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UNIQUE HAHN-BANACH THEOREMS FOR SPACES OF HOMOGENEOUS POLYNOMIALS

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Abstract

We investigate certain norm and continuity conditions that provide us with 'unique Hahn-Banach Theorems' from $\mathscr{P}(^n c_0)$ to $\mathscr{P}(^n \ell_\infty)$ and from $\mathscr{P}_N(^n E)$ to $\mathscr{P}_N(^n E'')$. We show that there is a unique norm-preserving extension for norm-attaining 2-homogeneous polynomials on complex c_0 to ℓ_∞ but there is no unique norm-preserving extension from $\mathscr{P}(^3 c_0)$ to $\mathscr{P}(^3 \ell_\infty)$.

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1. Introduction

The problem of extending (continuous) homogeneous polynomials from a subspace of a Banach space to the entire space was first studied by the first author and Berner [1] in 1978. (The definition of homogeneous polynomial on a Banach space and other related concepts are reviewed below.) They showed that in contrast to linear functionals, extensions of homogeneous polynomials may not always exist. However, it was also shown that for all n, every n-homogeneous polynomial P on a Banach space E extends to an n-homogeneous polynomial $\tilde{P}$ on its bidual E''. It is this class of extensions which has received most attention to date (see [1, 4, 5, 8, 9, 13, 17]).

It was not until 1989 that Davie and Gamelin [5] showed that a 'true' Hahn-Banach extension theorem holds in this situation, by proving that $\|P\|_E = \|\tilde{P}\|_{E''}$. The purpose of this article is to examine the following question: Under what conditions do we have a *unique* extension for spaces of homogeneous polynomials? We will look at

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extending homogeneous polynomials from c_0 to ℓ_∞ in Section 2. In Section 3, we will show that in order to have a unique Hahn-Banach Theorem it is necessary not only that the norm of a homogeneous polynomial over ℓ_∞ be equal to its norm over c_0 but that the norms of all its derivatives at every point when taken over ℓ_∞ coincide with the norms of the corresponding polynomials when taken over c_0. Finally, in Section 4, we examine norm-preserving extensions of nuclear polynomials from an M-ideal E to its bidual.

We collect here some basic definitions which will be needed in the rest of the paper. Most of this material can be found, for example, in [7]. Given a Banach space E we shall use B_E to denote the closed unit ball of E. We say that $P : E \to K = \mathbb{R}$ or $\mathbb{C}$ is an *n-homogeneous* polynomial if there is a continuous n-linear form $A : E \times \cdots \times E \to K$ such that $P(x) = A(x, \dots, x)$ for all $x \in E$. We shall use $\mathscr{P}(^nE)$ to denote the space of all n-homogeneous polynomials on E. Note that a 1-homogeneous polynomial is just a linear form. An application of the polarization formula yields the fact that there is a one-to-one correspondence between n-homogeneous polynomials P and *symmetric* n-linear forms $\check{P}$ such that $P(x) = \check{P}(x, \dots, x)$. The *canonical extension* $\tilde{P} : E'' \to K$ of an n-homogeneous polynomial $P : E \to K$ is given by means of the extension of the corresponding n-linear form A: For an n-tuple $(z_1, \dots, z_n) \in E'' \times \cdots \times E''$, define $\tilde{A} : E'' \times \cdots \times E'' \to K$ by $\tilde{A}(z_1, \dots, z_n) = \lim_{\alpha_1} \cdots \lim_{\alpha_n} A(x_{\alpha_1}, \dots, x_{\alpha_n})$, where each (x_{α_j}) is a net in E which converges to z_j in the weak* topology. Although the definition of $\tilde{A}(z_1, \dots, z_n)$ depends on the order in which one calculates the limits, the definition of the extended polynomial, $\tilde{P}(x) \equiv \tilde{A}(x, \dots, x)$, is independent of the order used (see [7, Section 6.2] for further details). An n-homogeneous polynomial P is said to be *nuclear* if there exists a bounded sequence $(\phi_j) \subset E'$ and a sequence $(\lambda_j) \in \ell_1$ such that $P(x) = \sum_{j=1}^\infty \lambda_j \phi_j^n(x)$. Given a nuclear polynomial, we define its nuclear norm $\|P\|_N \equiv \inf\left\{\sum_{j=1}^\infty |\lambda_j| \|\phi_j\|^n : P(x) = \sum_{j=1}^\infty \lambda_j \phi_j^n(x) \text{ all } x \text{ in } E\right\}$. We use $\mathscr{P}_N(^nE)$ to denote the space of all n-homogeneous nuclear polynomials on E.

2. Unique norm-preserving extensions from c_0 to ℓ_∞

A Banach space E is said to be an M-ideal in its bidual, E'', if E''' is the ℓ_1 sum of E' and $E^\perp = \{\phi \in E''' : \phi|_E \equiv 0\}$; that is, every continuous linear functional ϕ on E'' can be written as

$$\phi = \phi_1 + \phi_2,$$

$\phi_1 \in E'$, $\phi_2 \in E^\perp$, with

$$\|\phi\| = \|\phi_1\| + \|\phi_2\|.$$

It is immediate from this definition that when E is an M-ideal in its bidual that every continuous linear functional on E has a unique norm preserving extension to E''.

In particular, since c_0 is an M-ideal in ℓ_∞ (see [11]), every continuous *linear* functional on c_0 has a unique norm preserving extension to ℓ_∞. In this section we examine the extent to which this result carries over to *n-homogeneous* polynomials on c_0 when $n > 1$. We shall distinguish between the real and complex cases.

In the real case for every $n > 1$ the polynomial $P(x) = x_1^n - x_1^{n-2}\phi^2$, where $x = (x_1, x_2, \ldots), \phi \in c_0^\perp, 0 < \|\phi\| \leq 1$, is a norm-preserving extension of $Q(x) = x_1^n$, $x = (x_1, x_2, \ldots)$, to ℓ_∞ which is different from x_1^n on ℓ_∞. In particular, $x_1^2 - \phi^2$ is a norm-preserving extension of x_1^2 to ℓ_∞.

In the remainder of this section, we shall study the more complicated complex case, with different results depending on the degree of the space of polynomials. We shall begin by considering homogeneous polynomials of degree at least three, and for this the following lemma will be useful.

LEMMA 1. *Let $n \geq 3$ be a positive integer. Then $|1+w|^n + 2|1-w|^{n-1} \leq 2^n$, for all $w \in \mathbb{C}$, $|w| = 1$.*

PROOF. Let $x = |1+w|$ and $y = |1-w|$. Then $0 \leq x \leq 2$ and $x^2 + y^2 = 4$ (consider the triangle with vertices $1, -1$ and w in the unit circle). Then

$$|1+w|^n + 2|1-w|^{n-1} = x^n + 2(4-x^2)^{(n-1)/2}.$$

Hence it is enough to show that

$$x^n + 2(4-x^2)^{(n-1)/2} \leq 2^n,$$

for $0 \leq x \leq 2$. By dividing both sides by 2^n and setting $t = x/2$, it is equivalent to show that

$$f(t) = t^n + (1-t^2)^{(n-1)/2} \leq 1,$$

for $0 \leq t \leq 1$. Since

$$t^n + (1-t^2)^{(n-1)/2} \leq t^3 + (1-t^2),$$

for every $t \in [0, 1]$ and since

$$\max_{t \in [0,1]} t^3 + (1-t^2) = 1,$$

we conclude that

$$f(t) = t^n + (1-t^2)^{(n-1)/2} \leq 1. \qquad \square$$

Let $n \geq 3$ be an integer and consider the polynomial P on c_0 defined by

$$P(x) = (x_1 + x_2)^n,$$

$x = (x_1, x_2, \ldots)$. Then $\|P\| = 2^n = P(e_1 + e_2)$.

We define $\hat{P}_1$ on ℓ_∞ by

$$\hat{P}_1(x) = (x_1 + x_2)^n,$$

$x = (x_1, x_2, \dots)$. It is clear that $\hat{P}_1$ is a norm-preserving extension of P.

Define $\hat{P}_2 \in \mathscr{P}(^n\ell_\infty)$ by

$$\hat{P}_2(x) = (x_1 + x_2)^n + 2(x_1 - x_2)^{n-1}\phi(x),$$

$x = (x_1, x_2 \dots,)$, where ϕ is a Banach limit functional on ℓ_∞ of norm at most 1 such that $\phi(1, -1, 1, \dots,) = 1$. Then clearly we have that $\hat{P}_2|_{c_0} = P$ and since

$$\hat{P}_2(1, -1, 1, 1, \dots) = 2^n,$$

we have that $\hat{P}_1 \neq \hat{P}_2$. Fix $x = (x_k)_k$ in B_{ℓ_∞}. By the maximum modulus theorem we have

$$\begin{aligned}
&\sup_{|z|\le 1, |w|\le 1} |\hat{P}_2(z, w, x_3, x_4, \dots)| \\
&\quad = \sup_{|z|=|w|=1} |\hat{P}_2(z, w, x_3, x_4, \dots)| \le \sup_{|z|=|w|=1} |z+w|^n + 2|z-w|^{n-1} \\
&\quad = \sup_{|z|=|w|=1} \frac{|z+w|^n}{|z|^n} + \frac{2|z-w|^{n-1}}{|z|^{n-1}} = \sup_{|z|=|w|=1} \left|1 + \frac{w}{z}\right|^n + 2\left|1 - \frac{w}{z}\right|^{n-1} \\
&\quad = \sup_{|w|=1} |1+w|^n + 2|1-w|^{n-1}.
\end{aligned}$$

Applying Lemma 1 we see that $\|\hat{P}_2\|$ is also equal to 2^n. This shows that for each $n \geq 3$ on complex c_0 we are able to find an n-homogeneous polynomial which does not have unique norm-preserving extensions to ℓ_∞.

Let us turn to the case of 2-homogeneous polynomials on complex c_0. Since every n-homogeneous polynomial on c_0 is weakly continuous on bounded sets [3, 14] it follows from [6] that the monomials of degree n, with the square ordering, are a Schauder basis for $\mathscr{P}(^n c_0)$. Thus any P in $\mathscr{P}(^2 c_0)$ can be written as

$$P(x) = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} a_{ij} x_i x_j \right).$$

We shall say that P is *finite* if there is an integer n so that

$$P(x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} a_{ij} x_i x_j \right).$$

PROPOSITION 2. *A 2-homogeneous polynomial P on complex c_0 is norm-attaining if and only if it is finite.*

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PROOF. If P is finite we can regard it as a 2-homogeneous polynomial on ℓ_∞^n and so it will attain its norm. Conversely suppose P attains its norm at $x_0 = (\lambda_j)_{j\in\mathbb{N}}$. Without loss of generality we shall assume that $\|P\| = 1$. Let $J = \{j \in \mathbb{N} : |\lambda_j| = 1\}$. Since $x_0 \in c_0$, J is finite. By change of variable and rearrangement of indices we may assume that $J = \{1, 2, \dots, n\}$, $\lambda_1 = \lambda_2 = \dots = \lambda_n = 1$ and that $|\lambda_j| \geq |\lambda_{j+1}|$ for all j. Given any y in B_{c_0} of the form

$$y = (\underbrace{0, \dots, 0}_{n\text{-times}}, y_{n+1}, y_{n+2}, \dots),$$

for every $\lambda \in \mathbb{C}$, $|\lambda| = 1 - |\lambda_{n+1}|$, we have that

$$\begin{aligned} P(x_0 \pm \lambda y) = \check{P}(x_0 \pm \lambda y, x_0 \pm \lambda y) &= \check{P}(x_0, x_0) \pm 2\check{P}(x_0, \lambda y) + \check{P}(\lambda y, \lambda y) \\ &= 1 \pm 2\lambda \check{P}(x_0, y) + \lambda^2 P(y), \end{aligned}$$

has modulus at most 1, and thus

$$|1 + \lambda^2 P(y)| \leq 1.$$

Choosing λ so that $\lambda^2 P(y)$ is purely imaginary we conclude that $P(y) = 0$. In particular we get that $P(0, 0, \dots, 0, \lambda_{n+1}, \lambda_{n+2}, \dots) = 0$. We also have that $\check{P}(x_0, y)$ is 0 for all y of the above form. Taking $y = (0, 0, \dots, 0, \lambda_{n+1}, \lambda_{n+2}, \dots)$ we see that $\check{P}((1, 1, \dots, 1, 0, 0, \dots), (0, 0, \dots, 0, \lambda_{n+1}, \lambda_{n+2}, \dots)) = 0$ and therefore we have that $P(1, 1, \dots, 1, 0, 0, \dots) = 1$. Let $z_1 = (\underbrace{1, 1, \dots, 1}_{n\text{-times}})$ and define $z_2, z_3, \dots, z_n$ by

$$\begin{aligned} z_2 &= (1, -n+1, 1, 1, \dots, 1), \\ z_3 &= (1, 1, -n+1, 1, \dots, 1), \\ &\vdots \\ z_n &= (1, 1, 1, \dots, 1, -n+1). \end{aligned}$$

Since

$$(x_1, x_2, \dots, x_n) = \frac{1}{n}(x_1 + x_2 + \dots + x_n)\, z_1 + \frac{1}{n}\sum_{j=2}^{n}(x_1 - x_j) z_j,$$

$\{z_1, z_2, \dots, z_n\}$ forms a basis for $\mathbb{C}^n$. For $j = 1, \dots, n$, define $\tilde{z}_j$ in c_0 by $\tilde{z}_j = (z_j, 0, 0, \dots)$. Repeating the argument given above we see that $\check{P}(\tilde{z}_1, y) = 0$.

For any $(x_1, x_2, \dots, x_n)$ in $\mathbb{C}^n$ we have

$$\begin{aligned} &P(x_1, x_2, \dots, x_n, y_{n+1}, y_{n+2}, \dots) \\ &\quad = P(x_1, x_2, \dots, x_n, 0, \dots) + \frac{2}{n}\sum_{j=2}^{n}(x_1 - x_j)\check{P}(\tilde{z}_j, y). \end{aligned}$$

Setting $2/n\check{P}(\tilde{z}_j, \cdot) = \psi_j \in c_0'$ we can write this as

$$\begin{aligned} &P(x_1, x_2, \dots, x_n, y_{n+1}, y_{n+2}, \dots) \\ &\quad = P(x_1, x_2, \dots, x_n, 0, \dots) + \sum_{j=2}^{n}(x_1 - x_j)\psi_j(y). \end{aligned}$$

We will show now that $\psi_2 \equiv 0$. The same argument will work for any $j \geq 2$.

Set $x_1 = 1$, $x_2 = e^{i\theta}$, and $x_j = 1$ for $j > 2$. Then

$$P(1, e^{i\theta}, 1, \dots, 1, y_{n+1}, y_{n+2}, \dots) = P(1, e^{i\theta}, 1, \dots, 1, 0, 0, \dots) + (1 - e^{i\theta})\psi_2(y).$$

Since $\|P\| = 1$ and since we can vary the argument of y independent of θ so that

$$|P(1, e^{i\theta}, 1, \dots, 1, 0, \dots)| + |1 - e^{i\theta}||\psi_2(y)| \leq 1$$

for all y of norm at most 1, we obtain that

$$|P(1, e^{i\theta}, 1, \dots, 1, 0, \dots)| + |1 - e^{i\theta}|\|\psi_2\| \leq 1.$$

Let $f(\theta) = |P(1, e^{i\theta}, 1, \dots, 1, 0, \dots)|$ and $g(\theta) = |1 - e^{i\theta}|$. Then we have

$$\|\psi_2\| \leq \frac{1 - f(\theta)}{g(\theta)}$$

for all θ. Since P is a continuous 2-homogeneous polynomial it is (complex) differentiable. Therefore we have that the functions

$$\theta \in \mathbb{R} \to (\operatorname{Re} P(1, e^{i\theta}, 1, \dots, 1, 0, \dots), \operatorname{Im} P(1, e^{i\theta}, 1, \dots, 1, 0, \dots)) \in \mathbb{R}^2$$

and

$$(x, y) \in \mathbb{R}^2 \setminus \{0\} \to \sqrt{x^2 + y^2} \in \mathbb{R}$$

are differentiable. Since $P(1, e^{i\theta}, 1, \dots, 1, 0, \dots) \to 1$ as $\theta \to 0$ their composition, f, is also differentiable at $\theta = 0$. Since P has a local maximum on the unit sphere at $\theta = 0$ we have $\lim_{\theta\to 0} f'(\theta) = 0$. Since $g(\theta) = [2(1 - \cos\theta)]^{1/2} = 2\sin(\theta/2)$, $\lim_{\theta\to 0} g'(\theta) = 1$. Applying L'Hôspital's rule we have $\|\psi_2\| \leq \lim_{\theta\to 0} f'(\theta)/g'(\theta) = 0$. Thus ψ_j is 0 for $j = 2, \dots, n$ and hence P depends only on finitely many variables. □

REMARK. The corresponding result for homogeneous polynomials on c_0 of degree greater than 2 fails. Lemma 1 can be used to show that the k-homogeneous polynomial

$$P(x) = (x_1 + x_2)^k + 2(x_1 - x_2)^{k-1} \sum_{j=3}^{\infty} \left(\frac{x_j}{2^j}\right),$$

$k \geq 3$, attains it norm, yet is not finite.

Let us now consider norm-preserving extensions of 2-homogeneous polynomials from complex c_0 to ℓ_∞. A straightforward adaptation of the argument given in Proposition 2 yields the following result.

PROPOSITION 3. *Every 2-homogeneous norm-attaining polynomial on complex c_0 has a unique norm-preserving extension to ℓ_∞.*

A Banach space E with the property that every linear functional on E has a unique norm preserving extension to E'' is said to be Hahn-Banach smooth (see [11, 16]). In [15] Smith and Sullivan introduce the weaker concept of weak Hahn-Banach smoothness by requiring that every norm attaining linear functional on E has a unique norm preserving extension to E''. Thus we see that complex c_0 is 'weakly Hahn-Banach smooth' of degree 2 but not of degree 3 or higher.

The argument in Proposition 2 can be easily modified to show that every 2-homogeneous polynomial on ℓ_∞^k has a unique norm-preserving extension to ℓ_∞^l, where $2 \leq k < l$. Using Lemma 1 we can show that for $n \geq 3$, the two polynomials $\hat{P}_1$ and $\hat{P}_2$ on ℓ_∞^l, defined by

$$\hat{P}_1(x) = (x_1 + x_2)^n$$

and

$$\hat{P}_2(x) = (x_1 + x_2)^n + 2(x_1 - x_2)^{n-1}x_3,$$

are distinct norm preserving extensions of the polynomial

$$P(x) = (x_1 + x_2)^n$$

on ℓ_∞^k.

We are unable to determine if every 2-homogeneous polynomial on c_0 has a unique norm-preserving extension to ℓ_∞. However, we do know that there are 2-homogeneous polynomials on c_0 which do not attain their norm and yet do have unique norm-preserving extensions to ℓ_∞. For example, consider any polynomial P on c_0 of the form

$$P(x) = \sum_{n=1}^{\infty} a_n x_n^2.$$

Let us suppose that this polynomial does not have a unique norm-preserving extension to ℓ_∞. We may suppose that P has norm 1 and, by change of variable $x_k \to e^{i\theta_k} x_k$, that all of the a_n's are real and non-negative. Then there is $Q \in \mathcal{P}(^2\ell_\infty)$, Q vanishing on c_0, such that $\tilde{P} + Q$ is a norm-preserving extension of P to ℓ_∞. Suppose that $\|Q\| > \delta > 0$. Let $\epsilon = \delta/20$ and choose an integer n_0 so that $\|P|_{\mathbb{C}^n}\| > 1 - \epsilon$ for all

$n \geq n_0$. Let u be a vector in $\mathbb{C}^n$ of the form $u = \pm e_1 \pm e_2 \pm \cdots \pm e_n$. Then for any y in B_{ℓ_∞} of the form $(0, \ldots, 0, y_{n+1}, y_{n+2}, \ldots)$ we have

$$(*) \qquad \tilde{P}(u \pm \lambda y) = P(u) \pm \lambda 2\check{\tilde{P}}(u, y) + \lambda^2 \tilde{P}(y).$$

The standard 'trick' (pick λ so that $\lambda^2 \tilde{P}(y)$ is real and positive) gives that $|\tilde{P}(y)| \leq \epsilon$. Replacing λ by $i\lambda$ and adding to $(*)$ gives that

$$P(u) + (1 \pm i)\lambda \check{\tilde{P}}(u, y)$$

has modulus at most 1. Thus $|\check{\tilde{P}}(u, y)| \leq \epsilon$. We also have

$$(\tilde{P} + Q)(u \pm \lambda y) = P(u) \pm 2\lambda \check{\tilde{P}}(u, y) \pm 2\lambda \check{Q}(u, y) + \lambda^2 \tilde{P}(y) + \lambda^2 Q(y).$$

Since $\|\tilde{P} + Q\| \leq 1$ we have that $|\check{\tilde{P}}(u, y) + \check{Q}(u, y)| \leq \epsilon$ and $|\tilde{P}(y) + Q(y)| \leq \epsilon$. Hence $|\check{Q}(u, y)| \leq 2\epsilon$ and $|Q(y)| \leq 2\epsilon$.

We now make two observations:

1. Suppose that $B : \mathbb{R}^{n_0} \times \ell_\infty \to \mathbb{R}$ is a continuous bilinear form (where $\mathbb{R}^{n_0}$ has the supremum norm), such that for all choices $u = \pm e_1 \pm e_2 \pm \cdots \pm e_{n_0}$ and for all $y \in \ell_\infty$, $\|y\| \leq 1$, we have $|B(u, y)| \leq \epsilon$. Then, in fact for all $x \in \mathbb{R}^{n_0}$, $\|x\| \leq 1$ and for all $y \in \ell_\infty$, $\|y\| \leq 1$, $|B(x, y)| \leq \epsilon$.

2. Suppose that $B : \mathbb{C}^{n_0} \times \ell_\infty \to C$ is a continuous bilinear form such that for all $x \in \mathbb{R}^{n_0}$, $\|x\|_{\ell_\infty^{n_0}} \leq 1$ and for all $y \in \ell_\infty$, $\|y\| \leq 1$, $|B(x, y)| \leq \epsilon$. Then $\|B\| \leq 2\epsilon$ (see for example [12]).

To conclude the argument, let z be a point of ℓ_∞, of norm at most 1. We can write z as $z = x + y$, where x is in $B_{\ell_\infty}^{n_0}$ and y is of the above form. Then

$$|Q(z)| \leq 2|\check{Q}(x, y)| + |Q(y)| \leq 10\epsilon = \delta/2$$

which contradicts our assumption that $\|Q\| > \delta$.

3. Characterizations of the canonical extension and norm-preserving extensions

We have seen in the previous section that, at least for homogeneous polynomials of degree 3 or greater, the canonical norm-preserving extension from $\mathcal{P}({}^n c_0)$ to $\mathcal{P}({}^n \ell_\infty)$ is not unique. In this section we shall examine other properties that characterise the canonical extension from $\mathcal{P}({}^n c_0)$ to $\mathcal{P}({}^n \ell_\infty)$.

Let us begin with the following question. If P is an n-homogeneous polynomial on ℓ_∞ and $\|P\| = \|P|_{c_0}\|$, what can we conclude about P?

PROPOSITION 4. *If $P \in \mathscr{P}({}^n\ell_\infty)$ satisfies $\|P\| = \|P|_{c_0}\|$, then P is weak* continuous on bounded sets at* 0.

PROOF. Given $y \in \ell_\infty$ and $k \in \mathbb{N}$ we define $\alpha_k(y) = (\alpha_k(y)_i)_{i=1}^\infty \in \ell_\infty$ by

$$\alpha_k(y)_i = \begin{cases} y_i & \text{if } i \leq k; \\ 0 & \text{if } i > k \end{cases}$$

and $\omega_k(y) = (\omega_k(y)_i)_{i=1}^\infty \in \ell_\infty$ by

$$\omega_k(y)_i = \begin{cases} 0 & \text{if } i \leq k; \\ y_i & \text{if } i > k. \end{cases}$$

We begin by showing that if $\|P\| = \|P|_{c_0}\|$, then $\lim_{k\to\infty} \|P|_{\omega_k(B_{\ell_\infty})}\| = 0$. Let us suppose that this condition does not hold. Then there is $C > 0$, an increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ and a sequence of points $(y_n)_{n\in\mathbb{N}}$ in B_{ℓ_∞} so that $|P(\omega_{k_n}(y_n))| > C$. Choose $\epsilon > 0$ so that $\epsilon < C^2/\left(4\|P|_{c_0}\|\right)$. Since $(e_n)_{n\in\mathbb{N}}$ is a Schauder basis for c_0, we may choose $a \in B_{c_0}$ and a positive integer n_0 so that $|P(\alpha_{n_0}(a))| > \|P|_{c_0}\| - \epsilon$. Then $\{\alpha_{n_0}(a) + \lambda\omega_{k_{n_0}}(y_{n_0}) : |\lambda| \leq 1\}$ is contained in B_{ℓ_∞}. Furthermore, by [7, Lemma 1.9 (b)], we have that

$$\begin{aligned}\|P\|^2 &\geq \sup_{|\lambda|\leq 1} |P(\alpha_{n_0}(a) + \lambda\omega_{k_{n_0}}(y_{n_0}))|^2 \geq |P(\alpha_{n_0}(a))|^2 + |P(\omega_{k_{n_0}}(y_{n_0}))|^2 \\ &> (\|P|_{c_0}\| - \epsilon)^2 + C^2 > \|P|_{c_0}\|^2 + C^2/2,\end{aligned}$$

a contradiction.

Now let us use this condition to show that P is weak* continuous on bounded sets at 0. Suppose that $(x_\beta)_\beta$ is a bounded weak* null net in ℓ_∞, which we may suppose without loss of generality is contained in B_{ℓ_∞}. Given $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ so that $\|P|_{\omega_{n_0}(B_{\ell_\infty})}\| < \epsilon/2$. Then

$$|P(x_\beta)| = |P(\alpha_{n_0}(x_\beta) + \omega_{n_0}(x_\beta))| \leq \sum_{j=0}^{n} \binom{n}{j} |\check{P}(\alpha_{n_0}(x_\beta))^j (\omega_{n_0}(B_{\ell_\infty}))^{n-j}|.$$

Since $(x_\beta)_\beta$ is weak* null, $(\alpha_{n_0}(x_\beta))_\beta$ is a null sequence in ℓ_∞. Therefore, we may choose β_0 so that

$$\binom{n}{j} |\check{P}(\alpha_{n_0}(x_\beta))^j (\omega_{n_0}(B_{\ell_\infty}))^{n-j}| \leq \frac{\epsilon}{2^{j+2}}$$

for $\beta > \beta_0$ and all $j, 0 < j \leq n$. This implies that $|P(x_\beta)| \leq \epsilon$ for $\beta > \beta_0$ and so P is weak* continuous on bounded sets at 0. $\square$

Using Proposition 4 we get the following 'unique Hahn-Banach Theorem' for the space of n-homogeneous polynomials on c_0 extended to ℓ_∞. The conditions in this theorem are analogous to those given in [17, Theorem 2], in which a characterization is given of when an analytic function on E'' is the canonical extension of an analytic function on E. Our condition is a weak* continuity condition on P whereas Theorem 2 of [17] is a weak* continuity condition on the first derivative of P.

THEOREM 5. *For $P \in \mathscr{P}(^n\ell_\infty)$ the following properties are equivalent:*

(i) *P is the canonical extension of an n-homogeneous polynomial on c_0.*

(ii) *P is weak* continuous on bounded subsets.*

(iii) $\left\|\frac{\hat{d}^j P}{j!}(x)\right\| = \left\|\frac{\hat{d}^j P}{j!}(x)\Big|_{c_0}\right\|$ *for every x in ℓ_∞ and every integer j, $1 \leq j \leq n$.*

PROOF. It follows from the Littlewood-Bogdanowicz-Pełczyński Theorem (see [3, 14]) that every n-homogeneous polynomial Q on c_0 is weakly continuous on bounded sets. Applying [2] we see that Q is also weakly uniformly continuous on bounded sets and in particular on B_{c_0}. It therefore follows from Goldstine's Theorem that there is a unique weak* continuous extension, $\hat{Q}$, of Q to B_{ℓ_∞}. Given $\xi \in B_{\ell_\infty}$ we apply [5] to get a net $(x_\alpha)_\alpha$ in B_{c_0} such that for each positive integer k and for each R in $\mathscr{P}(^k c_0)$, $(R(x_\alpha))_\alpha$ converges to $\tilde{R}(\xi)$. In particular, (x_α) converges weak* to ξ and $(Q(x_\alpha))_\alpha$ converges to $\tilde{Q}(\xi)$. Since $\hat{Q}$ is weak* continuous on B_{ℓ_∞}, $(Q(x_\alpha))_\alpha$ converges to $\hat{Q}(\xi)$ and hence $\tilde{Q} = \hat{Q}$. Thus (i) and (ii) are equivalent.

Now suppose that (ii) holds. By the Polarization Formula it follows that $(\hat{d}^j P/j!)(x)$ is weak* continuous on bounded sets for every x in ℓ_∞ and each positive integer j, $1 \leq j \leq n$. By Goldstine's Theorem, B_{c_0} is weak* dense in B_{ℓ_∞}, from which (iii) follows.

If (iii) holds, then it follows from Proposition 4 that $(\hat{d}^j P/j!)(x)$ is weak* continuous on bounded sets at 0 for every x in ℓ_∞ and for every integer j, $1 \leq j \leq n$. If $(x_\alpha)_\alpha$ is a bounded net which converges weak* to x in ℓ_∞, then $(x - x_\alpha)_\alpha$ is weak* null. We now see that

$$P(x) - P(x_\alpha) = \sum_{j=1}^{n} \binom{n}{j} \frac{\hat{d}^j P}{j!}(x)(x - x_\alpha)^{n-j}$$

converges to 0 proving that P is weak* continuous on bounded sets. □

We have seen that there are homogeneous polynomials on c_0 which do not have unique norm preserving extensions to ℓ_∞. Motivated by Godefroy [10] we shall now give a criterion for a polynomial to have a unique norm preserving extension. In fact, this characterization holds for every Banach space E such that E'' has the metric approximation property. (Recall that E is said to have the metric approximation property if for every $\epsilon > 0$ and every compact set $K \subset E$, there is a finite rank continuous linear operator $T : E \to E$, $\|T\| \leq 1$, such that $\|Tx - x\| < \epsilon$ for every $x \in K$.) The linear version of the following theorem is due to Godefroy [10] (see also [11]).

THEOREM 6. *Let E be a Banach space such that E'' has the metric approximation property and $P \in \mathscr{P}({}^nE)$ have norm* 1. *Then the following are equivalent*:

(i) *P has a unique norm preserving extension to E''.*

(ii) *If $(P_\alpha)_\alpha$ is a net in $B_{\mathscr{P}({}^nE)}$ which converges pointwise to P, then $(\tilde{P}_\alpha(x))_\alpha$ converges to $\tilde{P}(x)$ for every x in E''.*

PROOF. Suppose that (i) holds and that $(P_\alpha)_\alpha$ is a net in $B_{\mathscr{P}({}^nE)}$ which converges pointwise to P. Since $B_{\mathscr{P}({}^nE'')}$ is compact for the compact-open topology, for every subnet $(P_\beta)_\beta$ of $(P_\alpha)_\alpha$ we can find $Q \in B_{\mathscr{P}({}^nE'')}$ and a subnet $(P_\gamma)_\gamma$ so that $\tilde{P}_\gamma(x)$ converges to $Q(x)$ for every x in E''. Since $(\tilde{P}_\gamma)_\gamma$ is in $B_{\mathscr{P}({}^nE'')}$ and $(P_\gamma)_\gamma$ converges pointwise to P, Q is a norm preserving extension of P and therefore by our assumption Q must be equal to $\tilde{P}$. In particular, $(\tilde{P}_\alpha(x))_\alpha$ converges to $\tilde{P}(x)$ for all x in $B_{E''}$ and so (ii) holds.

Conversely, suppose that (ii) holds and that $\hat{P} \in \mathscr{P}({}^nE'')$ is a norm preserving extension of P. We may write $\hat{P}$ as $\hat{P} = \tilde{P} + Q$ where $Q \in \{R \in \mathscr{P}({}^nE'') : R|_E \equiv 0\}$. Since E'' has the metric approximation property, using [4, Theorem 4.4] we can find a net $(P_\alpha)_\alpha$ in $B_{\mathscr{P}({}^nE)}$ so that $(\tilde{P}_\alpha(x))_\alpha$ converges to $\hat{P}(x)$ for every x in E''. Clearly $(P_\alpha)_\alpha$ converges pointwise to P on $B_{\mathscr{P}({}^nE)}$. By our assumption $(\tilde{P}_\alpha(x))$ converges to $\tilde{P}(x)$ for all x in E'' and so $\hat{P} = \tilde{P}$. □

The canonical extension $P \to \tilde{P}$ may be viewed as an isometry from $\mathscr{P}({}^nE)$ into $\mathscr{P}({}^nE'')$. Condition (ii) of Theorem 6 may be regarded as saying that the restriction of this function to $B_{\mathscr{P}({}^nE)}$ is pointwise-to-pointwise continuous at P.

4. Unique norm-preserving extensions of nuclear polynomials

In this section, the Banach space E will be either real or complex, and we will examine the question of unique norm-preserving extensions of *nuclear* polynomials from a E to E''. Let us begin with the observation that the canonical extension is a norm preserving extension from $(P_N({}^nE), \|\cdot\|_N)$ to $(P_N({}^nE''), \|\cdot\|_N)$. To show this, given any P in $(P_N({}^nE), \|\cdot\|_N)$ and $\epsilon > 0$, choose a representation $\sum_{k=1}^\infty \lambda_k\phi_k^n$ of P so that $\sum_{k=1}^\infty |\lambda_k|\|\phi_k\|^n \leq \|P\|_N + \epsilon$. Since $\sum_{k=1}^\infty \lambda_k\phi_k^n$ is a representation of the canonical extension $\tilde{P}$, we have that

$$\|\tilde{P}\|_N \leq \sum_{k=1}^{\infty} |\lambda_k|\|\phi_k\|^n \leq \|P\|_N + \epsilon.$$

Hence $\|\tilde{P}\|_N \leq \|P\|_N$. Conversely, we can choose a representation $\sum_{k=1}^{\infty} \mu_k \psi_k^n$ of $\tilde{P}$ so that $\sum_{k=1}^{\infty} |\mu_k| \|\psi_k\|^n \leq \|\tilde{P}\|_N + \epsilon$, where $\psi_k \in E'''$. Then we have

$$\|\tilde{P}\|_N + \epsilon \geq \sum_{k=1}^{\infty} |\mu_k| \|\psi_k\|^n \geq \sum_{k=1}^{\infty} |\mu_k| \|\psi_k|_E\|^n \geq \|P\|_N$$

and therefore $\|\tilde{P}\|_N \geq \|P\|_N$.

The following is a partial answer to the question: When do we have a unique norm-preserving extension from $\mathscr{P}_N({}^nE)$ to $\mathscr{P}_N({}^nE'')$?

PROPOSITION 7. *Let E be a Banach space which is an M-ideal in its bidual. Then the canonical extension is the unique norm preserving extension from $P_N({}^nE)$ to $P_N({}^nE'')$.*

PROOF. Let Q be an extension of P which is not equal to $\tilde{P}$. We will show that $\|Q\|_N > \|P\|_N$. Since $Q \neq \tilde{P}$, there is $y \in E''$, $\|y\| = 1$, so that

$$|Q(y) - \tilde{P}(y)| = \delta > 0.$$

Choose a representation $\sum_{k=1}^{\infty} \lambda_k \phi_k^n$ of Q so that $\sum_{k=1}^{\infty} |\lambda_k| \|\phi_k\|^n \leq \|Q\|_N + \delta/2$, where each $\phi_k \in E'''$. Then

$$\delta = |Q(y) - \tilde{P}(y)| = \left| \sum_{k=1}^{\infty} \lambda_k \phi_k(y)^n - \sum_{k=1}^{\infty} \lambda_k y(\phi_k|_E)^n \right|$$
$$\leq \sum_{k=1}^{\infty} |\lambda_k| |\phi_k(y)^n - y(\phi_k|_E)^n|.$$

Since E is an M-ideal in E'' we have $\phi_k = \phi_k|_E + \phi_k^\perp$, with $\phi_k^\perp$ in $E^\perp$ and $\|\phi_k\| = \|\phi_k|_E\| + \|\phi_k^\perp\|$. Therefore,

$$\begin{aligned}
\delta &\leq \sum_{k=1}^{\infty} |\lambda_k| \left| \left(y(\phi_k|_E) + \phi_k^\perp(y)\right)^n - y(\phi_k|_E)^n \right| \\
&\leq \sum_{k=1}^{\infty} |\lambda_k| \left(\sum_{j=1}^{n} \binom{n}{j} \|\phi_k|_E\|^{n-j} \|\phi_k^\perp\|^j \right) \\
&= \sum_{k=1}^{\infty} |\lambda_k| \left[\left(\|\phi_k|_E\| + \|\phi_k^\perp\| \right)^n - \|\phi_k|_E\|^n \right] \\
&= \sum_{k=1}^{\infty} |\lambda_k| \|\phi_k\|^n - \sum_{k=1}^{\infty} |\lambda_k| \|\phi_k|_E\|^n \leq \|Q\|_N + \delta/2 - \|\tilde{P}\|_N.
\end{aligned}$$

This means we have that

$$\|Q\|_N - \|\tilde{P}\|_N \geq \delta/2 > 0$$

which completes the proof. □

In particular, for each $n \in \mathbb{N}$, each nuclear polynomial on c_0 has a unique nuclear norm preserving extension to ℓ_∞.

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