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A FAMILY OF STRONGLY SINGULAR OPERATORS

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Abstract

Let ψ be a positive function defined near the origin such that $\lim_{t\to 0^+}\psi(t) = 0$. We consider the operator $T_z f$, defined as the principal value of the convolution of a function f and a kernel $K(t) = e^{i\gamma(t)}t^{-z}/\psi(t)^{1-z}$, where z is a complex number, $0 \le \operatorname{Re}(z) \le 1$, $0 < t \le 1$ and γ is a real function. Assuming certain regularity conditions on ψ and γ and certain relations between ψ and γ we show that T_θ is a bounded operator on $L^p(\mathbb{R})$ for $1/p = (1+\theta)/2$ and $0 \le \theta < 1$, and T_1 is bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$.

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1. Introduction

Consider the following operator, defined for functions in $C_0^\infty(\mathbb{R})$

$$T_{\alpha\beta}f(x) = \lim_{\epsilon\to 0^+}\int_\epsilon^1 e^{it^{-\alpha}} f(x-t)\frac{dt}{t^\beta},$$

where $\alpha > 0$ and $\beta > 1$. Hirschman studied this operator and proved the following theorem in [2].

THEOREM 1.1. *Let $\alpha > 0$ and $\beta > 1$. Whenever $\alpha + 2 \ge 2\beta$ the following holds*:

(i) *$T_{\alpha\beta}$ extends to a bounded operator on $L^2(\mathbb{R})$.*

(ii) *If $|1/2 - 1/p| < 1/2 - (\beta-1)/\alpha$ then $T_{\alpha\beta}$ extends to a bounded operator on $L^p(\mathbb{R})$.*

(iii) *If* $|1/2 - 1/p| > 1/2 - (\beta - 1)/\alpha$ *then* $T_{\alpha\beta}$ *is not a bounded operator on* $L^p(\mathbb{R})$.

Fefferman and Stein considered the case $|1/2 - 1/p| = 1/2 - (\beta - 1)/\alpha$ in [1]. Their results complete Theorem 1.1.

THEOREM 1.2. *Let* $\alpha > 0$ *and* $\beta > 1$ *such that* $\alpha + 2 \geq 2\beta$. *If*

$$\left|\frac{1}{2} - \frac{1}{p}\right| \leq \frac{1}{2} - \frac{\beta - 1}{\alpha}$$

then $T_{\alpha\beta}$ *extends to a bounded operator on* $L^p(\mathbb{R})$, *for* $1 < p < \infty$.

In the present work we are interested in studying the operator when the singularity at zero is worse than a power. An example would be

$$Tf(x) = \lim_{\epsilon \to 0^+} \int_\epsilon^1 e^{i\gamma(t)} f(x - t) \frac{e^{1/t}}{t} dt, \tag{1}$$

where γ is a real-valued function.

To compensate for the singularity at the origin, the phase function γ should approach infinity fast as the argument tends to zero. For f in $C_0^\infty(\mathbb{R})$ we consider the operator

$$Tf(x) = \lim_{\epsilon \to 0^+} \int_\epsilon^1 e^{i\gamma(t)} f(x - t) \frac{dt}{\Phi(t)}, \tag{2}$$

where $\lim_{t \to 0^+} \Phi(t) = 0$ and $\lim_{t \to 0^+} \Phi'(t) = 0$.

The first thing to do is to understand under what conditions T is a bounded operator on $L^2(\mathbb{R})$.

When Φ approaches zero faster than a power, γ has to approach infinity faster than the reciprocal of a power. It is clear that when Φ is supported in the interval $[0, 1]$, the behavior of γ near zero should be dictated by that of Φ.

In the example given by equation (1) it turns out that if γ is such that $\gamma'(t) = e^{2/t}$, for example $\gamma(t) = -\int_t^1 e^{2/s} ds$, then T is bounded on $L^2(\mathbb{R})$.

If we now choose γ so that $\gamma''(t) = t^{2b-2} e^{2/t}$, for $b > 0$, T will not be bounded on $L^2(\mathbb{R})$.

For this choice of Φ and γ we have that

$$\lim_{t \to 0^+} \frac{1}{|\Phi(t)|\sqrt{|\gamma''(t)|}} = \lim_{t \to 0^+} t^{-b} = \infty.$$

Assuming some regularity conditions on γ and Φ we will show that if

$$\left|\frac{1}{\Phi(t)}\right| \leq C\sqrt{|\gamma''(t)|}, \tag{3}$$

then T is bounded on $L^2(\mathbb{R})$.

On the other hand if

$$\lim_{t\to 0^+} \frac{1}{|\Phi(t)|\sqrt{|\gamma''(t)|}} = \infty \tag{4}$$

then T will not be bounded on $L^2(\mathbb{R})$.

Notice that when $\Phi(t) = t^{\beta}$ and $\gamma(t) = t^{-\alpha}$ the above statements imply that $T_{\alpha\beta}$ is bounded on $L^2(\mathbb{R})$ only when $\alpha/2 + 1 - \beta \geq 0$, which is Hirschman's result for $p = 2$.

We now turn our attention to the study of T on $L^1(\mathbb{R})$. The only way T can be bounded on $L^1(\mathbb{R})$ is when the function $1/\Phi$ is integrable near zero. Since $1/\Phi(t) = t^{-1}$ just fails to be integrable near zero, there is some hope that

$$Tf(x) = \lim_{\epsilon\to 0^+} \int_{\epsilon}^{1} e^{i\gamma(t)} f(x-t)\frac{dt}{t}, \tag{5}$$

is a bounded operator from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$, due to the oscillatory factor $e^{i\gamma(t)}$.

Fefferman and Stein proved this statement when γ is the reciprocal of a power (see [1]). Theorem 1.2 follows as a corollary.

When γ approaches zero faster than the reciprocal of a power we will also have that T, as defined in (5), is a bounded operator from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$.

The above discussion leads us to consider a family of operators

$$T_z f(x) = \lim_{\epsilon\to 0^+} \int_{\epsilon}^{1} e^{i\gamma(t)} f(x-t)\frac{t^{-z}}{\psi(t)^{1-z}}dt, \tag{6}$$

where $z = a + ib$, $0 \leq a \leq 1$ and ψ is a positive function that satisfies

$$\frac{1}{\psi(t)} \leq C\sqrt{|\gamma''(t)|}.$$

In this setting one of our tasks is to prove that T_0 is a bounded operator on $L^2(\mathbb{R})$ and T_1 is a bounded operator from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$. To do so we will impose the following regularity conditions on γ and ψ.

1.1. Assumptions and results **(a.1)** We will assume that γ and γ' are monotone, guaranteeing the existence of the inverse of γ', denoted γ'^{-1}. Without loss of generality we will take $\gamma'(t) > 0$, with γ' decreasing on $(0, 1]$.

(a.2) As discussed before, for the operator to be bounded on $L^2(\mathbb{R})$ we will need to assume $1/\psi(t) \leq C\sqrt{|\gamma''(t)|}$, for $t > 0$ close to zero and C a constant. However to prove that the operator in bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$ we will assume the stronger condition

$$\left|\frac{\psi'(t)}{\psi(t)}\right| \leq \frac{1}{2}\left|\frac{\gamma'''(t)}{\gamma''(t)}\right|, \tag{7}$$

for $t > 0$ close to zero and C a constant. We will take $\psi \in C^2[0, 1]$ and $\gamma \in C^3(0, 1]$.

The fact that (7) implies that $1/\psi(t) \leq C\sqrt{|\gamma''(t)|}$ will be proved below.

(a.3) Several growth conditions on γ'' and γ' will be assumed. One of these is that γ'' is to be roughly constant where γ' 'doubles'. We write that as follows:

$$|\gamma''(\gamma'^{-1}(2s))| \leq C|\gamma''(\gamma'^{-1}(s))|,$$

where C is a constant bigger than one independent of s.

(a.4) We will assume that there exists a constant A and $\epsilon > 0$ with $A > 1 + \epsilon$ such that

$$\gamma'(t) \geq A\gamma'((1+\epsilon)t),$$

for $t > 0$ close to zero.

(a.5) The last condition on the growth of γ is the following. There exists a λ such that $1/2 < \lambda < 1$, and a constant C such that

$$|\gamma''(t)| \leq C\gamma'(t)^{2\lambda},$$

for $t > 0$ close to zero.

(a.6) Finally we will assume that ψ, ψ' and γ'' are monotone.

Unless otherwise noted, we will assume throughout that ψ and γ satisfy assumptions (a.1) through (a.6) in Theorem 1.3 through Theorem 1.5 below.

Examples: $\gamma'(t) = e^{1/t}, e^{e^{1/t}}, t^{-\alpha}, e^{(\ln(1/t))^\alpha}$ for $\alpha > 1$ satisfy (a.1) through (a.6).

Let us prove now that inequality (7) implies that $1/\psi(t) \leq C\sqrt{|\gamma''(t)|}$. For $0 < t < 1$ integrate both sides of (7) from t to 1.

$$\int_t^1 \left|\frac{\psi'(s)}{\psi(s)}\right| ds \leq \frac{1}{2}\int_t^1 \left|\frac{\gamma'''(s)}{\gamma''(s)}\right| ds.$$

Since ψ and γ'' are monotone we get

$$\ln\left(\frac{\psi(1)}{\psi(t)}\right) \leq \frac{1}{2}\ln\left(\frac{|\gamma''(t)|}{|\gamma''(1)|}\right),$$

and so $1/\psi(t) \leq C\sqrt{|\gamma''(t)|}$.

Assuming that the limit in (6) exists we now state the results on T_z. If $z = \theta$, for $0 \leq \theta \leq 1$, we have

$$T_\theta f(x) = \lim_{\epsilon \to 0^+} \int_\epsilon^1 e^{i\gamma(t)} f(x-t) \frac{t^{-\theta}}{\psi(t)^{1-\theta}} dt.$$

The main theorem for T_θ is the following.

THEOREM 1.3. (i) *T_θ is a bounded operator on $L^p(\mathbb{R})$ for $1/p = (1+\theta)/2$ and $0 \le \theta < 1$, with $\|T_\theta\|_{L^p \to L^p} \le A_\theta$ where A_θ depends only on θ.*
(ii) *T_1 is a bounded operator from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$.*

Theorem 1.3 will be obtained from the two following results.

THEOREM 1.4. *T_{ib} extends to a bounded operator on $L^2(\mathbb{R})$ with $\|T_{ib}\|_{L^2\to L^2} \le A_b$, where $A_b = O(|b|+1)$.*

THEOREM 1.5. *T_{1+ib} is a bounded operator from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$ with $\|T_{1+ib}\|_{H^1\to L^1} \le A_b$, where $A_b = O(|b|+1)$.*

Theorem 1.1 says that $T_{\alpha\beta}$ will not be bounded on $L^2(\mathbb{R})$ when $2+\alpha < 2\beta$. This result is generalized as follows.

THEOREM 1.6. *Suppose that γ, γ', γ'' and ψ are monotone and that*

$$\lim_{\epsilon\to 0^+} \frac{\gamma''(\epsilon)}{\gamma'^2(\epsilon)} = 0.$$

Also suppose there is a constant $A > 1$ such that

$$\gamma'(|\gamma|^{-1}(2s)) \le A\gamma'(|\gamma|^{-1}(s))$$

for all large $s > 0$. Then if $\lim_{t\to 0^+} 1/\left(|\psi(t)|\sqrt{|\gamma''(t)|}\right) = \infty$, T_0 is not bounded on $L^2(\mathbb{R})$.

In what follows C will denote a constant that may change from line to line.

1.2. Existence Let f be a function in $C_0^\infty(\mathbb{R})$. To see that under assumptions (a.1) through (a.6) the limit in (6) exists, we integrate by parts. For $0 < \epsilon' \le \epsilon$ we write

$$\begin{aligned}\int_{\epsilon'}^{\epsilon} e^{i\gamma(t)} f(x-t)\frac{t^{-z}}{\psi(t)^{1-z}}dt &= \frac{1}{i}\int_{\epsilon'}^{\epsilon}\frac{d}{dt}\left(e^{i\gamma(t)}\right)\frac{f(x-t)}{\gamma'(t)}\frac{t^{-z}}{\psi(t)^{1-z}}dt \\ &= \frac{1}{i}e^{i\gamma(t)}\frac{f(x-t)}{\gamma'(t)}\frac{t^{-z}}{\psi(t)^{1-z}}\bigg|_{\epsilon'}^{\epsilon} - \frac{1}{i}\int_{\epsilon'}^{\epsilon} e^{i\gamma(t)}\frac{d}{dt}\left(\frac{f(x-t)}{\gamma'(t)}\frac{t^{-z}}{\psi(t)^{1-z}}\right)dt \\ &= \mathrm{I}_\epsilon + \mathrm{II}_\epsilon .\end{aligned}$$

Let $z = a + ib$ with $0 \le a \le 1$. Then

$$|\,\mathrm{I}_\epsilon\,| \le \frac{|f(x-\epsilon')|\epsilon'^{-a}}{\gamma'(\epsilon')\psi(\epsilon')^{1-a}} + \frac{|f(x-\epsilon)|\epsilon^{-a}}{\gamma'(\epsilon)\psi(\epsilon)^{1-a}}.$$

Since $\lim_{t\to 0^+} \psi'(t) = 0$, we have that for t close to zero, $\psi(t) \le t$. Using this and the fact that $1/\psi(t) \le C\sqrt{|\gamma''(t)|}$ we have that

$$\frac{\epsilon^{-a}}{\gamma'(\epsilon)\psi(\epsilon)^{1-a}} \le C\frac{\sqrt{|\gamma''(\epsilon)|}}{\gamma'(\epsilon)}\epsilon^{-a}\psi(\epsilon)^a \le C\frac{\sqrt{|\gamma''(\epsilon)|}}{\gamma'(\epsilon)}.$$

Assumption (a.5) implies that $\lim_{\epsilon\to 0^+} \sqrt{|\gamma''(\epsilon)|}/\gamma'(\epsilon) = 0$. Hence

$$\lim_{\epsilon\to 0^+} |\, \mathrm{I}_\epsilon \,| = 0.$$

Let's now estimate II_ϵ.

$$\begin{aligned}
|\, \mathrm{II}_\epsilon \,| &\le \int_{\epsilon'}^{\epsilon} \frac{|f'(x-t)|t^{-a}}{\gamma'(t)\psi(t)^{1-a}}dt + \int_{\epsilon'}^{\epsilon} |z|\frac{|f(x-t)|t^{-a-1}}{\gamma'(t)\psi(t)^{1-a}}dt \\
&\quad + \int_{\epsilon'}^{\epsilon} \frac{|f(x-t)|t^{-a}|\gamma''(t)|}{\gamma'^2(t)\psi(t)^{1-a}}dt + \int_{\epsilon'}^{\epsilon} |z-1|\frac{|f(x-t)|t^{-a}|\psi'(t)|}{\gamma'(t)\psi(t)^{2-a}}dt \\
&= \mathrm{II}_{\epsilon 1} + \mathrm{II}_{\epsilon 2} + \mathrm{II}_{\epsilon 3} + \mathrm{II}_{\epsilon 4}\,.
\end{aligned}$$

As before we have that $\lim_{\epsilon\to 0^+} \epsilon^{-a}/\big(\gamma'(\epsilon)\psi(\epsilon)^{1-a}\big) = 0$ and hence

$$\lim_{\epsilon\to 0^+} |\, \mathrm{II}_{\epsilon 1} \,| = 0.$$

Since $1/\psi(t) \le C\sqrt{|\gamma''(t)|}$ we see that

$$\begin{aligned}
|\, \mathrm{II}_{\epsilon 2} \,| &\le \max_{0\le t\le 1} |f(x-t)||z| \int_{\epsilon'}^{\epsilon} \frac{t^{-a-1}}{\gamma'(t)\psi(t)^{1-a}}dt \\
&\le C \max_{0\le t\le 1} |f(x-t)||z| \int_{\epsilon'}^{\epsilon} \frac{\sqrt{|\gamma''(t)|}t^{-a-1}\psi(t)^a}{\gamma'(t)}dt.
\end{aligned}$$

Since $\lim_{t\to 0^+} \gamma''(t)/\gamma'(t)^2 = 0$, we have that $1/\gamma'(t) \le Ct$, for t small. Using this together with assumption (a.5), $\sqrt{|\gamma''(t)|} \le C\gamma'^{\lambda}(t)$ for some $1/2 < \lambda < 1$, and $\psi(t) \le t$ we see that

$$\begin{aligned}
|\, \mathrm{II}_{\epsilon 2} \,| &\le C \max_{0\le t\le 1} |f(x-t)||z| \int_{\epsilon'}^{\epsilon} \gamma'(t)^{\lambda-1}t^{-1}dt \\
&\le C \max_{0\le t\le 1} |f(x-t)||z| \int_{\epsilon'}^{\epsilon} t^{-\lambda}dt.
\end{aligned}$$

Hence

$$\lim_{\epsilon\to 0^+} |\, \mathrm{II}_{\epsilon 2} \,| = 0.$$

In a similar way we have

$$|\,\mathrm{II}_{\epsilon 3}\,| \leq C \max_{0\leq t\leq 1} |f(x-t)| \int_{\epsilon'}^{\epsilon} \frac{|\gamma''(t)|^{3/2}}{\gamma'^2(t)} dt.$$

If k' is such that $\gamma'^{-1}(2^{k'+1}) < \epsilon \leq \gamma'^{-1}(2^{k'})$ then we see that

$$\begin{aligned}
|\,\mathrm{II}_{\epsilon 3}\,| &\leq C \max_{0\leq t\leq 1} |f(x-t)| \sum_{k=k'}^{\infty} \int_{\gamma'^{-1}(2^{k+1})}^{\gamma'^{-1}(2^k)} \frac{|\gamma''(t)|}{\gamma'(t)} \frac{\sqrt{|\gamma''(t)|}}{\gamma'(t)} dt \\
&\leq C \max_{0\leq t\leq 1} |f(x-t)| \sum_{k=k'}^{\infty} \int_{\gamma'^{-1}(2^{k+1})}^{\gamma'^{-1}(2^k)} \frac{|\gamma''(t)|}{\gamma'(t)} \gamma'(t)^{\lambda-1} dt \\
&\leq C \max_{0\leq t\leq 1} |f(x-t)| \sum_{k=k'}^{\infty} (2^k)^{\lambda-1} \ln(2).
\end{aligned}$$

As $\epsilon \to 0^+$, $k' \to \infty$ and $\lambda - 1 < 0$ hence we can conclude that

$$\lim_{\epsilon\to 0^+} |\,\mathrm{II}_{\epsilon 3}\,| = 0.$$

To bound $\mathrm{II}_{\epsilon 4}$ just notice that

$$|\,\mathrm{II}_{\epsilon 4}\,| \leq C \max_{0\leq t\leq 1} |f(x-t)||z-1| \int_{\epsilon'}^{\epsilon} \frac{1}{\gamma'(t)} \frac{|\psi'(t)|}{\psi^2(t)} dt.$$

After an integration by parts, methods used before show that

$$\lim_{\epsilon\to 0^+} |\,\mathrm{II}_{\epsilon 3}\,| = 0.$$

This shows that the limit in (6) exists.

2. Preliminaries

Let $0 < \epsilon < 1$, and define

$$T_{\epsilon,z} f(x) = \int_{\epsilon}^{1} e^{i\gamma(t)} f(x-t) \frac{t^{-z}}{\psi(t)^{1-z}} dt$$

so that for f in $C_0^\infty(\mathbb{R})$ we have that $T_z f(x) = \lim_{\epsilon\to 0^+} T_{\epsilon,z} f(x)$. Let $K_{\epsilon,z}$ be such that $T_{\epsilon,z} f(x) = K_{\epsilon,z} * f(x)$. To prove the L^2-boundedness of T_{ib} as well as the L^1-boundedness of T_{1+ib} on $H^1(\mathbb{R})$ we need some estimations of $\widehat{K_{\epsilon,ib}}$ and $\widehat{K_{\epsilon,1+ib}}$, respectively. We devote this section to the proof of such estimates, which are contained in the following theorem.

THEOREM 2.1. *If $z = a + ib$ and $0 \le a \le 1$ then*

$$\left|\widehat{K_{\epsilon,z}(\xi)}\right| \le \frac{C(1+|b|)}{\left(\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\gamma'^{-1}(|\xi|)\right)^a}$$

if $|\xi|$ is large, and

$$\left|\widehat{K_{\epsilon,z}(\xi)}\right| \le C(1+|b|)$$

otherwise.

We start with a basic result on oscillatory integrals due to van der Corput.

LEMMA 2.1. *Suppose that ϕ is real-valued and smooth on (a, b), and that $|\phi^k(t)| \ge \lambda > 0$ for all $t \in (a, b)$. Then*

$$\left|\int_a^b e^{i\phi(t)}dt\right| \le C_k\lambda^{-1/k}$$

holds when:

(i) $k \ge 2$; *or*
(ii) $k = 1$ *and* $\phi'(t)$ *is monotonic.*

C_k depends only on k.

The proof of Lemma 2.1 can be found in [4].
We now state some propositions needed to prove Theorem 2.1.

PROPOSITION 2.1. *If $|\xi|$ is large, $0 < \epsilon < \gamma'^{-1}(|\xi|)$, and $0 \le a \le 1$ then*

$$\int_\epsilon^{\gamma'^{-1}(|\xi|)}\left(\frac{t^{-1-a}}{\gamma'(t)\psi(t)^{1-a}} + \frac{t^{-a}|\psi'(t)|}{\gamma'(t)\psi(t)^{2-a}}\right)dt \le \frac{C}{\left(\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\gamma'^{-1}(|\xi|)\right)^a}$$

where C is a constant independent of $|\xi|$ and ϵ.

PROPOSITION 2.2. *If $|\xi|$ is large and $0 \le a \le 1$ then*

$$\frac{1}{|\xi|}\int_{\gamma'^{-1}(|\xi|)}^1\left(\frac{|\psi'(t)|t^{-a}}{\psi(t)^{2-a}} + \frac{t^{-1-a}}{\psi(t)^{1-a}}\right)dt \le \frac{C}{\left(\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\gamma'^{-1}(|\xi|)\right)^a}$$

where C is a constant independent of $|\xi|$.

Before we proceed to the proof of these propositions the following remarks are in order.

REMARKS. (1) Given $u < v$ let $G(t) = \int_u^t e^{i(\gamma(s)-\xi s)}ds$ for $u < t < v$. Then $G'(t) = e^{i(\gamma(t)-\xi t)}$. If $\rho(s) = \gamma(s) - \xi s$ then for any $u < s < t$, $|\rho''(s)| = |\gamma''(s)| \geq |\gamma''(t)|$. Hence by Lemma 2.1 we have that $|G(t)| \leq C/\sqrt{|\gamma''(t)|}$.

(2) Since $\lim_{t\to 0^+} d\big(1/\gamma'(t)\big)/dt = 0$, for $t > 0$ close to zero $1/\gamma'(t) \leq Ct$.

(3) Since $\gamma'(x) \geq A\gamma'((1+\epsilon)x)$ for x small we have that $1/(\gamma'^{-1}(A^{k+1}|\xi|)) \leq (1+\epsilon)/(\gamma'^{-1}(A^k|\xi|))$, for $|\xi|$ large and k any positive integer.

(4) For $0 \leq a \leq 1$ and t small we have that $|\gamma''(t)|^{a/2} \leq C\gamma'(t)^{a\lambda} \leq C\gamma'^{1-\lambda(1-a)}(t)$. Hence $|\gamma''(\gamma'^{-1}(|\xi|))|^{a/2} \leq C|\xi|^{1-\lambda(1-a)}$ for $|\xi|$ large.

(5) If t is small then $1/t \leq \gamma'(t)$. Hence if $|\xi|$ is large we must have that $1/\gamma'^{-1}(|\xi|) \leq |\xi|$.

(6) For $|\xi|$ large, $1/(\psi(\gamma'^{-1}(|\xi|))) \leq C|\xi|$, since $1/\psi(t) \leq C\sqrt{|\gamma''(t)|} \leq C\gamma'(t)$.

(7) There exists a constant C such that for $t > 0$ close to zero, $1/t \leq C|\gamma''(t)|/\gamma'(t)$.

Remark 7 is a consequence of assumption (a.4):

$$\epsilon a(-\gamma''(a)) \geq \int_a^{(1+\epsilon)a} -\gamma''(t)dt = \gamma'(a) - \gamma'((1+\epsilon)a).$$

Since

$$-\gamma'((1+\epsilon)a) \geq -\frac{\gamma'(a)}{A}$$

we have that

$$|\gamma''(a)| \geq \frac{\gamma'(a) - \gamma'((1+\epsilon)a)}{\epsilon a} \geq \frac{\gamma'(a)}{a}\left(\frac{A-1}{A}\right)\frac{1}{\epsilon} > \frac{\gamma'(a)}{a}\frac{1}{A}$$

which is Remark 7.

Let us prove the propositions.

PROOF OF PROPOSITION 2.1. Since $1/\psi(t) \leq C\sqrt{|\gamma''(t)|} \leq C\gamma'(t)^\lambda$ we see that

$$\begin{aligned}
\int_\epsilon^{\gamma'^{-1}(|\xi|)} \frac{t^{-1-a}}{\gamma'(t)\psi(t)^{1-a}}dt &\leq C\sum_{k=0}^{\infty}\int_{\gamma'^{-1}(A^{k+1}|\xi|)}^{\gamma'^{-1}(A^k|\xi|)} \frac{t^{-1-a}\gamma'(t)^{\lambda(1-a)}}{\gamma'(t)}dt \\
&\leq C\sum_{k=0}^{\infty}\frac{\left(A^k|\xi|\right)^{\lambda(1-a)}}{\left(\gamma'^{-1}(A^{k+1}|\xi|)\right)^a A^k|\xi|}\ln\left(\frac{\gamma'^{-1}(A^k|\xi|)}{\gamma'^{-1}(A^{k+1}|\xi|)}\right) \\
&\leq C\sum_{k=0}^{\infty}\frac{\left(A^k|\xi|\right)^{\lambda(1-a)}}{\left(\gamma'^{-1}(A^{k+1}|\xi|)\right)^a A^k|\xi|},
\end{aligned}$$

since by Remark 3 we have that $\ln\left(\gamma'^{-1}(A^k|\xi|)/\gamma'^{-1}(A^{k+1}|\xi|)\right) \leq \ln(1+\epsilon)$.

Hence iterating Remark 3 k times we see that

$$\int_{\epsilon}^{\gamma'^{-1}(|\xi|)} \frac{t^{-1-a}}{\gamma'(t)\psi(t)^{1-a}} dt \leq C \frac{|\xi|^{\lambda(1-a)}}{\left(\gamma'^{-1}(|\xi|)\right)^{a} |\xi|} \sum_{k=0}^{\infty} \left(\frac{1+\epsilon}{A}\right)^{ka} A^{k(\lambda-1)}$$
$$\leq C\left(\gamma'^{-1}(|\xi|)\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\right)^{-a}.$$

The last inequality is due to Remark 4 and the facts that $1-\lambda > 0$ and $1+\epsilon < A$.

Using assumption (a.2) we have that

$$\int_{\epsilon}^{\gamma'^{-1}(|\xi|)} \frac{t^{-a}|\psi'(t)|}{\gamma'(t)\psi(t)^{2-a}} dt \leq C \sum_{k=0}^{\infty} \frac{\left(A^k|\xi|\right)^{\lambda(1-a)}}{\left(\gamma'^{-1}(A^{k+1}|\xi|)\right)^{a} A^k|\xi|} \int_{\gamma'^{-1}(A^{k+1}|\xi|)}^{\gamma'^{-1}(A^k|\xi|)} \left|\frac{\psi'(t)}{\psi(t)}\right| dt$$
$$\leq C \sum_{k=0}^{\infty} \frac{\left(A^k|\xi|\right)^{\lambda(1-a)}}{\left(\gamma'^{-1}(A^{k+1}|\xi|)\right)^{a} A^k|\xi|} \ln\left(\frac{|\gamma''(\gamma'^{-1}(A^{k+1}|\xi|))|}{|\gamma''(\gamma'^{-1}(A^k|\xi|))|}\right).$$

Using assumption (a.3) we see that there is a constant C independent of s so that

$$\left|\gamma''(\gamma'^{-1}(As))\right| \leq C \left|\gamma''(\gamma'^{-1}(s))\right|,$$

and hence

$$\ln\left(\frac{|\gamma''(\gamma'^{-1}(A^{k+1}|\xi|))|}{|\gamma''(\gamma'^{-1}(A^k|\xi|))|}\right) \leq C.$$

As before we have that

$$\sum_{k=0}^{\infty} \frac{\left(A^k|\xi|\right)^{\lambda(1-a)}}{\left(\gamma'^{-1}(A^{k+1}|\xi|)\right)^{a} A^k|\xi|} \leq C\left(\gamma'^{-1}(|\xi|)\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\right)^{-a},$$

and hence we have proved that

$$\int_{\epsilon}^{\gamma'^{-1}(|\xi|)} \left(\frac{t^{-1-a}}{\gamma'(t)\psi(t)^{1-a}} + \frac{t^{-a}|\psi'(t)|}{\gamma'(t)\psi(t)^{2-a}}\right) dt \leq \frac{C}{\left(\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\gamma'^{-1}(|\xi|)\right)^{a}}.$$

□

PROOF OF PROPOSITION 2.2. Since ψ is monotone and $1/\psi(t) \leq C\sqrt{|\gamma''(t)|}$ we have that

$$\frac{1}{|\xi|}\int_{\gamma'^{-1}(|\xi|)}^{1} \frac{|\psi'(t)|t^{-a}}{\psi(t)^{2-a}} dt \leq \frac{C}{|\xi|} \frac{\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}^{1-a}}{\left(\gamma'^{-1}(|\xi|)\right)^{a}} \ln\left(\frac{\psi(1)}{\psi(\gamma'^{-1}(|\xi|))}\right)$$
$$\leq C\left(\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\gamma'^{-1}(|\xi|)\right)^{-a}$$

since

$$\frac{\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\ln\left(\psi(1)/\psi(\gamma'^{-1}(|\xi|))\right)}{|\xi|} \le C\frac{|\xi|^{\lambda}\ln(|\xi|)}{|\xi|} \le C$$

if $|\xi|$ is large.

Similarly

$$\begin{aligned}\frac{1}{|\xi|}\int_{\gamma'^{-1}(|\xi|)}^{1}\frac{t^{-1-a}}{\psi(t)^{1-a}}dt &\le \frac{C}{|\xi|}\frac{\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}^{1-a}}{\left(\gamma'^{-1}(|\xi|)\right)^{a}}\ln\left(\frac{1}{\gamma'^{-1}(|\xi|)}\right)\\ &\le C\left(\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\gamma'^{-1}(|\xi|)\right)^{-a}\end{aligned}$$

since $\ln\left(1/\gamma'^{-1}(|\xi|)\right) \le C\ln(|\xi|)$.

Hence we have that

$$\frac{1}{|\xi|}\int_{\gamma'^{-1}(|\xi|)}^{1}\left(\frac{|\psi'(t)|t^{-a}}{\psi(t)^{2-a}}+\frac{t^{-1-a}}{\psi(t)^{1-a}}+\right)dt \le \frac{C}{\left(\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\gamma'^{-1}(|\xi|)\right)^{a}}.$$

□

We can now proceed to the proof of Theorem 2.1.

PROOF OF THEOREM 2.1. We need to estimate

$$\widehat{K_{\epsilon,z}}(\xi) = \int_{\epsilon}^{1} e^{i(\gamma(t)-\xi t)}\frac{t^{-a-ib}}{\psi(t)^{1-a-ib}}dt$$

where $z = a + ib$. In order to take advantage of the behavior of γ near zero we would like to do an integration by parts by writing in the notation of Remark 1

$$e^{i(\gamma(t)-\xi t)} = G'(t),$$

and then integrate by parts.

Case I: ξ is small so that $|\gamma'(t) - \xi| \ge C\gamma'(t)$, for some constant C.

For $\epsilon < t < 1$ we write $G(t) = \int_{\epsilon}^{t} e^{i(\gamma(s)-\xi s)}ds$. Since $|\rho'(s)| = |\gamma'(s) - \xi| \ge C\gamma'(s)$, using van der Corput's Lemma we see that $|G(t)| \le C/\gamma'(t)$.

We now have $\widehat{K_{\epsilon,z}}(\xi) = \int_{\epsilon}^{1} G'(t)t^{-a-ib}/\psi(t)^{1-a-ib}dt$. We integrate by parts to get $|\widehat{K_{\epsilon,z}}(\xi)| \le |\mathrm{I}| + |\mathrm{II}|$ where

$$|\mathrm{I}| \le \frac{C}{\gamma'(1)\psi(1)} \le C$$

For II we have that

$$\mathrm{II} = \int_\epsilon^1 G(t) \frac{d}{dt}\left(\frac{t^{-a-ib}}{\psi(t)^{1-a-ib}}\right) dt$$

and so

$$|\,\mathrm{II}\,| \le C(1+|b|) \int_\epsilon^1 \left[\frac{t^{-a-1}}{\gamma'(t)\psi(t)^{1-a}} + \frac{t^{-a}|\psi'(t)|}{\psi(t)^{2-a}\gamma'(t)}\right] dt.$$

Using Proposition 2.1 we see that $|\,\mathrm{II}\,| \le C(1+|b|)$.

Case II: $|\xi|$ is large.

Let $t_0 = \gamma'^{-1}(2|\xi|)$ and $t_1 = \gamma'^{-1}(|\xi|/2)$. We write

$$\begin{aligned}\widehat{K_{\epsilon,z}}(\xi) &= \int_\epsilon^{t_0} + \int_{t_0}^{t_1} + \int_{t_1}^1 e^{i(\gamma(t)-\xi t)} \frac{t^{-a-ib}}{\psi(t)^{1-a-ib}} dt \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

We now treat II, III, and I separately.

For $t_0 < t < t_1$, define G as in Remark 1, $G(t) = \int_{t_0}^t e^{i(\gamma(s)-\xi s)} ds$. Then we have $|G(t)| \le B/\sqrt{|\gamma''(t)|}$ and $\mathrm{II} = \int_{t_0}^{t_1} G'(t) t^{-a-ib}/\psi(t)^{1-a-ib} dt$. We integrate by parts to get

$$|\,\mathrm{II}\,| \le C(|\,\mathrm{II}_1\,| + |\,\mathrm{II}_2\,|),$$

where

$$|\,\mathrm{II}_1\,| \le \frac{C}{\left(\sqrt{|\gamma''(t_1)|}t_1\right)^a}$$

and

$$\mathrm{II}_2 = \int_{t_0}^{t_1} G(t) \frac{d}{dt}\left(\frac{t^{-a-ib}}{\psi(t)^{1-a-ib}}\right) dt.$$

So

$$|\,\mathrm{II}_2\,| \le C(1+|b|) \left(\int_{t_0}^{t_1} \frac{t^{-a}|\psi'(t)|}{\psi(t)^{2-a}\sqrt{|\gamma''(t)|}} dt + \int_{t_0}^{t_1} \frac{t^{-1-a}}{\psi(t)^{1-a}\sqrt{|\gamma''(t)|}} dt\right).$$

Since $1/\psi(t) \le C\sqrt{|\gamma''(t)|}$ we see that

$$\int_{t_0}^{t_1} \frac{t^{-1-a}}{\psi(t)^{1-a}\sqrt{|\gamma''(t)|}} dt \le C \frac{1}{\left(t_0\sqrt{|\gamma''(t_1)|}\right)^a} \ln\left(\frac{t_1}{t_0}\right).$$

Since

$$\frac{1}{\gamma'^{-1}(A|\xi|)} \leq \frac{1+\epsilon}{\gamma'^{-1}(|\xi|)},$$

if k is such that $2/A^k < 1$ we must have that

$$\frac{1}{\gamma'^{-1}(2|\xi|)} \leq \frac{(1+\epsilon)^k}{\gamma'^{-1}(2|\xi|/A^k)} \leq \frac{(1+\epsilon)^k}{\gamma'^{-1}(|\xi|)},$$

and so $\ln(t_1/t_0) \leq C$. Hence

$$\int_{t_0}^{t_1} \frac{t^{-1-a}}{\psi(t)^{1-a}\sqrt{|\gamma''(t)|}} dt \leq C \frac{1}{\left(t_0\sqrt{|\gamma''(t_1)|}\right)^a}.$$

Similarly, using assumptions (a.2) and (a.3) we get

$$\int_{t_0}^{t_1} \frac{|\psi'(t)|t^{-a}}{\psi(t)^{2-a}\sqrt{|\gamma''(t)|}} dt \leq \frac{1}{\left(t_0\sqrt{|\gamma''(t_0)|}\right)^a} \left|\ln\left(\frac{\gamma''(t_0)}{\gamma''(t_1)}\right)\right| \leq C\left(\frac{1}{\sqrt{|\gamma''(t_0)|}}\frac{1}{t_0}\right)^a.$$

Finally, since

$$\frac{1}{\gamma'^{-1}(2|\xi|)} \leq \frac{(1+\epsilon)^k}{\gamma'^{-1}(|\xi|)},$$

and

$$|\gamma''(\gamma'^{-1}(2s))| \leq C|\gamma''(\gamma'^{-1}(s))|$$

we get

$$|\,\mathrm{II}\,| \leq \frac{C(1+|b|)}{\left(\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\gamma'^{-1}(|\xi|)\right)^a}.$$

Let's now estimate

$$\mathrm{III} = \int_{t_1}^{1} e^{i(\gamma(t)-\xi t)} \frac{t^{-a-ib}}{\psi(t)^{1-a-ib}} dt.$$

For $t_1 \leq t \leq 1$ we define

$$G(t) = \int_{t_1}^{t} e^{i(\gamma(s)-\xi s)} ds.$$

For $t_1 \leq s \leq t$ we have that $|\gamma'(s)-\xi| \geq C|\xi|$ and $\gamma'(s)-\xi$ is monotone, hence by van der Corput's Lemma we have that $|G(t)| \leq C/|\xi|$.

We write

$$\mathrm{III} = \int_{t_1}^{1} G'(t)\frac{t^{-a-ib}}{\psi(t)^{1-a-ib}}dt$$

and after an integration by parts we see that $\mathrm{III} = \mathrm{III}_1 + \mathrm{III}_2$ with

$$|\mathrm{III}_1| \leq \frac{C}{|\xi|\psi(1)^{1-a}} \leq \frac{C}{|\xi|}.$$

Using Remark 4 we see that

$$|\mathrm{III}_1| \leq C\left(\gamma'^{-1}(|\xi|)\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\right)^{-a}.$$

Since for $t_1 \leq t \leq 1$, $|G(t)| \leq C/|\xi|$ we have that

$$|\mathrm{III}_2| \leq \frac{C(1+|b|)}{|\xi|}\int_{t_1}^{1}\left[\frac{t^{-1-a}}{\psi(t)^{1-a}} + \frac{|\psi'(t)|t^{-a}}{\psi(t)^{2-a}}\right]dt.$$

By Proposition 2.2 we have that

$$|\mathrm{III}_2| \leq C\left(\gamma'^{-1}(|\xi|)\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\right)^{-a}.$$

We still have to deal with

$$\mathrm{I} = \int_{\epsilon}^{t_0} e^{i(\gamma(t)-\xi t)}\frac{t^{-a-ib}}{\psi(t)^{1-a-ib}}dt.$$

For $\epsilon \leq t \leq t_0$ we define

$$G(t) = \int_{\epsilon}^{t} e^{i(\gamma(s)-\xi s)}ds.$$

In this case we have that $|\gamma'(s) - \xi| \geq C\gamma'(s)$ and hence we now have that $|G(t)| \leq C/\gamma'(t)$. We write

$$\mathrm{I} = \int_{\epsilon}^{t_0} G'(t)\frac{t^{-a-ib}}{\psi(t)^{1-a-ib}}dt$$

and integrate by parts to get

$$\begin{aligned}|\mathrm{I}| &\leq \frac{C\sqrt{|\gamma''(t_0)|}}{\gamma'(t_0)(\sqrt{|\gamma''(t_0)|}t_0)^a} + \left|\int_{\epsilon}^{t_0} G(t)\frac{d}{dt}\left(\frac{t^{-a-ib}}{\psi(t)^{1-a-ib}}\right)dt\right| \\ &= \mathrm{I}_1 + \mathrm{I}_2 .\end{aligned}$$

As before, we have that

$$\mathrm{I}_1 \leq \frac{C}{\left(\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\gamma'^{-1}(|\xi|)\right)^a}.$$

Since for $\epsilon \leq t \leq t_0$ we have that $|G(t)| \leq \frac{C}{\gamma'(t)}$, we see that

$$\mathrm{I}_2 \leq C(1+|b|)\int_\epsilon^{t_0}\left[\frac{t^{-1-a}}{\gamma'(t)\psi(t)^{1-a}} + \frac{|\psi'(t)|t^{-a}}{\gamma'(t)\psi(t)^{2-a}}\right]dt .$$

In view of Proposition 2.1 we can conclude that

$$\mathrm{I}_2 \leq C\left(\gamma'^{-1}(|\xi|)\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\right)^{-a}.$$

So for $|\xi|$ small

$$\left|\widehat{K_{\epsilon,z}}(\xi)\right| \leq C(1+|b|)$$

and for $|\xi|$ large

$$\left|\widehat{K_{\epsilon,z}}(\xi)\right| \leq \frac{C(1+|b|)}{\left(\sqrt{|\gamma''(\gamma'^{-1}|\xi|)|}\gamma'^{-1}(|\xi|)\right)^a}$$

and Theorem 2.1 is proved. $\square$

3. $L^p(\mathbb{R})$-boundedness of T_θ

In this section we want to prove the statement of the L^p-boundedness of T_θ, that is Theorem 1.3. We start with the proof of Theorem 1.4 and Theorem 1.5. Then using an interpolation argument we will be able to prove Theorem 1.3.

3.1. $L^2(\mathbb{R})$-boundedness of T_{ib} To prove the L^2-boundedness of T_{ib} it is enough to prove the following theorem.

THEOREM 3.1. *$\|T_{\epsilon,ib}f\|_{L^2} \leq C(1+|b|)\|f\|_{L^2}$ for every f in $C_0^\infty(\mathbb{R})$. The constant C is independent of ϵ.*

To prove this statement we just have to see that $|\widehat{K_{\epsilon,ib}}(\xi)| \leq C(1+|b|)$, where C is independent of ξ and ϵ. Since this is Theorem 2.1 when $a = 0$, there is nothing to prove. $\square$

However it is important to mention the following. To prove that T_{ib} is bounded on $L^2(\mathbb{R})$ it is enough to assume that $1/\psi(t) \leq C\sqrt{|\gamma''(t)|}$.

The stronger assumption (a.2)

$$\left|\frac{\psi'(t)}{\psi(t)}\right| \leq C\left|\frac{\gamma'''(t)}{\gamma''(t)}\right|,$$

was used twice during the proof of Theorem 2.1, to estimate a pair of integrals, namely

$$\int_{\epsilon}^{\gamma'^{-1}(|\xi|)} \frac{t^{-a}|\psi'(t)|}{\gamma'(t)\psi(t)^{2-a}}dt \quad \text{and} \quad \int_{t_0}^{t_1} \frac{t^{-a}|\psi'(t)|}{\sqrt{|\gamma''(t)|}\psi(t)^{2-a}}dt.$$

It can be easily seen that when $a = 0$ the desired estimate for these two integrals follows from the monotonicity of ψ and the weaker assumption $1/\psi(t) \leq C\sqrt{|\gamma''(t)|}$.

3.2. $L^1(\mathbb{R})$-boundedness of T_{1+ib} on $H^1(\mathbb{R})$ Again, to prove Theorem 1.5 it is enough to prove the following theorem.

THEOREM 3.2. *$\|T_{\epsilon,1+ib}f\|_{L^1} \leq C(1+|b|)\|f\|_{H^1}$ for every f in $H_1(\mathbb{R})$. The constant C is independent of ϵ.*

Since $K_{\epsilon,1+ib}$ is the kernel of $T_{\epsilon,1+ib}$ we have that

$$K_{\epsilon,1+ib}(x) = \begin{cases} \dfrac{e^{i\gamma(x)}}{x^{1+ib}\psi(x)^{-ib}} & \text{if } \epsilon \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.2 will be a consequence of the following Calderón-Zygmund type lemmas. The first one is Theorem 2.1 for $a = 1$. We restate it here for convenience.

LEMMA 3.1. *For ξ large $|\widehat{K_{\epsilon,1+ib}}(\xi)| \leq (C(1+|b|))/\left(\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\gamma'^{-1}(|\xi|)\right)$, and for ξ small $|\widehat{K_{\epsilon,1+ib}}(\xi)| \leq C(1+|b|)$. The constant C is independent of ϵ.*

LEMMA 3.2. *For L small and $|y| \leq L$*

$$\int_{|x|\geq 2\gamma'^{-1}(|y|^{-1})} \left|K_{\epsilon,1+ib}(x-y) - K_{\epsilon,1+ib}(x)\right|dx \leq C(1+|b|).$$

Let's assume for a moment Lemma 3.2 in order to prove Theorem 3.2.

PROOF OF THEOREM 3.2. It is enough to do it for atoms. So let $a(x)$ be an atom. Without loss of generality we can assume that a is supported in $(-L, L)$. Since a is an atom we have that $|a(x)| \leq 1/L$ and $\int a(x)dx = 0$.

Case I: $L \geq 1$. Using Schwarz's inequality we have

$$\begin{aligned}\int |T_{\epsilon,1+ib}a(x)|dx &\leq \int_{-L-1}^{L+1} |T_{\epsilon,1+ib}a(x)|dx \\ &\leq C(L+1)^{1/2}\left(\int |T_{\epsilon,1+ib}a(x)|^2 dx\right)^{1/2} \\ &\leq C(1+|b|)(L+1)^{1/2}\left(\int |a(x)|^2 dx\right)^{1/2}.\end{aligned}$$

The last inequality follows from the fact that $T_{\epsilon,1+ib}$ is bounded in $L^2(\mathbb{R})$, by Lemma 3.1.

Since $\int |a(x)|^2 dx \leq 1/L$ and $L \geq 1$ we get

$$\int |T_{\epsilon,1+ib}a(x)|dx \leq C(1+|b|)\left(\frac{L+1}{L}\right)^{1/2} \leq C(1+|b|).$$

Case II: $L < 1$. Let L be as in Lemma 3.2 and write

$$\begin{aligned}\int |T_{\epsilon,1+ib}a(x)|dx &= \int_{|x|\leq 2\gamma'^{-1}(1/L)} |T_{\epsilon,1+ib}a(x)|dx + \int_{|x|\geq 2\gamma'^{-1}(1/L)} |T_{\epsilon,1+ib}a(x)|dx \\ &= \mathrm{I} + \mathrm{II}.\end{aligned}$$

We will first estimate I. Again using Schwarz's inequality we have

$$\begin{aligned}\mathrm{I}^2 &\leq 2\gamma'^{-1}\left(\frac{1}{L}\right)\int |T_{\epsilon,1+ib}a(x)|^2 dx \\ &= 2\gamma'^{-1}\left(\frac{1}{L}\right)\int |\widehat{K_{\epsilon,1+ib}}(\xi)\widehat{a}(\xi)|^2 d\xi \\ &= 2\gamma'^{-1}\left(\frac{1}{L}\right)\left[\int_{|\xi|\leq 1} |\widehat{K_{\epsilon,1+ib}}(\xi)\widehat{a}(\xi)|^2 d\xi\right. \\ &\quad \left. + \int_{1\leq|\xi|\leq\frac{1}{L}} |\widehat{K_{\epsilon,1+ib}}(\xi)\widehat{a}(\xi)|^2 d\xi + \int_{|\xi|\geq\frac{1}{L}} |\widehat{K_{\epsilon,1+ib}}(\xi)\widehat{a}(\xi)|^2 d\xi\right] \\ &= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.\end{aligned}$$

Since $|\widehat{K_{\epsilon,1+ib}}(\xi)| \leq C(1+|b|)$ for ξ small and $|\widehat{a}(\xi)| \leq \|a\|_{L^1} \leq 1$ we have

$$\begin{aligned}|\mathrm{I}_1| &\leq 2\gamma'^{-1}\left(\frac{1}{L}\right) C(1+|b|)^2 \int_{|\xi|\leq 1} |\widehat{a}(\xi)|^2 d\xi \\ &\leq 2\gamma'^{-1}\left(\frac{1}{L}\right) C(1+|b|)^2 \leq C(1+|b|)^2.\end{aligned}$$

By Lemma 3.1 and the fact that $|\widehat{a}(\xi)| \leq 1$ we see that

$$|\,\mathrm{I}_2\,| \leq 2\gamma'^{-1}\left(\frac{1}{L}\right) C(1+|b|)^2 \int_{1\leq|\xi|\leq\frac{1}{L}} \frac{d\xi}{|\gamma''(\gamma'^{-1}(|\xi|))|(\gamma'^{-1}(|\xi|))^2}.$$

Make the change of variable $u = \gamma'^{-1}(\xi)$ to get

$$\begin{aligned}|\,\mathrm{I}_2\,| &\leq 2\gamma'^{-1}\left(\frac{1}{L}\right) C(1+|b|)^2 \int_{\gamma'^{-1}(1/L)\leq u\leq\gamma'^{-1}(1)} u^{-2}du \\ &\leq C(1+|b|)^2\gamma'^{-1}\left(\frac{1}{L}\right)\left[\frac{1}{\gamma'^{-1}(1)} + \frac{1}{\gamma'^{-1}(1/L)}\right] \leq C(1+|b|)^2.\end{aligned}$$

Since for ξ large

$$|\widehat{K_{\epsilon,1+ib}}(\xi)| \leq \frac{C(1+|b|)}{\sqrt{|\gamma''(\gamma'^{-1}(|\xi|))|}\gamma'^{-1}(|\xi|)}$$

and for t small $\gamma'(t)/t \leq C|\gamma''(t)|$, we see that for ξ large

$$|\widehat{K_{\epsilon,1+ib}}(\xi)|^2 \leq \frac{C(1+|b|)^2}{|\xi|\gamma'^{-1}(|\xi|)}.$$

This together with the fact that $\int |\widehat{a}(\xi)|^2 d\xi = \int |a(x)|^2 dx \leq 1/L$ gives us

$$\begin{aligned}|\,\mathrm{I}_3\,| &\leq 2\gamma'^{-1}\left(\frac{1}{L}\right)\sum_{k=0}^{\infty}\int_{\frac{A^k}{L}\leq|\xi|\leq\frac{A^{k+1}}{L}} |\widehat{K_{\epsilon,1+ib}}(\xi)\widehat{a}(\xi)|^2 d\xi \\ &\leq 2C(1+|b|)^2\gamma'^{-1}\left(\frac{1}{L}\right)\sum_{k=0}^{\infty}\frac{L}{A^k\gamma'^{-1}\,(A^{k+1}/L)}\int |\widehat{a}(\xi)|^2 d\xi \\ &\leq 2C(1+|b|)^2\gamma'^{-1}\left(\frac{1}{L}\right)\sum_{k=0}^{\infty}\frac{1}{A^k\gamma'^{-1}\,(A^{k+1}/L)}.\end{aligned}$$

Now by hypothesis we have that for x small $\gamma'(x) \geq A\gamma'((1+\epsilon)x)$. For t large let $x = \gamma'^{-1}(t)$. We then get

$$\begin{aligned} &\frac{t}{A} \geq \gamma'((1+\epsilon)\gamma'^{-1}(t)) \\ \text{if and only if}\quad &\gamma'^{-1}\left(\frac{t}{A}\right) \leq (1+\epsilon)\gamma'^{-1}(t) \\ \text{if and only if}\quad &\gamma'^{-1}(\xi) \leq (1+\epsilon)\gamma'^{-1}(A\xi).\end{aligned}$$

So finally we have

$$\begin{aligned}|\,\mathrm{I}_3\,| &\leq 2C(1+|b|)^2\gamma'^{-1}\left(\frac{1}{L}\right)\sum_{k=0}^{\infty}\frac{(1+\epsilon)^{k+1}}{A^k\gamma'^{-1}(1/L)} \\ &= 2C(1+|b|)^2(1+\epsilon)\sum_{k=0}^{\infty}\left(\frac{1+\epsilon}{A}\right)^k \leq C(1+|b|)\end{aligned}$$

since $1+\epsilon < A$.

So we do have that $|\mathrm{I}| \leq C(1+|b|)$. It remains to see that $|\mathrm{II}| \leq C(1+|b|)$, where $\mathrm{II} = \int_{|x|\geq 2\gamma'^{-1}(1/L)} |T_{\epsilon,1+ib}a(x)|dx$.

Since $\int a(x)dx = 0$ we have

$$T_{\epsilon,1+ib}a(x) = \int K_{\epsilon,1+ib}(x-y)a(y)dy = \int \left[K_{\epsilon,1+ib}(x-y) - K_{\epsilon,1+ib}(x)\right]a(y)dy.$$

Hence

$$\begin{aligned}
|\mathrm{II}| &\leq \int_{|x|\geq 2\gamma'^{-1}(1/L)} \left| \int [K_{\epsilon,1+ib}(x-y) - K_{\epsilon,1+ib}(x)]a(y)dy \right| dx \\
&\leq \int |a(y)|dy \int_{|x|\geq 2\gamma'^{-1}(1/L)} |K_{\epsilon,1+ib}(x-y) - K_{\epsilon,1+ib}(x)|dx \\
&\leq \int |a(y)|dy \int_{|x|\geq 2\gamma'^{-1}(|y|^{-1})} |K_{\epsilon,1+ib}(x-y) - K_{\epsilon,1+ib}(x)|dx \\
&\leq C(1+|b|) \int |a(y)|dy \leq C(1+|b|).
\end{aligned}$$

The next to last inequality is due to Lemma 3.2.

Altogether we have that $\int |T_{\epsilon,1+ib}a(x)|dx \leq C(1+|b|)$ and Theorem 3.2 is proved. $\square$

Let's now prove Lemma 3.2.

PROOF OF LEMMA 3.2. Recall that

$$K_{\epsilon,1+ib}(x) = \begin{cases} \dfrac{e^{i\gamma(x)}}{x^{1+ib}\psi(x)^{-ib}} & \text{if } \epsilon \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $0 \leq t \leq 1$ let $f(t) = K_{\epsilon,1+ib}(x-ty)$. Then

$$K_{\epsilon,1+ib}(x-y) - K_{\epsilon,1+ib}(x) = \int_0^1 f'(t)dt = \int_0^1 -yK'_{\epsilon,1+ib}(x-ty)dt.$$

So

$$\begin{aligned}
|K_{\epsilon,1+ib}(x-y) - K_{\epsilon,1+ib}(x)| \leq{}& |y|(1+|b|) \int_0^1 \left| \frac{\gamma'(x-ty)}{x-ty} \right| dt \\
&+ |y|(1+|b|) \int_0^1 \frac{1}{|x-ty|^2} dt \\
&+ |y|(1+|b|) \int_0^1 \left| \frac{\psi'(x-ty)}{\psi(x-ty)} \right| \frac{1}{|x-ty|} dt.
\end{aligned}$$

And hence

$$\begin{aligned}\int \left|K_{\epsilon,1+ib}(x-y) - K_{\epsilon,1+ib}(x)\right|dx &\le |y|(1+|b|)\iint_0^1 \left|\frac{\gamma'(x-ty)}{x-ty}\right| dtdx + |y|(1+|b|)\iint_0^1 \frac{1}{|x-ty|^2}dtdx \\ &+ |y|(1+|b|)\iint_0^1 \left|\frac{\psi'(x-ty)}{\psi(x-ty)}\right| \frac{1}{|x-ty|}dtdx.\end{aligned}$$

Making the change of variable $z = x - ty$ and interchanging the order of integration we get

$$\begin{aligned}\int_{|x|\ge 2\gamma'^{-1}(|y|^{-1})} &|K_{\epsilon,1+ib}(x-y) - K_{\epsilon,1+ib}(x)|dx \\ &\le |y|(1+|b|)\int_{1\ge|z|\ge 2\gamma'^{-1}(|y|^{-1})-|y|} \gamma'(z)\frac{dz}{|z|} + |y|(1+|b|)\int_{1\ge|z|\ge 2\gamma'^{-1}(|y|^{-1})-|y|} \frac{dz}{|z|^2} \\ &+ |y|(1+|b|)\int_{1\ge|z|\ge 2\gamma'^{-1}(|y|^{-1})-|y|} \left|\frac{\psi'(z)}{z\psi(z)}\right| dz = \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

For L small and $|y| \le L$ we have that $1/|y| \le \gamma'(|y|)$. Hence $|y|/\gamma'^{-1}(|y|^{-1}) \le 1$ and $2\gamma'^{-1}\left(|y|^{-1}\right) - |y| \ge \gamma'^{-1}\left(|y|^{-1}\right)$. So

$$\mathrm{II} \le C|y|(1+|b|)\left(\frac{1}{2} + \frac{1}{2\gamma'^{-1}\left(|y|^{-1}\right) - |y|}\right) \le C(1+|b|).$$

Since $\gamma'(x)/x \le C|\gamma''(x)|$ we see that

$$\begin{aligned}\mathrm{I} &\le |y|(1+|b|)\int_{1\ge|z|\ge 2\gamma'^{-1}(|y|^{-1})-|y|} (-\gamma''(z))dz \\ &\le |y|(1+|b|)\left[\gamma'\left(2\gamma'^{-1}\left(|y|^{-1}\right) - |y|\right) + \gamma'(1)\right].\end{aligned}$$

Since $2\gamma'^{-1}\left(|y|^{-1}\right) - |y| \ge \gamma'^{-1}\left(|y|^{-1}\right)$ we have that $\gamma'\left(2\gamma'^{-1}\left(|y|^{-1}\right) - |y|\right) \le |y|^{-1}$.
So $\mathrm{I} \le C(1+|b|)$. To estimate III it is sufficient to note that

$$\left|\frac{\psi'(z)}{z\psi(z)}\right| \le C\frac{1}{|z\psi(z)|} \le C\frac{\sqrt{|\gamma''(z)|}}{|z|} \le C\frac{\gamma'(z)}{z}.$$

So $\mathrm{III} \le C\,\mathrm{I} \le C(1+|b|)$. Altogether we have

$$\int_{|x|\ge 2\gamma'^{-1}(|y|^{-1})} |K_{\epsilon,1+ib}(x-y) - K_{\epsilon,1+ib}(x)|dx \le C(1+|b|),$$

for $|y| < L$ and Lemma 3.2 is proved. □

3.3. $L^p(\mathbb{R})$-boundedness of T_θ To prove Theorem 1.3 we will need a theorem on interpolation of analytic families of operators. Here we will formulate the version needed to prove the desired L^p-estimate. The proof of this particular case can be found in [1]. For the general version see [5].

Let S be the open strip of complex numbers z such that $0 < \mathrm{Re}(z) < 1$. Consider the mapping taking z to T_z from the closure of S to bounded operators on $L^2(\mathbb{R})$. Suppose this mapping is analytic in S and continuous and bounded in the closure of S. Then we have the following theorem.

THEOREM 3.3. *Suppose $\|T_{iy}f\|_{L^1} \le M_0(y)\|f\|_{H^1}$ for $f \in L^2(\mathbb{R}) \cap H^1(\mathbb{R})$ and $\|T_{1+iy}f\|_{L^2} \le M_1(y)\|f\|_{L^2}$ for $f \in L^2(\mathbb{R})$, where $M_i(y) \le A_i(1+|y|)^N$ for some N, and $i = 0, 1$. Then $\|T_t f\|_{L^p} \le M_t\|f\|_{L^p}$ for $f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ whenever $0 < t < 1$ and $1/p = 1 - t/2$. M_t depends only on t, A_0 and A_1.*

Taking $\theta = 1 - t$ in Theorem 3.3 we see that

$$T_\theta f(x) = \lim_{\epsilon \to 0^+} \int_\epsilon^1 e^{i\gamma(t)} f(x-t) \frac{t^{-\theta}}{\psi(t)^{1-\theta}} dt$$

is bounded on $L^p(\mathbb{R})$ for $1/p = (1+\theta)/2$, $0 \le \theta < 1$. This concludes the proof of the L^p-boundedness of T_θ. $\square$

4. Sharp $L^2(\mathbb{R})$ result

The purpose of this section is to prove that if

$$\lim_{t \to 0^+} \frac{1}{|\gamma''(t)|\psi^2(t)} = \infty$$

and if there is a constant $A \ge 1$ such that for all large $s > 0$

$$\gamma'(|\gamma|^{-1}(2s)) \le A\gamma'(|\gamma|^{-1}(s))$$

then T_0 cannot be bounded on $L^2(\mathbb{R})$.

In order to prove this we will need to find a lower bound for $|T_0 f(x)|$ for certain points x and an appropriate function f in $L^2(\mathbb{R})$. These points x will be such that for t close to x, the oscillation $e^{i\gamma(t)}$ will not vary too much. Those points will lie in the intervals built in the following lemma.

LEMMA 4.1. *Suppose there is a constant $A \ge 1$ such that for all large $s > 0$*

$$\gamma'(|\gamma|^{-1}(2s)) \le A\gamma'(|\gamma|^{-1}(s)).$$

Then there is a constant B_0 such that whenever $\epsilon \le B_0 \le 1$ and $k \le |\gamma|(\epsilon)$, the following is true:

(i)

$$\frac{B}{\gamma'(\epsilon)} \leq |\gamma|^{-1}\left(2k\pi - \frac{\pi}{3}\right) - |\gamma|^{-1}\left(2k\pi + \frac{\pi}{3}\right),$$

where $B = 2\pi/(3A^3)$;

(ii) *Let* $I_k = \left[|\gamma|^{-1}(2k\pi + \pi/3) + B/(3\gamma'(\epsilon)), |\gamma|^{-1}(2k\pi - \pi/3) - B/(3\gamma'(\epsilon))\right]$ *and* $J_k = \left[|\gamma|^{-1}(2(k+1)\pi - \pi/3) - B/(3\gamma'(\epsilon)), |\gamma|^{-1}(2k\pi + \pi/3) + B/(3\gamma'(\epsilon))\right]$. *If* k' *is such that* $k' \leq |\gamma|(\epsilon) < k' + 1$ *then*

$$(\epsilon, |\gamma|^{-1}(7)) \subseteq I_{k'} \cup \left[\bigcup_{k=1}^{k'-1} J_k \cup I_k\right].$$

(iii) *There is a constant* D *such that* $|J_k| \leq D|I_{k+1}|$.

The construction of these intervals when the phase function is the inverse of a power can be found in [3].

We will also need the following fact. If γ is such that $\gamma'(t) \geq 0$ for $t > 0$, γ' is decreasing, $\lim_{\epsilon\to 0^+} \gamma'(\epsilon) = \infty$ and $\lim_{\epsilon\to 0^+} \gamma''(\epsilon)/\gamma'(\epsilon)^2 = 0$, then

$$\lim_{\epsilon\to 0^+} \gamma'(\epsilon + 1/\gamma'(\epsilon))/\gamma'(\epsilon) = 1.$$

To see this we write

$$\begin{aligned}\gamma'\left(\epsilon + \frac{1}{\gamma'(\epsilon)}\right) &= \gamma'\left(\epsilon + \frac{1}{\gamma'(\epsilon)}\right) + \gamma'(\epsilon) - \gamma'(\epsilon)\\ &= \gamma'(\epsilon) + \int_\epsilon^{\epsilon + 1/\gamma'(\epsilon)} \gamma''(t)\,dt.\end{aligned}$$

Since

$$\left|\int_\epsilon^{\epsilon+1/\gamma'(\epsilon)} \gamma''(t)\,dt\right| \leq \frac{|\gamma''(\epsilon)|}{\gamma'(\epsilon)}$$

and $\lim_{\epsilon\to 0^+} \gamma''(\epsilon)/\gamma'(\epsilon)^2 = 0$ we get

$$\lim_{\epsilon\to 0^+} \frac{\gamma'(\epsilon + 1/\gamma'(\epsilon))}{\gamma'(\epsilon)} = 1.$$

Let's assume Lemma 4.1 and prove Theorem 1.6.

PROOF OF THEOREM 1.6. We are going to argue by contradiction. Suppose that T_0 is bounded on $L^2(\mathbb{R})$ and that $\lim_{t\to 0^+} 1/\left(|\psi(t)|\sqrt{|\gamma''(t)|}\right) = \infty$. Then there exists a positive, decreasing function g such that $1/\psi^2(t) \geq |\gamma''(t)|g(t)$ and $\lim_{t\to 0^+} g(t) = \infty$.

Let $f(x) = 1$ for $x \in (-1, 1)$ and 0 otherwise so that $\int |f(x)| dx = 2$ and $\int |f(x)|^2 dx = 2$. Let ϵ and B be the same as in Lemma 4.1 and define

$$f_\epsilon(x) = f\left(\frac{x}{B/(3\gamma'(\epsilon))}\right).$$

In the notation of Lemma 4.1 let $x \in I_k$. If $x \in I_k$ and $|x - t| \leq B/(3\gamma'(\epsilon))$ then

$$2k\pi - \frac{\pi}{3} \leq |\gamma|(t) \leq 2k\pi + \frac{\pi}{3}.$$

Hence for $0 \leq \epsilon' \leq \epsilon$ and $x \in I_k$ we have

$$\begin{aligned} \left|\int_{\epsilon'}^{1} e^{i\gamma(t)} f_\epsilon(x-t)\frac{dt}{\psi(t)}\right| &\geq \frac{1}{2}\int_{\epsilon'}^{1} f_\epsilon(x-t)\frac{dt}{\psi(t)} \\ &\geq \frac{1}{2}\int_{\epsilon'}^{1} f_\epsilon(x-t)\frac{dt}{\psi\,(x + B/(3\gamma'(\epsilon)))}. \end{aligned}$$

The last inequality is due to the fact that since

$$t = |t| \leq |x - t| + x \leq \frac{B}{3\gamma'(\epsilon)} + x$$

and ψ is monotone then $\psi(t) \leq \psi\,(B/(3\gamma'(\epsilon)) + x)$.

So we have

$$\begin{aligned} \left|\int_{\epsilon'}^{1} e^{i\gamma(t)} f_\epsilon(x-t)\frac{dt}{\psi(t)}\right| &\geq \frac{1}{2}\frac{1}{\psi\,(x + B/(3\gamma'(\epsilon)))}\int f_\epsilon(x-t)dt \\ &= \frac{B}{6\psi\,(x + B/(3\gamma'(\epsilon)))\,\gamma'(\epsilon)}. \end{aligned}$$

Thus for $x \in I_k$ we have

$$|T_0 f_\epsilon(x)| \geq \frac{C}{\gamma'(\epsilon)\psi\,(B/(3\gamma'(\epsilon)) + x)}.$$

Hence we get

$$\int_{I_k} |T_0 f_\epsilon(x)|^2 dx \geq \frac{C}{\gamma'(\epsilon)^2}\int_{I_k}\frac{dx}{\psi\,(B/(3\gamma'(\epsilon)) + x)^2}.$$

And so we see that

$$\int |T_0 f_\epsilon(x)|^2 dx \geq \frac{C}{\gamma'(\epsilon)^2}\sum_{k=1}^{k'}\int_{I_k}\frac{dx}{\psi\,(B/(3\gamma'(\epsilon)) + x)^2}.$$

For any $1 \le l \le k'-1$, we have $|J_l| \le D|I_{l+1}|$. Since ψ is monotone and I_{l+1} lies to the left of J_l we have that $\max_{J_l} 1/\psi(t) \le \min_{I_{l+1}} 1/\psi(t)$ and hence

$$\int_{J_l} \frac{dx}{\psi\,(B/(3\gamma'(\epsilon))+x)^2} \le D \int_{I_{l+1}} \frac{dx}{\psi\,(B/(3\gamma'(\epsilon))+x)^2}.$$

This implies that

$$\begin{aligned}\int |T_0 f_\epsilon(x)|^2 dx &\ge \frac{C}{\gamma'(\epsilon)^2}\left(\int_{I_{k'}} \frac{dx}{\psi(B/(3\gamma'(\epsilon))+x)^2}\right.\\ &\qquad \left.+ \int_{\bigcup_{k=1}^{k'-1}(I_k\cup J_k)} \frac{dx}{\psi(B/(3\gamma'(\epsilon))+x)^2}\right)\\ &\ge \frac{C}{\gamma'(\epsilon)^2}\int_\epsilon^{|\gamma|^{-1}(7)} \frac{dx}{\psi(B/(3\gamma'(\epsilon))+x)^2}.\end{aligned}$$

For $\epsilon \le \delta \le |\gamma|^{-1}(7)$ we have

$$\int |T_0 f_\epsilon(x)|^2 dx \ge \frac{C}{\gamma'(\epsilon)^2}\int_\epsilon^{\delta} \frac{dx}{\psi(B/(3\gamma'(\epsilon))+x)^2}.$$

Since we are assuming that $T_0 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ we have

$$\int |T_0 f_\epsilon|^2(x)dx \le \|T_0\|^2_{L^2\to L^2}\|f_\epsilon\|_2^2 \le \|T_0\|^2_{L^2\to L^2}\frac{B}{3\gamma'(\epsilon)}.$$

So finally we have

$$\|T_0\|^2_{L^2\to L^2}\frac{B}{3\gamma'(\epsilon)} \ge \frac{C}{\gamma'(\epsilon)^2}\int_\epsilon^{\delta} \frac{dx}{\psi(B/(3\gamma'(\epsilon))+x)^2}$$

or

$$\begin{aligned}(8)\qquad 1 &\ge \frac{C}{\gamma'(\epsilon)}\int_\epsilon^\delta \frac{dx}{\psi\,(B/(3\gamma'(\epsilon))+x)^2}\\ &\ge \frac{C}{\gamma'(\epsilon)}\int_\epsilon^\delta -\gamma''\left(\frac{B}{3\gamma'(\epsilon)}+x\right) g\left(\frac{B}{3\gamma'(\epsilon)}+x\right)dx\\ &\ge \frac{C}{\gamma'(\epsilon)} g\left(\frac{B}{3\gamma'(\epsilon)}+\delta\right)\int_\epsilon^\delta -\gamma''\left(\frac{B}{3\gamma'(\epsilon)}+x\right)dx\\ &= \frac{C}{\gamma'(\epsilon)} g\left(\frac{B}{3\gamma'(\epsilon)}+\delta\right)\left[\gamma'\left(\frac{B}{3\gamma'(\epsilon)}+\epsilon\right)-\gamma'\left(\frac{B}{3\gamma'(\epsilon)}+\delta\right)\right].\end{aligned}$$

Now, $g(B/(3\gamma'(\epsilon))+\delta) \ge g(2\delta)$ by also requiring $\delta > B/(3\gamma'(\epsilon))$.

The proof that

$$\lim_{\epsilon\to 0^+} \frac{\gamma'\big(\epsilon + 1/\gamma'(\epsilon)\big)}{\gamma'(\epsilon)} = 1$$

gives us that

$$\lim_{\epsilon\to 0^+} \frac{\gamma'\big(\epsilon + B/(3\gamma'(\epsilon))\big)}{\gamma'(\epsilon)} = 1$$

and hence we have that

$$\frac{1}{\gamma'(\epsilon)}\left[\gamma'\left(\epsilon + \frac{B}{3\gamma'(\epsilon)}\right) - \gamma'\left(\delta + \frac{B}{3\gamma'(\epsilon)}\right)\right] \to 1, \quad \text{as } \epsilon \to 0^+.$$

So letting $\epsilon \to 0^+$ in inequality (8) we get $1 \geq Cg(2\delta)$ and letting $\delta \to 0^+$ we have a contradiction. This concludes the proof of Theorem 1.6. □

Now let us prove Lemma 4.1.

PROOF OF LEMMA 4.1. (i) Using the mean value theorem we have:

$$|\gamma|^{-1}\left(2k\pi - \frac{\pi}{3}\right) - |\gamma|^{-1}\left(2k\pi + \frac{\pi}{3}\right) = \left(-\frac{2\pi}{3}\right)(|\gamma|^{-1})'(d) = \frac{2\pi}{3}\frac{1}{\gamma'(|\gamma|^{-1}(d))}$$

for some d such that $2k\pi - \pi/3 \leq d \leq 2k\pi + \pi/3$.

Since there is an $A \geq 1$ such that $\gamma'(|\gamma|^{-1}(2k)) \leq A\gamma'(|\gamma|^{-1}(k))$ and $\gamma'(|\gamma|^{-1}(t))$ is increasing we have that

$$\begin{aligned}\frac{1}{\gamma'(|\gamma|^{-1}(d))} &\geq \frac{1}{\gamma'(|\gamma|^{-1}(2k\pi + \pi/3)} \geq \frac{1}{\gamma'(|\gamma|^{-1}(8k)}\\ &\geq A^{-3}\frac{1}{\gamma'(|\gamma|^{-1}(k))} \geq \frac{1}{A^3\gamma'(\epsilon)}.\end{aligned}$$

Hence we have $|\gamma|^{-1}(2k\pi - \pi/3) - |\gamma|^{-1}(2k\pi + \pi/3) \geq 2\pi/(3A^3\gamma'(\epsilon))$.

(ii) It is enough to prove

(a) $\epsilon \geq |\gamma|^{-1}(2k'\pi + \pi/3) + B/(3\gamma'(\epsilon))$ and

(b) $|\gamma|^{-1}(7) \leq |\gamma|^{-1}(2\pi - \pi/3) - B/(3\gamma'(\epsilon))$.

(a) Since $k' \leq |\gamma|(\epsilon) < k' + 1 \leq 2k'\pi - \pi/3$ we have that

$$\begin{aligned}\epsilon &\geq |\gamma|^{-1}\left(2k'\pi - \frac{\pi}{3}\right) \geq |\gamma|^{-1}\left(2k'\pi + \frac{\pi}{3}\right) + \frac{B}{\gamma'(\epsilon)}\\ &\geq |\gamma|^{-1}\left(2k'\pi + \frac{\pi}{3}\right) + \frac{B}{3\gamma'(\epsilon)}.\end{aligned}$$

(b) Let

$$B_0 = \min\left[|\gamma|^{-1}(7), \gamma'^{-1}\left(\frac{B}{3(|\gamma|^{-1}(2\pi - \pi/3) - |\gamma|^{-1}(7))}\right)\right].$$

Since $\epsilon \le B_0$ we have

$$\begin{aligned}
& \epsilon \le \gamma'^{-1}\left(\frac{B}{3(|\gamma|^{-1}(2\pi - \pi/3) - |\gamma|^{-1}(7))}\right) \\
\Leftrightarrow \quad & \gamma'(\epsilon) \ge \frac{B}{3(|\gamma|^{-1}(2\pi - \pi/3) - |\gamma|^{-1}(7))} \\
\Leftrightarrow \quad & \frac{B}{3\gamma'(\epsilon)} \le |\gamma|^{-1}\left(2\pi - \frac{\pi}{3}\right) - |\gamma|^{-1}(7) \\
\Leftrightarrow \quad & |\gamma|^{-1}(7) \le |\gamma|^{-1}\left(2\pi - \frac{\pi}{3}\right) - \frac{B}{3\gamma'(\epsilon)}.
\end{aligned}$$

(iii) Let $\alpha = 2k\pi + \pi/3$ and $\beta = 2\pi/3$, so that $2(k+1)\pi + \pi/3 = \alpha + 3\beta$. Since

$$|I_{k+1}| + |J_k| = |\gamma|^{-1}\left(2k\pi + \frac{\pi}{3}\right) - |\gamma|^{-1}\left(2(k+1)\pi + \frac{\pi}{3}\right)$$

then

$$|I_{k+1}| + |J_k| = (-3\beta)(|\gamma|^{-1})'(t_1) = 3\beta\frac{1}{\gamma'(|\gamma|^{-1}(t_1))}$$

for some t_1 such that $\alpha \le t_1 \le \alpha + 3\beta$.

Also

$$\begin{aligned}
|I_{k+1}| + \frac{2B}{3\gamma'(\epsilon)} &= |\gamma|^{-1}\left(2(k+1)\pi - \frac{\pi}{3}\right) - |\gamma|^{-1}\left(2(k+1)\pi + \frac{\pi}{3}\right) \\
&= |\gamma|^{-1}(\alpha + 2\beta) - |\gamma|^{-1}(\alpha + 3\beta) \\
&= -\beta(|\gamma|^{-1})'(t_2) \\
&= \beta\frac{1}{\gamma'(|\gamma|^{-1}(t_2))}
\end{aligned}$$

for some t_2 with $\alpha + 2\beta \le t_2 \le \alpha + 3\beta$.

Since $t_2 \le \alpha + 3\beta \le 2\alpha \le 2t_1$, we have $t_2/2 \le t_1$. Using the doubling property of $\gamma'(|\gamma|^{-1}(t))$ we get $\gamma'(|\gamma|^{-1}(t_2/2)) \ge \frac{1}{A}\gamma'(|\gamma|^{-1}(t_2))$.

So finally we see that

$$\begin{aligned}
|I_{k+1}| + |J_k| &= 3\beta\frac{1}{\gamma'(|\gamma|^{-1}(t_1))} \le 3\beta\frac{1}{\gamma'(|\gamma|^{-1}(t_2/2))} \\
&\le 3\beta A\frac{1}{\gamma'(|\gamma|^{-1}(t_2))} = 3A\left(|I_{k+1}| + \frac{2B}{3\gamma'(\epsilon)}\right) \\
&\le 3A\left(|I_{k+1}| + 2|I_{k+1}|\right).
\end{aligned}$$

So $|J_k| \le (9A-1)|I_{k+1}|$. This ends the proof of Lemma 4.1. $\square$

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References

[1] C. Fefferman and E. M. Stein, 'H^p-spaces of several variables', *Acta Math* **129** (1972), 137–193.
[2] I. I. Hirschman, 'On multiplier transformations', *Duke Math. J.* **26** (1959), 221–242.
[3] B. H. Shayya, *A class of strongly singular integral operators* (Ph.D. Thesis, University of Wisconsin, 1996).
[4] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals* (Princeton Univ. Press, Princeton NJ, 1993).
[5] E. M. Stein and G. Weiss, 'On the interpolation of analytic families of operators acting on H^p-spaces', *Tôhoku Math. J.* **9** (1957), 318–339.

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