1 - H_1)(A_1 - H_1)^d](A_1 - H_1)^n H_2 (A_2^d)^{n+2} \\ 0 & 0 \end{bmatrix}, \\ (I + R^d H)(I - F)A^d = \begin{bmatrix} 0 & (A_1 - H_1)^d H_2 A_2^d \\ 0 & A_2^d \end{bmatrix}$$

and

$$(R^d)^{n+2} H(I - F)A^n(I - AA^d) = \begin{bmatrix} 0 & ((A_1 - H_1)^d)^{n+2} H_2 A_2^n (I - A_2 A_2^d) \\ 0 & 0 \end{bmatrix}.$$

Hence, the proof is completed.  $\square$



Campbell and Meyer (see [2]) investigated the continuity properties of the Drazin inverse of complex square matrices, but they did not establish the norm estimates for the perturbation of the Drazin inverse. Their results are extended to infinite dimensional settings by Rakočević (see [4]) and Koliha and Rakočević (see [15]), but norm estimates are not established there. It is interesting to mention that special cases of these perturbation results are already known. For example, see [20] for complex square matrices, [5] for complex Banach algebras and [8] for unbounded operators on Banach spaces. See also Case (4) of the following section. Hence, in this paper we partially solve the previous problem of Campbell and Meyer and extend some well-known results from previous papers.

### 3. Special cases

Many interesting special cases of our Theorem 2.4 are considered in [13] for matrices. Some of them are generalizations of well-known results.

**Case (1).** If  $HF = 0$ , then  $(A - H)^n FH = A^n H$  for  $n \geq 0$ . Consequently,

$$(A - H)^d = A^d F - \sum_{n=0}^{\infty} (A^d)^{n+2} H A^n (I - A A^d) \\ + (I - F + A^d H) A^d - \sum_{n=0}^{\infty} A^n (I - A A^d) H (A^d)^{n+2}.$$

**Case (1a).** If  $HF = 0$  and  $F = A A^d$ , then  $H A^d = H A A^d = 0$  and  $(A - H)^d = A^d - \sum_{n=0}^{\infty} (A^d)^{n+2} H A^n$ .

**Case (1b).** If  $HF = 0$  and  $F = I - A A^d$ , then  $A^d H = 0$  and  $(A - H)^d = A^d - \sum_{n=0}^{\infty} A^n H (A^d)^{n+2}$ .

**Case (2).** If  $F = A A^d$ , then  $(A - H)^d = R^d - \sum_{n=0}^{\infty} (R^d)^{n+2} H A^n (I - A A^d)$ .

**Case (2a).** Let  $F = A A^d$  and let  $U = I - A^d H A A^d$  be invertible. Recall notation from Theorem 2.4, noticing that  $A_1$  is invertible and  $A_2$  is quasinilpotent. Then  $U = \begin{bmatrix} I - A_1^{-1} H_1 & 0 \\ 0 & I \end{bmatrix}$ . Since  $U$  is invertible, then  $I - A_1^{-1} H_1$  is invertible and  $A_1 - H_1 = A_1(I - A_1^{-1} H_1)$  is invertible. Hence,  $\text{ind}(R) \leq 1$ ,

$$R^\# = \begin{bmatrix} (I - A_1^{-1} H_1)^{-1} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and  $(A - H)^d = R^\# - \sum_{n=0}^{\infty} (R^\#)^{n+2} H A^n (I - A A^d)$ .

Let  $V = I - A A^d H A^d = \begin{bmatrix} I - H_1 A_1^{-1} & 0 \\ 0 & I \end{bmatrix}$  and  $W = I - A^d H = \begin{bmatrix} I - A_1^{-1} H_1 & -A_1^{-1} H_2 \\ 0 & I \end{bmatrix}$ . Notice that  $U$  is invertible if and only if  $1 \notin \sigma(A_1^{-1} H_1)$ , if and only if  $1 \notin \sigma(H_1 A_1^{-1})$ , since it is well-known that for  $T \in \mathcal{L}(X, Y)$  and  $S \in \mathcal{L}(Y, X)$

$$(3) \quad \sigma(ST) \cup \{0\} = \sigma(TS) \cup \{0\}.$$

Hence,  $U$  is invertible if and only if  $V$  is invertible if and only if  $W$  is invertible. We also easily verify  $R^* = U^{-1}A^d = A^dV^{-1}$ . Hence, this result generalizes the main results in [16, 18, 20, 19].

**Case (2b).** If  $F = I - AA^d$ , then  $A^dH = A^dF = 0$ ,

$$R = A(I - AA^d) - (I - AA^d)H(I - AA^d)$$

and  $(A - H)^d = R^d + (I + R^dH)A^d - (I - RR^d) \sum_{n=0}^{\infty} (A - H)^n H(A^d)^{n+2}$ .

**Case (3).** Let  $AA^dHF = HFAA^d = HF$ , let  $U = I - A^dHF$  be invertible and  $(AF)^*$  exists. Recall notation from Theorem 2.4. Then  $AF = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$ , implying  $\text{ind}(A_1) \leq 1$ . From  $AA^dHF = HFAA^d = HF$  we conclude  $A_1A_1^*H_1 = H_1A_1A_1^* = H_1$ . We also have

$$U = \begin{bmatrix} I - A_1^*H_1 & 0 \\ 0 & I \end{bmatrix}.$$

Let

$$V = I - HFA^d = V = \begin{bmatrix} I - H_1A_1^* & 0 \\ 0 & I \end{bmatrix}.$$

By (3) we conclude that  $U$  is invertible if and only if  $V$  is invertible. Now a matrix computation shows  $R = AFU = VFA$ . From  $A_1^*(I - H_1A_1^*) = (I - A_1^*H_1)A_1^*$  we get the equality  $A^dFV = UA^dF$ . Let  $Y = U^{-1}A^dF = A^dFV^{-1}$ . Then it is easy to verify  $Y = R^*$ . Finally, we have  $(A - H)^d = R^* - \sum_{n=0}^{\infty} (R^*)^{n+2}H(I - AA^d)A^n$ .

**Case (4).** If  $FH = HF = F$ , then  $(A - H)^d = R^d + (I - F)A^d$ . Moreover, if  $F = AA^d$  and  $U = I - A^dH$  is invertible, then  $(A - H)^d = R^d = U^{-1}A^d$ .

Case (4) shows that results of this paper are more general than the corresponding results in [20]. Analogous results are proved in complex Banach algebras in [5]. Results of this paper are more general than results in [5] if we consider a Banach algebra of all operators on a fixed complex Banach space. We expect that all results of this paper should be valid in an arbitrary complex Banach algebra with a unit, but this will be a matter of further investigations. It is interesting to mention results related to this Case (4), concerning the Drazin inverse for closed linear operators, as it is done in [8].

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