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SHARKOVSKII'S ORDER AND THE STABILITY OF PERIODIC POINTS OF MAPS OF THE INTERVAL

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Abstract

Let f be a C^r ($r \geq 0$) map from the interval [0, 1] into itself and m be a positive integer. This paper gives a sufficient and necessary condition under which the set of periodic points of period m disappears after a certain small C^r-perturbation.

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1. Introduction

In this paper, we denote by I the interval [0, 1], by $C^0(I, I)$ the space of all continuous maps from I into I, by $\mathbb{N}$ the set of positive integers and by $\lhd$ the Sharkovskii's order of $\mathbb{N}$:

$$\begin{array}{ccccc}
 & 3 \lhd & 5 \lhd \cdots \lhd & 2k+1 \lhd & 2k+3 \lhd \cdots \\
\cdots \lhd & 2\cdot 3 \lhd & 2\cdot 5 \lhd \cdots \lhd & 2(2k+1) \lhd & 2(2k+3) \lhd \cdots \\
\cdots \lhd & 2^2 3 \lhd & 2^2 5 \lhd \cdots \lhd & 2^2(2k+1) \lhd & 2^2(2k+3) \lhd \cdots \\
\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots \lhd & 2^n 3 \lhd & 2^n 5 \lhd \cdots \lhd & 2^n(2k+1) \lhd & 2^n(2k+3) \lhd \cdots \\
\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots
\end{array}$$

$$\cdots \lhd 2^k \lhd 2^{k-1} \lhd \cdots \lhd 2^3 \lhd 2^2 \lhd 2 \lhd 1.$$

For $f \in C^0(I, I)$ and $k \in \mathbb{N}$, f^k is the k-th iterate of f defined as $f^0 = \mathrm{id}$, $f^1 = f, \ldots, f^k = f \circ f^{k-1}$ successively, where id is the identity. For $p \in I$, if

$f^k(p) = p$ but $f^i(p) \neq p$ for any positive integer $i < k$, then p is called a periodic point of period k or k-periodic point of f. We denote by $P(f, k)$ the set of all k-periodic points of f and by $PP(f)$ the set $\{n \in \mathbb{N}; f$ has an n-periodic point$\}$. One of the most striking results in the theory of one dimensional dynamical systems is Sharkovskii's theorem.

THEOREM 1.1. *Let $f \in C^0(I, I)$. If $m \in PP(f)$ and $m \lhd n$, then $n \in PP(f)$.*

This theorem, however, gives no information about the stability of periodic points. On this problem, Block [B] obtained a remarkable result.

THEOREM 1.2. *Let $f \in C^0(I, I)$ and $m \in PP(f)$. Then there is a positive number ε_0 such that any map $g \in C^0(I, I)$ with $\max_{x \in I} |g(x) - f(x)| < \varepsilon_0$ has an n-periodic point if $m \lhd n$.*

The case $n = m$ is not involved in Block's theorem. In fact, the conclusion no longer holds in this case. If $f_r(x) = 1 - rx$, $x \in [0, 1]$, $r \in [0, 1]$, for example, then $PP(f_1) = \{1, 2\}$ but $PP(f_r) = \{1\}$ for $r \neq 1$ though f_r tends to f_1 on $[0, 1]$ uniformly as $r \to 1$.

Now, let $m \in \mathbb{N}$, $r \in \{0, \infty\} \cup \mathbb{N}$ and $f \in C^r(I, I)$. Suppose that $P(f, m) \neq \emptyset$ and let J be a component of $P(f, m)$. If there exist a sequence $\{f_n\} \subset C^r(I, I)$ and a relatively open subset V of I containing J such that $f_n \xrightarrow{C^r} f$ as $n \to \infty$ and $V \cap P(f_n, m) = \emptyset$ for all n, then we say that J is removable under C^r-perturbation. If there exists a sequence $\{f_n\} \subset C^r(I, I)$ such that $f_n \xrightarrow{C^r} f$ as $n \to \infty$ and $P(f_n, m) = \emptyset$ for all n, then we say that $P(f, m)$ is removable under C^r-perturbation. Here $f_n \xrightarrow{C^r} f$ as $n \to \infty$ means that, if $r < \infty$, then

$$\|f_n - f\|_{C^r(I)} = \max_{0 \leq i \leq r} \max_{x \in I} \left| f_n^{(i)}(x) - f^{(i)}(x) \right| \to 0 \quad \text{as } n \to \infty,$$

and if $r = \infty$, then for each $r' < \infty$,

$$\|f_n - f\|_{C^{r'}(I)} \to 0 \quad \text{as } n \to \infty.$$

For a subset J of I, we always use the term neighbourhood of J to indicate any relatively open subset of I containing J. This paper is to give a condition under which the set of periodic points of a certain period is removable. The main results of this paper are the following theorems.

THEOREM 1.3. *Let $m \in \mathbb{N}$, $r \in \{0, \infty\} \cup \mathbb{N}$, $f \in C^r(I, I)$ and let J be a component of $P(f, m)$. Then J is removable under C^r-perturbation if and only if: $J \neq [0, 1]$ and J has a neighborhood V_J such that either $f^m(x) - x \geq 0$ on V_J, or $f^m(x) - x \leq 0$ on V_J. Moreover, $f^m(x) - x \geq 0$ on V_J if $0 \in J$ and $f^m(x) - x \leq 0$ on V_J if $1 \in J$.*

It follows from this theorem that J is removable under C^r-perturbation if and only if J is removable under C^0-perturbation.

THEOREM 1.4. *Let $m \in \mathbb{N}$, $r \in \{0, \infty\} \cup \mathbb{N}$ and $f \in C^r(I, I)$. Then $P(f, m)$ is removable under C^r-perturbation if and only if every component of $P(f, m)$ is removable under C^0-perturbation.*

By Theorem 1.3 and Theorem 1.4, it is reasonable to replace the term *removable under C^r-perturbation* by the term *removable*.

2. Preparations

For simplicity, we denote by $\langle a, b\rangle$ any interval with endpoints a and b (a may be larger than b).

LEMMA 2.1. *Let $f \in C^0(I, I)$ and J be a component of $P(f, m)$. Then $J, f(J), \dots, f^{m-1}(J)$ are pairwise disjoint components of $P(f, m)$.*

PROOF. It is clear that $J, f(J), \dots, f^{m-1}(J)$ are components of $P(f, m)$ and $J = f^m(J)$. If $f^{m_1}(J) \cap f^{m_2}(J) \neq \emptyset$ for some non-negative integers m_1 and m_2 with $m_1 < m_2 < m$, then $J \cap f^{m_2-m_1}(J) \neq \emptyset$, and then $J = f^{m_2-m_1}(J)$, which implies that there is a point $p \in J$ such that $f^{m_2-m_1}(p) = p$. This contradicts $p \in J \subset P(f, m)$. $\square$

LEMMA 2.2. *Let $f \in C^0(I, I)$ and $J = \langle a, c\rangle$ ($a \neq c$) be a component of $P(f, m)$. If $c \notin P(f, m)$ then m must be even and*

(a) *$c \in P(f, m/2)$;*
(b) *$a \in P(f, m)$; and*
(c) *there is another component $J^* = \langle b, c\rangle$ of $P(f, m)$ such that $b \in P(f, m)$, $J^* \cap J = \emptyset$, $f^{m/2}(J^*) = J$ and $f^{m/2}(J) = J^*$.*

PROOF. Clearly $f^m(c) = c$, and then c is a periodic point of period m', m' is some positive integer with $m' < m$ and $m' \mid m$. By Lemma 2.1, $J^* = f^{m'}(J)$ is another component of $P(f, m)$ which does not join J, but J and J^* have the same endpoint c since $f^{m'}(c) = c$, and then we may assume $J^* = \langle b, c\rangle$. On the other hand, it follows from Lemma 2.1 again that $f^{m'}(J^*) \cap J^* = \emptyset$ and $f^{m'}(J^*)$ is a component of $P(f, m)$. Thus $f^{m'}(J^*) = J$, which implies $f^{2m'}(J) = J$ and $m' = m/2$, and then (a) is proved.

If $a \notin P(f, m)$, then we have $a \in P(f, m/2)$ again. Thus, considering that $f^{m/2}$ is one-to-one on J, we have $f^{m/2}(J) = J$, which is a contradiction. Thus $a \in P(f, m)$ and, for the same reason, $b \in P(f, m)$. Hence (b) and (c) hold. $\square$

LEMMA 2.3. *Let* $f \in C^0(I, I)$, *and* J *be a component of* $P(f, m)$. *If* a *is an endpoint of* J *and* $a \in P(f, m) \cap (0, 1)$, *then there is a sequence* $\{a_n\} \subset I$ *tending to* a *such that* $f^m(a_n) \neq a_n$, $n = 1, 2, \dots$.

PROOF. Without loss of generality, assume $J = [a, b]$ and $a \in (0, 1)$. Since $a \in P(f, m)$, the points $a, f(a), \dots, f^{m-1}(a)$ are pairwise distinct. Thus every point x near a is either an m-periodic point of f or $f^m(x) \neq x$. But J is a component of $P(f, m)$, and then for every $n \in \mathbb{N}$, there exists a point $a_n \in (a - 1/n, a)$ such that $f^m(a_n) \neq a_n$. This completes the proof. □

LEMMA 2.4. *Let* a *and* b *be real numbers with* $a < b$. *Then there exists a sequence* $\{h_n\} \subset C^\infty(R, I)$ *such that*

(a) $h_n(x) \equiv 0$ *for* $x \in R \setminus (a, b)$ *but* $h_n(x) > 0$ *on* (a, b);
(b) $a < x + h_n(x) < b$ *for* $x \in (a, b)$; *and*
(c) $\|h_n\|_{C^n(I)} = \|h_n\|_{C^n([a,b])} < 1/n$.

PROOF. Without loss of generality, assume $[a, b] = [-1, 1]$. Let

$$h_n(x) = \begin{cases} \dfrac{\lambda_n}{2} \exp\left\{\dfrac{1}{x^2 - 1}\right\}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where

$$\lambda_n = \left(n \left\| \exp\left\{\frac{1}{x^2 - 1}\right\} \right\|_{C^n([-1,1])}\right)^{-1}.$$

Then h_n has the desired properties. □

LEMMA 2.5. *Let* T *be a topological space. If* T *has a countable basis, then any open covering of* T *has a countable subcovering* [D].

3. Proofs of Theorem 1.3 and Theorem 1.4

By Lemma 2.2, it is easy to see that the sufficient and necessary condition in Theorem 1.3 is equivalent to the following condition (i) or condition (ii):

(i) J is a semi-open interval and has a neighborhood V_J such that either $f^m(x) - x \geq 0$ on V_J, or $f^m(x) - x \leq 0$ on V_J;

(ii) J is a closed interval, $J \neq [0, 1]$ and has a neighborhood V_J such that either $f^m(x) - x \geq 0$ on V_J, or $f^m(x) - x \leq 0$ on V_J; moreover, if $0 \in J$ then $f^m(x) - x \geq 0$ on V_J and if $1 \in J$ then $f^m(x) - x \leq 0$ on V_J.

PROOF OF THEOREM 1.3. To prove the necessity, we assume that J is a component of $P(f, m)$ that is removable under C^r-perturbation and show that J satisfies the above condition (i) or (ii). By Lemma 2.2, J is either a semi-open interval or a closed interval.

If $J = [0, 1]$, then $f = \mathrm{id}$. Since any map in $C^0(I, I)$ has at least one fixed point, as a component of $P(\mathrm{id}, m)$, J can not be removable under C^r-perturbation. Thus $J \neq [0, 1]$

If $J \neq [0, 1]$ and J does not satisfy (i) or (ii), then by Lemma 2.1 and Lemma 2.2 we can find two sequences $\{r_n\}$ and $\{r_n'\}$ contained in $[0, 1]$, such that each sequence converges to an endpoint of J, that $(f^m(r_n) - r_n)(f^m(r_n') - r_n') < 0$ and that $f^i([r_n, r_n'])$, $i = 0, 1, \ldots, m-1$, are pairwise disjoint. By continuity, for any $n \in \mathbb{N}$ and any $g \in C^0(I, I)$, if $\|g - f\|_{C^0(I)}$ is small enough, then g has at least one periodic point of period m in (r_n, r_n') (note that $f^l(x) - x \neq 0$ on $[r_n, r_n']$ for $l = 1, 2, \ldots m-1$), that is to say, J cannot be removable under C^0-perturbation. Thus J cannot be removable under C^r-perturbation, which is a contradiction.

To prove the sufficiency, we only prove Proposition 3.1 which clearly implies the sufficiency and will be used to prove Theorem 1.4. □

PROPOSITION 3.1. *Let* $m \in \mathbb{N}$, $r \in \{0, \infty\} \cup \mathbb{N}$, $f \in C^r(I, I)$ *and let* J *be a component of* $P(f, m)$ *satisfying* (i) *or* (ii). *Then there exist a sequence* $\{h_n\} \subset C^\infty(I, I)$ *and a neighbourhood* V_J *of* J *such that*

(a) $h_n |_{I \setminus V_J} \equiv 0$, $\|h_n\|_{C^n(I)} < 1/n$ *and* $h_n + \mathrm{id}$ *bijectively maps* V_J *onto a subset of* V_J, *where* id *is the identity*;

(b) *for* $f_n = f \circ (h_n + \mathrm{id})$, *if* J *satisfies* (i), *then* f_n^m *has only one fixed point* c *in* $\overline{V_J}$, *furthermore,* c *is an* $m/2$*-periodic point of both* f_n *and* f; *and if* J *satisfies* (ii), *then* f_n^m *has no fixed point on* $\overline{V_J}$; *and*

(c) $P(f_n^m, 1) \subset P(f^m, 1)$, $P(f_n, m) \subset P(f, m)$ *and each component* J' *of* $P(f_n, m)$ *is a component of* $P(f, m)$ *and satisfies* (i) *or* (ii) *for* f_n *if* J' *satisfies* (i) *or* (ii) *for* f.

PROOF. We first assume that J satisfies (i). Without loss of generality, we additionally assume $J = [a, c)$. Thus by Lemma 2.2, $f^{m/2}(c) = c$ and there exists a number $b > c$ such that $J_0 = (c, b]$ is another component of $P(f, m)$.

If $a \neq 0$, then by Lemma 2.3, there exists a point $a' \in [0, a)$ such that $f^m(a') - a' \neq 0$ and $\mathrm{sgn}\{(f^m(a') - a')(f^m(x) - x)\} \geq 0$, for all $x \in [a', c]$. Without loss of generality, assume

$$f^m(a') > a' \quad \text{and} \quad f^m(x) \geq x, \quad x \in [a', c]. \tag{3.1}$$

Since $a \in P(f, m)$, by Lemma 2.1, we can chose a' such that

$$[a', c) \cap f^i([a', c)) = \emptyset, \quad \text{for } i = 1, 2, \ldots, m-1. \tag{3.2}$$

It follows from Lemma 2.4 that there exists a sequence $\{h_n\} \subset C^\infty(I, I)$ such that

$$h_n(x) > 0 \quad \text{for } x \in (a', c), \qquad h_n(x) \equiv 0 \quad \text{for } x \in I \setminus (a', c), \qquad \|h_n\|_{C^n(I)} < 1/n,$$

and $h_n(x) + x$ bijectively maps $[a', c]$ onto $[a', c]$.

Setting $f_n(x) = f(x + h_n(x))$, $x \in I$, we have that $f_n \in C^r(I, I)$,

$$\text{(3.3)} \qquad \begin{aligned} &f_n(x) - f(x)\,|_{I\setminus(a',c)} \equiv 0, \qquad \text{and} \\ &f_n([a', c)) = f([a', c)). \end{aligned}$$

Thus by (3.2) we infer that

$$[a', c) \cap f_n^i([a', c)) = \emptyset, \quad i = 1, 2, \dots, m-1,$$

and then by (3.2) and (3.3), $f_n^i(x) = f^i(x)$ for $x \in f([a', c))$ $(= f_n([a, c)))$ and $i = 1, 2, \dots, m-1$, which implies

$$f_n^m(x) = f^{m-1}(f_n(x)) = f^m(x + h_n(x)), \quad x \in [a', c).$$

Hence it follows from (3.1) and the property of h_n that

$$f_n^m(x) = f^m(x + h_n(x)) \geq x + h_n(x) > x \quad \text{for} \quad x \in [a', c).$$

So, f_n^m has no fixed point in $[a', c)$.

It follows from (3.2) and (3.3) that f_n and f coincide on the set $\{c, f(c), \dots, f^{m/2-1}(c)\}$, and then $c \in P(f_n, m/2)$.

By Lemma 2.2, $f^{m/2}((c, b)) = (a, c)$, and then (3.2), (3.3), and Lemma 2.1 imply

$$f_n^{m/2}((c, b)) = f^{m/2}((c, b)) = (a, c),$$

which implies that f_n^m has no fixed point in (c, b) since we have proved that f_n^m has no fixed point in (a', c) $(\supset (a, c))$. Thus, putting $V_J = (a', (c+b)/2)$, we obtain (a) and (b), which imply (c).

If $a = 0$, the proof is just slightly different. By Lemma 2.4, for any $n \in \mathbb{N}$ there exists a map $h_n \in C^\infty(I, I)$ such that

$$h_n(x) > 0 \quad \text{for } x \in [0, c), \qquad h_n(x) \equiv 0 \quad \text{for } x \in [c, 1], \qquad \|h_n\|_{C^n(I)} < 1/n,$$

and $h_n(x) + x$ bijectively maps $[0, c]$ onto $[h(0), c]$.

Repeating the argument in the case $J = [a, c)$, one can prove that conditions (a)–(c) hold for $\{h_n\}$ and $V_J = [0, (c+b)/2)$.

Now we assume that $J = [a, b]$ is a closed interval and satisfies (ii). If $a \neq 0$ and $b \neq 1$, then by condition (ii), we can assume that there exist a point $a' \in [0, a)$ and a point $b' \in (b, 1]$ such that

$$\text{(3.4)} \qquad f^m(x) - x \geq 0, \quad x \in (a', b').$$

Since $J = [a, b]$ is a component of $P(f, m)$, it follows from (3.4), and Lemma 2.1 that a' and b' can be chosen so that

$$[a', b'] \cap f^i([a', b']) = \emptyset, \quad i = 1, 2, \dots, m-1. \tag{3.5}$$

By (3.4) and (3.5), and using similar argument as in the case $J = [a, c)$, we can construct a sequence $\{h_n\} \subset C^\infty(I, I)$ such that $V_J = (a', b')$ and $\{h_n\}$ satisfy (a)–(c).

If $a = 0$ or $b = 0$, similarly to the case of $J = [a, b]$ with $a \neq 0$ and $b \neq 1$ and to the case of $J = [a, c)$, we can prove the existence of h_n and V_J. So (a)–(c) hold in any case. This completes the proof. □

PROOF OF THEOREM 1.4. The necessity is obvious.

To prove the sufficiency, suppose that every component of $P(f, m)$ is removable under C^0-perturbation. Then by Theorem 1.3, every component of $P(f, m)$ satisfies condition (i) or (ii).

We first assume that $r < \infty$.

By Proposition 3.1, every component J of $P(f, m)$ has a neighbourhood V_J such that to any positive number ε corresponds a C^∞-map $h_\varepsilon \in C^\infty(I, I)$ such that $g_\varepsilon = f \circ h_\varepsilon$ satisfies the following (V_J, f, ε)-conditions:

(1) $(g_\varepsilon - f)|_{I\setminus V_J} \equiv 0$ and $\|g_\varepsilon - f\|_{C^r(I)} < \varepsilon$;

(2) $P(g_\varepsilon, m) \subset P(f, m)$ and any component J' of $P(g_\varepsilon, m)$ is also a component of $P(f, m)$ and satisfies (i) or (ii) for g_ε and $V_{J'}$; and

(3) If J satisfies (i), then g_ε^m has only one fixed point c_J in $\overline{V}_J$, which is an $m/2$-periodic point of both g_ε and f; if J satisfies (ii), then g_ε^m has no fixed point in $\overline{V}_J$.

The union of all V_Js is an open covering of $P(f, m)$. Since I has a countable basis, the covering has a countable subcovering according to Lemma 2.5. Thus there exists a sequence $\{J_i\}$ of components of $P(f, m)$, $i = 1, 2, \dots$, such that $\bigcup_{i=1}^\infty V_{J_i} \supset P(f, m)$.

Fix a positive number ε arbitrarily and put $\overline{V}_J = [a_J, b_J]$. Without loss of generality, assume that J_i and V_{J_i} satisfy condition (i) if i is odd and satisfy condition (ii) if i is even.

For $n \in \mathbb{N}$ and $k \in \mathbb{N}$ let

$$V_{J_{2k-1}}^n = V_{J_{2k-1}} \setminus \left(c_{J_{2k-1}} - 1/n, c_{J_{2k-1}} + 1/n\right) \quad \text{and} \quad V_{J_{2k}}^n = V_{J_{2k}}.$$

Then there exists a map $f_1 \in C^r(I, I)$ satisfying $(V_{J_1}, f, \varepsilon/2)$-conditions. By continuity, f_1 also satisfies the following condition (P_1):

(P_1) *There exists a positive number $\varepsilon_1 < \varepsilon/2$, such that for each $g \in C^r(I, I)$, g^m has no fixed point in $V_{J_1}^1$ if $\|f_1 - g\|_{C^r(I)} < \varepsilon_1$.*

For the same reason, there is a map $f_2 \in C^r(I, I)$ satisfying $(V_{J_2}, f_1, \varepsilon_1/2)$-conditions (if f_1 has no m-periodic point in V_{J_2} then define $f_2 = f_1$) and the following condition (P_2):

(P$_2$) *There exists a positive number* $\varepsilon_2 < \varepsilon_1/2$, *such that for each* $g \in C^r(I, I)$, g^m *has no fixed point in* $V^2_{J_1} \cup V^1_{J_2}$ *if* $\|f_2 - g\|_{C^r(I)} < \varepsilon_2$.

Now, repeating the same argument, we can successively obtain a sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive numbers and a sequence $\{f_n\}_{n=1}^\infty \subset C^r(I, I)$ such that $\varepsilon_{n+1} < \varepsilon_n/2$, $\|f_{n+1} - f_n\|_{C^r(I)} < \varepsilon_n/2$, f_{n+1} satisfies $(V_{J_{n+1}}, f_n, \varepsilon_n/2)$-conditions and

(P$_n$) *For each* $g \in C^r(I, I)$, g^m *has no fixed point in* $V^n_{J_1} \cup V^{n-1}_{J_2} \cup \cdots \cup V^1_{J_n}$ *if* $\|f_n - g\|_{C^r(I)} < \varepsilon_n$, $n = 1, 2, \ldots$.

Clearly, $f_n \xrightarrow{C^r} f^*$ for some $f^* \in C^r(I, I)$ and it is easy to see that $\|f^* - f\|_{C^r(I)} < \varepsilon$ and $\|f^* - f_n\|_{C^r(I)} < \varepsilon_n$, $n = 1, 2, \ldots$. Thus by condition (P$_n$), f^* has no m-periodic point in $V^n_{J_i} \cup V^{n-1}_{J_2} \cup \cdots \cup V^1_{J_n}$ for every $n \in \mathbb{N}$, which implies that f^* has no m-periodic point in $\bigcup_{n=1}^\infty \{V^n_{J_1} \cup V^{n-1}_{J_2} \cup \cdots \cup V^1_{J_n}\} = \left\{\bigcup_{n=1}^\infty V_{J_n}\right\} \setminus \{c_{J_{2k-1}} \mid k \in \mathbb{N}\}$. On the other hand, $(V_{J_{n+1}}, f_n, \varepsilon_n/2)$-conditions, $n = 1, 2, \ldots$, imply that

$$(f^* - f)|_{I \setminus \{\bigcup_{n=1}^\infty V_{J_n}\}} \equiv 0$$

and that $\{c_{J_{2k-1}} \mid k \in \mathbb{N}\} \subset P(f^*, m/2)$. Hence f^* has no periodic point of period m in $I \setminus \left\{\bigcup_{n=1}^\infty V_{J_n}\right\}$ and $\{c_{J_{2k-1}} \mid k \in \mathbb{N}\}$, and then f^* has no m-periodic point in I. Thus the sufficiency is proved in the case $r < \infty$.

If $r = \infty$, then the above g_ε is of class C^∞. Thus for each $k \in \mathbb{N}$, repeating the above arguments, we can construct a sequence $\{f_{nk}\} \subset C^\infty(I, I)$ such that $\|f - f_{1k}\|_{C^k(I)} < 1/(2k)$, $\|f_{nk} - f_{n+1,k}\|_{C^{n+k}(I)} < 1/(2^{n+1}k)$ for each n, $\lim_{n\to\infty} \|f_{nk} - f^*_k\|_{C^k(I)} = 0$ for some $f^*_k \in C^\infty(I, I)$ with $P(f^*_k, m) = \emptyset$. Clearly, $\|f - f^*_k\|_{C^k(I)} < 1/k$ and then $f^*_k \xrightarrow{C^\infty} f$ as $k \to \infty$. This completes the proof. $\square$

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