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ON THE VALUE DISTRIBUTION OF ITERATED ENTIRE FUNCTIONS

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Abstract

Let f be a transcendental entire function and denote the n-th iterate of f by f_n. For $n \geq 2$, we give an explicit estimate of the number of periodic points of f with period n, that is, fix-points of f_n which are not fix-points of f_k for $1 \leq k < n$.

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1. Introduction and results

We begin by introducing the following fundamental notation and definitions. Let $f(z)$ be a transcendental entire function. We denote by $f_n(z)$ the n-th iterate of $f(z)$ which is defined by $f_0(z) = z$, $f_1(z) = f(z)$, $f_n(z) = f(f_{n-1}(z)) = f_{n-1}(f(z))$. A point z_0 is said to be a periodic point with period n if $f_n(z_0) = z_0$ but for $0 < k < n$, $f_k(z_0) \neq z_0$. And according as the modulus of its multiplier, $\lambda = f_n'(z_0)$, satisfies $|\lambda| < 1$, $|\lambda| = 1$, or $|\lambda| > 1$, we classify the periodic point z_0 of period n into, respectively, attracting, indifferent or repelling. We denote by $\rho(f)$ the order of $f(z)$; by F a set on the positive real axis with finite logarithmic measure, not necessarily the same at each occurrence; and by $v(r, f)$ the central index of the power series of f expanded at $z = 0$. We shall use the standard notation of Nevanlinna theory, such as $T(r, f)$, $\overline{N}(r, f)$ and $N(r, f)$ (see [7]).

The following is the main result of this paper.

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THEOREM 1. *Let $f(z)$ be a transcendental entire function. Then for $n \geq 2$, there exists an unbounded sequence of r such that either*

$$(1) \qquad (1 - o(1))\frac{1}{1403}\log M(r, f_n) < \overline{N_n}\left((e+1)r, \frac{1}{f_n - z}\right),$$

or for every finite complex number a,

$$(1 - o(1))n\left(r^d, \frac{1}{f_n - a}\right) < \overline{n_n}\left(r, \frac{1}{f_n - z}\right),$$

where $\overline{N_n}$ $(r, 1/(f_n - z))$ is the counting function corresponding to the number $\overline{n_n}(r, 1/(f_n - z))$ of periodic points of f with period n, ignoring multiplicities, in $|z| \leq r$, and $d \geq (1/1500)^2$.

We remark that Theorem 1 gives an estimate of the number of periodic points of period n of f and confirms the conjecture, posed by Baker ([8, Problem 2.20]) and proved by Bergweiler [5], that for $n \geq 2$, there exist infinitely many periodic points of period n. For references of the background of this subject we refer the reader to [2], [3] and [5].

The method used in this paper, and which is in essence due to [5] and [11], enables us to prove the following theorem.

THEOREM 2. *Let $f(z)$ be a transcendental entire function and $P(z)$ a non-constant polynomial. Then for $n \geq 2$, there exists an unbounded sequence of r such that*

$$(2) \qquad (1 - o(1))\frac{1}{1403}\log M(r^d, f_n) < N\left(r, \frac{1}{f_n - P}\right),$$

where $d \geq 1/1500$.

We remark that under the assumption of $\rho(f) < 1/2$, Baker [2] showed that the inequality

$$\log M(r^d, f_n) < N\left(r, \frac{1}{f_n - z}\right) + O(\log r)$$

holds for all sufficiently large r, where d depends on n and $\rho(f)$, and $d \to 0$, as $n \to \infty$.

Finally, from the proof of the above theorems given below, we immediately deduce the following theorem.

THEOREM 3. *Let $f(z)$ and $g(z)$ be two transcendental entire functions and $P(z)$ a non-constant polynomial. Assume that $f(z)$ has a finite deficient value or a finite asymptotic value. Then we have*

$$\begin{aligned}\frac{1}{1083}\log M(r, f(g)) &< \overline{n}\left(\chi, \frac{1}{f(g)-P}\right) + O(\log r\nu(r, g)) \\ &\leq \overline{N}\left(e\chi, \frac{1}{f(g)-P}\right) + O(\log r\nu(r, g)), \quad r \notin F,\end{aligned} \tag{3}$$

where $\chi = r + cr/\nu(r, g)$, $c = 20(4+\pi)$.

Note that each of the conditions which $f(z)$ satisfies in Theorem 3 implies $\rho(f) \geq 1/2$.

THEOREM 4. *Let $f(z)$, $g(z)$ and $P(z)$ be given as in Theorem 3. Assume that $\rho(f) < 1/2$. Then there is a constant $c > 1$ such that for all sufficiently large r, we have*

$$(1 - o(1))\frac{1}{d}T(r^d, f(g)) < N\left(r, \frac{1}{f(g)-P}\right), \tag{4}$$

where $d = \min\{1/1085, c/(\cos\pi\rho - \varepsilon)\}$, *in which $\varepsilon > 0$ is chosen sufficiently small so that* $\cos\pi\rho - \varepsilon > 0$.

2. Some results needed in proofs

First of all, let us establish a different form of Nevanlinna's second fundamental theorem. It is well-known that the second fundamental theorem of Nevanlinna can be re-expressed as follows:

Let $F(z)$ be meromorphic in $|z| \leq R$. If $F(0) \neq 0, 1, \infty$ and $F'(0) \neq 0$, then for $0 < r < R$, we have

$$\begin{aligned}T(r, F) &< \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-1}\right) + m\left(r, \frac{F'}{F}\right) \\ &+ m\left(r, \frac{F'}{F-1}\right) + \log\left|\frac{F(0)(F(0)-1)}{F'(0)}\right| + \log 2.\end{aligned} \tag{5}$$

From (5) and by the same argument as in Yang [10, p. 64], we deduce the following lemma.

LEMMA 1. *Let $F(z)$ be meromorphic in $|z| \le R(< \infty)$. If $F(0) \ne 0, 1, \infty$ and $F'(0) \ne 0$, then for $0 < r < R$,*

$$T(r, F) < 2\left\{\overline{N}(R, F) + \overline{N}\left(R, \frac{1}{F}\right) + \overline{N}\left(R, \frac{1}{F-1}\right)\right\} + 191$$
$$+ 4\log^+|F(0)| + 2\log^+\frac{1}{R|F'(0)|} + 12\log^+\frac{R}{R-r}.$$

We further deduce the following consequence of Lemma 1.

LEMMA 2. *Let $F(z)$ be holomorphic in $|z| \le R$ and let*

$$N := \overline{n}\left(R, \frac{1}{F}\right) + \overline{n}\left(R, \frac{1}{F-1}\right),$$

where $\overline{n}(R, *)$ denotes the number of distinct poles of $*$ in $|z| < R$. Then for $0 < r < R$, we have

$$\log M(r, F) < \frac{20R}{R-r}\left(2N\log\frac{80eR}{R-r} + 195 + 4\log^+|F(z_0)| + 12\log^+\frac{5R}{R-r}\right),$$

for all z_0 in $|z| < (R-r)/5$, except possibly for the points in the union (γ) of certain disks, the total sum of whose radii does not exceed $(R-r)/20$.

PROOF. Let a_v $(v = 1, \dots, N)$ be all the distinct zeros and distinct 1-points of $F(z)$ in $|z| < R$. By the Boutroux-Cartan theorem, we have

$$\prod_{v=1}^{N}|z - a_v| > \mu^N, \qquad \mu = \frac{R-r}{40e}, \tag{6}$$

except for the points in the union (γ) of certain disks, the total sum of whose radii is at most $2e\mu < (R-r)/20$.

Let $z_0 \notin (\gamma)$ be a point in $|z| < (R-r)/5$, and in the annulus $I := \{r + \frac{2}{5}(R-r) \le |z - z_0| \le r + \frac{3}{5}(R-r)\}$, we can find a circle $|z - z_0| = t$ which does not intersect with (γ). We can do this because the distance between the inner and outer circles of I is $(R-r)/5 > 4e\mu$. Set

$$|F(w)| = \max_{|z-z_0|=t}|F(z)|$$

and draw a segment $\overline{z_0 w}$ connecting z_0 and w. Then we can construct a curve L from $\overline{z_0 w}$ by replacing the part of $\overline{z_0 w}$ in the interior of (γ) by the corresponding boundary arcs of the discs. Obviously, the length of L does not exceed

$$r + \frac{3}{5}(R-r) + \pi\frac{R-r}{10} < R.$$

We need to consider two cases.

Case(a). The inequality $|F'(z)| < 1/R$ holds uniformly on L. Obviously,

$$|F(w)| \le |F(z_0)| + \left| \int_L F'(z)dz \right|.$$

Since $\{z : |z| < r\} \subset \{z : |z - z_0| < t\}$, we have

$$\log M(r, F) \le \log M(t, z_0, F) \le \log^+ |F(z_0)| + \log 2.$$

Case(b). There exists a point z_1 on L such that $|F'(z_1)| = 1/R$ and $|F'(z)| < 1/R$ holds uniformly on L from z_0 to z_1. If $|F'(z_0)| \ge 1/R$, we define $z_1 = z_0$. Then

$$|F(z_1)| \le |F(z_0)| + 1.$$

Set $\gamma_0 = |w - z_1| + (R-r)/10$, $\gamma_1 = |w - z_1| + (R-r)/5$ and $\gamma_2 = |w - z_1| + 2(R-r)/5$. It is easy to see that $\{z : |z - z_1| \le \gamma_2\} \subset \{z : |z| < R\}$. Let a_v $(v = 1, 2, \dots, N_1)$ be all the distinct zeros and distinct 1-points of $F(z)$ in $|z - z_1| \le \gamma_2$. Since $z_1 \notin (\gamma)$, from (6) it follows that

$$(2R)^{N-N_1} \prod_{v=1}^{N_1} |z_1 - a_v| \ge \left(\prod_{v=1}^{N_1} |z_1 - a_v| \right) \left(\prod_{v=N_1+1}^{N} |z_1 - a_v| \right) > \mu^N,$$

and further

$$\left(\frac{80eR}{R-r} \right)^N \ge \prod_{v=1}^{N_1} \frac{R}{|z_1 - a_v|}.$$

Thus we have

$$\overline{N}\left(\gamma_2, z_1, \frac{1}{F}\right) + \overline{N}\left(\gamma_2, z_1, \frac{1}{F-1}\right) \le N \log \frac{80eR}{R-r}. \tag{7}$$

From (7), and using Lemma 1 in the disk $\{|z - z_1| < \gamma_2\}$, it follows that

$$\begin{aligned}
\log M(r, F) &\le \log |F(w)| \le \log M(\gamma_0, z_1, F) \\
&\le \frac{\gamma_1 + \gamma_0}{\gamma_1 - \gamma_0} T(\gamma_1, z_1, F) \\
&\le \frac{20R}{R-r} \left(2N \log \frac{80eR}{R-r} + 191 + 4 \log^+ |F(z_1)| + 12 \log^+ \frac{\gamma_2}{\gamma_2 - \gamma_1} \right) \\
&\le \frac{20R}{R-r} \left(2N \log \frac{80eR}{R-r} + 195 + 4 \log^+ |F(z_0)| + 12 \log^+ \frac{5R}{R-r} \right).
\end{aligned}$$

Thus Lemma 2 follows. □

LEMMA 3. *Let $g(z)$ be a transcendental entire function. Assume that $c > 0$, $K > 0$ and $\eta > 0$. Suppose that $|z_0| = r \notin F$, $|g(z_0)| \geq \eta M(r, g)$ and $|\sigma| < K$. Then there exists a unique s such that $|\nu(r, g)s - \sigma| = o(1)$ and*

$$g(z_0 e^s) = g(z_0)e^\sigma,$$

and a function $\tau(z)$ defined and analytic on $|z - z_0| \leq cr/\nu(r, g)$, and satisfying

$$|\tau(z)\nu(r, g) - 2\pi i| = o(1)$$

such that

$$g(ze^{\tau(z)}) = g(z).$$

Lemma 3 is in essence proved by using Wiman-Valiron theory [9] and was explicitly developed by Bergweiler [4]. The following lemma is due to Clunie [6].

LEMMA 4. *Let $f(z)$ and $g(z)$ be two transcendental entire functions. Then*

$$M(r, f(g)) = M((1 - o(1))M(r, g), f), \ r \notin F.$$

The following lemma is due to Baker [1]. It is often used in the proof of the main theorem of Bergweiler [5], as well as in this paper.

LEMMA 5. *Let $f(z)$ be an entire function. For any $B > A$, if $|f(z)| < R$ in $|z| < A$, but $|f(z)| > R$ on $|z| = B$, then there exists a simple curve Γ in $\{A \leq |z| < B\}$ going around the origin once such that $|f(z)| = R$ on Γ*

By analyzing the proof of [2, Theorem 1], we are immediately able to prove the following lemma.

LEMMA 6. *Let $F(z)$ be a transcendental entire function and $P(z)$ be a polynomial with the first term $a_m z^m$ ($a_m \neq 0$). Assume that, for some $\sigma > 1$ and some r satisfying*

$$2|a_m|r^{m\sigma} \geq |P(z)|, \quad \text{on} \quad |z| = r^\sigma,$$

there exists a simple closed curve $\Gamma \subset \{r < |z| < r^\sigma\}$ going around the origin once, on which

$$|F(z)| \geq M > 2|a_m|r^{m\sigma}.$$

Then

$$N\left(r^\sigma, \frac{1}{f - P}\right) > \log(M - 2|a_m|r^{m\sigma}) - O(\log r).$$

3. Proofs of theorems

We begin with the proof of Theorem 2. Put $g = f_{n-1}$, so that $f(g) = f_n$. Assume that for sufficiently large $r \notin F$, where F is the set arising from Lemmas 3 and 4, we have

$$\bar{n}\left(\chi, \frac{1}{f(g) - P}\right) \leq \frac{1}{1403} \log M(r, f(g)), \tag{8}$$

where $\chi = r + 2R$, $R = 20(4+\pi)r/v(r, g)$.

We want to prove the following

CLAIM. *In*

$$e^{-3}M(r, g) \leq |w| \leq e^{3}M(r, g), \tag{9}$$

there exists a circle $\Gamma_0 : |w| = \xi_r$ *on which we have*

$$(1 - o(1))\frac{1}{1403} \log M(r, f(g)) \leq \log|f(w)|. \tag{10}$$

The same inequality still holds with f *and* g *interchanged.*

Now we choose a point z_0 on $|z| = r$ such that

$$|f(g(z_0))| = M(r, f(g)) = M((1 - o(1))M(r, g), f),$$

then $M(r, g) \geq |g(z_0)| \geq (1 - o(1))M(r, g)$. Application of Lemma 3 to z_0 and g implies the existence of an analytic function $\tau(z)$ defined in $|z - z_0| \leq R$, where $R = 20(4+\pi)r/v(r, g)$, such that

$$|\tau(z)v(r, g) - 2\pi i| < 1 \tag{11}$$

and

$$g(ze^{\tau(z)}) = g(z). \tag{12}$$

Set $k(z) = ze^{\tau(z)}$ and

$$h(z) = \frac{f(g(z)) - P(z)}{P(k(z)) - P(z)}, \tag{13}$$

in $|z - z_0| \leq R$. It is easy to prove that $h(z)$ is analytic in $|z - z_0| \leq R$. Let (γ) be the union of exceptional disks, the total sum of whose radii does not exceed $R/80$, the existence of which follows from Lemma 2, taking $F = h$, $r = \frac{3}{4}R$ and the disk $|z - z_0| < R$.

For w satisfying (9) we can write $w = e^{\sigma} g(z_0)$, where $|\operatorname{Re}\sigma| \le 3$, $|\operatorname{Im}\sigma| \le \pi$. From Lemma 3, we can find a unique s such that $v(r, g)s = \sigma + o(1)$ and $g(z_0 e^s) = g(z_0)e^{\sigma} = w$. Put $u = z_0 e^s$, then

$$|u - z_0| = r|e^s - 1| \le \frac{(4+\pi)r}{v(r, g)} = \frac{R}{20}.$$

It is obvious that the mapping $w = e^{\sigma} g(z_0)$ maps the segment

$$L_1 = \{w : \arg w = \theta \text{ and } e^{-3}M(r, g) \le |w| \le e^3 M(r, g)\}$$

into a segment which contains the segment

$$L_2 = \left\{\sigma : \arg g(z_0) + \operatorname{Im}\sigma = \theta \text{ and } -\frac{8}{3} \le \operatorname{Re}\sigma \le \frac{8}{3}\right\},$$

and that L_2 is mapped by $u = z_0 e^s$, $v(r, g)s = \sigma + o(1)$, into a curve L_3, the diameter of which is at least $R/30$. And therefore we can find a circle $\Gamma_0 : |w| = \xi_r$ in $\{e^{-3}M(r, g) \le |w| \le e^3 M(r, g)\}$ which is such that u corresponding to w on Γ_0 is not contained in (γ).

Obviously, by (11), a simple calculation implies that

$$\{z : |z - z_0| < R\} \subset \{z : |z| < r + R\}$$

and

$$\{k(z) : |z - z_0| < R\} \subset \{z : |z| < r + 2R = \chi\}.$$

Thus it follows from (12) and (13) that

$$\overline{n}(R, z_0, h) + \overline{n}\left(R, z_0, \frac{1}{h}\right) + \overline{n}\left(R, z_0, \frac{1}{h-1}\right) \le \overline{n}\left(\chi, \frac{1}{f(g) - P}\right).$$

By applying Lemma 2 to $h(z)$ in $|z - z_0| < R$, for $u \notin (\gamma)$ satisfying $|u - z_0| < (R - \frac{3}{4}R)/5 = R/20$, we have

$$\begin{aligned}
\log|h(z_0)| &\le \log M\left(\frac{3}{4}R, z_0, h\right) \\
&< \frac{20R}{R - \frac{3}{4}R}\left\{2\overline{n}\left(\chi, \frac{1}{f(g) - P}\right)\log\frac{80eR}{R - \frac{3}{4}R} + 195 + 4\log^+|h(u)|\right. \\
&\qquad\left. + 12\log^+\frac{5R}{R - \frac{3}{4}R}\right\} \\
&< 1083\overline{n}\left(\chi, \frac{1}{f(g) - P}\right) + O(1) + 320\log^+|h(u)|.
\end{aligned}$$

On the other hand, from (13) it follows that

$$\log |h(z_0)| > \log M(r, f(g)) - O(\log rv(r, g)).$$

Hence, by (13) we have

$$\begin{aligned} \log M(r, f(g)) &\le 1083\overline{n}\left(\chi, \frac{1}{f(g)-P}\right) + O(\log rv(r, g)) + 320 \log^+ |h(u)| \\ &\le 1083\overline{n}\left(\chi, \frac{1}{f(g)-P}\right) + O(\log rv(r, g)) + 320 \log^+ |f(w)|, \qquad (14) \end{aligned}$$

and further from (8)

$$\frac{1}{1403} \log M(r, f(g)) \le \log^+ |f(w)| + O(\log rv(r, g)),$$

on Γ_0. Thus the claim is proved.

Now choose a $t \notin F$ in the interval $((M(r, f)/2)^{1/1425}, (M(r, f)/2)^q)$, where $q = \frac{1}{2}(1/1403 + 1/1425)$. Then from the claim, with alternation of f and g, and the fact that $\log M(r, g)$ is convex with respect to $\log r$, we have on $\Gamma_0 : |w| = \xi_r \subset \{e^{-3}M(r, f) \le |w| \le e^3 M(r, f)\}$

$$\begin{aligned} \log |g(w)| &\ge (1 - o(1)) \frac{1}{1403} \log M(r, g(f)) \\ &\ge (1 - o(1)) \frac{1}{1403} \log M(M(r, f)/2, g) \\ &> \log M(t, g) + 3, \end{aligned}$$

so that

$$|g(w)| > e^3 M(t, g) \ge \xi_t.$$

On the other hand, obviously for $|w| < t^\alpha$, $\alpha = 29/30$, we have

$$|g(w)| < e^{-3} M(t, g) \le \xi_t.$$

Then there exists a simple curve $\Gamma \subset \{t^\alpha < |w| < \xi_r\} \subset \{t^\alpha \le |w| < t^{1450}\} = \{\tilde{t} \le |w| < \tilde{t}^{1500}\}$, $\tilde{t} = t^\alpha$, which goes around the origin once and on which $|g(w)| = \xi_t$. Applying the claim once more implies that

$$\log^+ |f(g(w))| \ge (1 - o(1)) \frac{1}{1403} \log M(t, f(g)), \text{ on } \Gamma. \qquad (15)$$

Then by Lemma 6, we get (2).

If for an unbounded sequence of $r \notin F$, (8) does not hold, it is easy to deduce (2) from the convexity of $\log M(r, g)$ with respect to $\log r$ and the fact that

$$\bar{n}\left(\chi, \frac{1}{f(g) - P}\right) \leq \bar{N}\left(e\chi, \frac{1}{f(g) - P}\right).$$

Thus the proof of Theorem 2 is complete.

Now we are in position to prove Theorem 3. Conversely, suppose that for $r \notin F$, we have

$$\text{(16)} \quad \bar{n}\left(\chi, \frac{1}{f(g) - P}\right) \leq \frac{1}{1083}(\log M(r, f(g)) - O(\log rv(r, g)) - 320 \log r),$$

where $O(\log rv(r, g))$ is the quantity occurring in (14). By the same argument as in the proof of the claim , from (14) and (16), we can prove that there exists a circle $\Gamma_0 : |w| = \xi_r$ on which we have

$$\text{(17)} \qquad \log r \leq \log |f(w)|.$$

In fact, first of all, we deduce (14), and by (16) deduce (17). Obviously for any finite number a, it follows from (17) that for sufficiently large r

$$\log |f(w) - a| > 1, \text{ that is, } m\left(\xi_r, \frac{1}{f - a}\right) = 0,$$

where $|w| = \xi_r$, so $\delta(a, f) = 0$ and it is easy to see from (17) that f has no finite asymptotic values, which is a contradiction. Thus (16) does not hold, and Theorem 3 follows.

Before proving Theorem 4, we need a well-known result on transcendental entire function with order less than $1/2$.

LEMMA 7. [1, p. 131, formula (25)]. *Let $\varepsilon > 0$ be a given number and $h(z)$ an entire function of order $\rho < 1/2$. Then there exists a constant $c > 1$ such that, for all sufficiently large R, the interval (R, R^c) contains an R_0 with*

$$\tilde{m}(R_0, h) > \{M(R_0, h)\}^{-\varepsilon + \cos \pi\rho},$$

where $\tilde{m}(R_0, h)$ denotes the minimum modulus of $h(w)$ on $|w| = R_0$.

Now we prove Theorem 4. Let $\varepsilon > \varepsilon_0 > 0$ be sufficiently small such that $\cos \pi\rho - \varepsilon > 0$. Put $p = (\cos \pi\rho - \varepsilon)^{-1}$, $p_0 = (\cos \pi\rho - \varepsilon_0)^{-1}$. By the convexity of $\log M(r, g)$ as a function of $\log r$, we have

$$M(r^p, g) \geq (e^3 M(r, g))^{p_0}.$$

Application of Lemma 7 to ε_0 and g implies the existence of $c > 1$ which is such that the interval (r^p, r^{cp}) contains an R_0 such that on $|z| = R_0$,

$$\begin{aligned}|g(z)| &\geq \tilde{m}(R_0, g) \geq M(R_0, g)^{1/p_0}\\ &\geq M(r^p, g)^{1/p_0} \geq e^3 M(r, g)\\ &> \xi_r.\end{aligned}$$

On the other hand, it is obvious that $|g(z)| < M(r^{1-\delta}, g) < e^{-3}M(r, g) \leq \xi_r$ in $|z| < r^{1-\delta}$, where δ is a sufficiently small positive number. Then it follows from Lemma 5 that there exists a curve $\Gamma \subset \{r^{1-\delta} \leq |z| < r^{cp}\}$ which contains the origin in its interior and on which $|g(z)| = \xi_r$.

If for $r \notin F$, (16) with $320cp(\deg P + 1)\log r$ instead of $320 \log r$ does not hold, then (4) immediately follows. Now we assume such (16) holds, then we have (17) with $cp(\deg P + 1)\log r$ instead of $\log r$ on $|w| = \xi_r$, that is, $\log^+ |f(w)| \geq cp(\deg P + 1)\log r$, $|w| = \xi_r$. Then on Γ we have

$$\log^+ |f(g(z))| \geq cp(\deg P + 1)\log r > \log(|P(z)| + 1 + a),$$

where a is a complex number such that $N(r, 1/(f(g) - a)) = (1 - o(1))T(r, f(g))$. By Rouché's theorem, we deduce that $f(g) - P$ has as many zeros as $f(g) - a$ does in the interior of Γ, and therefore for all sufficiently large $r \geq r_0$,

$$\begin{aligned}n\left(r^{cp}, \frac{1}{f(g) - P}\right) &\geq n\left(\Gamma, \frac{1}{f(g) - P}\right) = n\left(\Gamma, \frac{1}{f(g) - a}\right)\\ &\geq n\left(r^{1-\delta}, \frac{1}{f(g) - a}\right),\end{aligned}$$

where $n(\Gamma, *)$ is the number of poles of $*$ in $\operatorname{int}\Gamma$. Further for $r \geq r_0$,

$$\begin{aligned}N\left(r^{cp}, \frac{1}{f(g) - P}\right) &\geq \frac{cp}{1-\delta} N\left(r^{1-\delta}, \frac{1}{f(g) - a}\right)\\ &= (1 - o(1))\frac{cp}{1-\delta} T(r^{1-\delta}, f(g)),\end{aligned}$$

since for arbitrary positive s,

$$\int_{r_0}^{r} \frac{n(t^s, *)}{t}\, dt = \frac{1}{s}\int_{r_0}^{r} \frac{n(t^s, *)}{t^s}\, dt^s = \frac{1}{s} N(r^s, *) + O(1).$$

By choosing a smaller ε than the one in the above, we deduce (4). Thus Theorem 4 follows.

Finally, we prove Theorem 1. For $0 < k < n$ and arbitrarily large K, we have

$$\begin{aligned}\log M(r, f_n) &\geq \log M\left(\frac{1}{2}M\left(\frac{1}{4}r, f_{n-k}\right), f_k\right)\\ &\geq \log M\left(((e+1)\,r)^{K+1}, f_k\right)\\ &\geq (1-o(1))K\,T((e+1)r, f_k),\end{aligned}$$

and hence

$$\begin{aligned}N\left((e+1)r, \frac{1}{f_k - z}\right) &\leq T((e+1)r, f_k) + O(\log r)\\ &= o(\log M(r, f_n)).\end{aligned}$$

On the other hand, it is easy to see that

$$\overline{N_n}\left(r, \frac{1}{f_n - z}\right) \geq \overline{N}\left(r, \frac{1}{f_n - z}\right) - \sum_{k=1}^{n-1} \overline{N}\left(r, \frac{1}{f_k - z}\right).$$

Set $g = f_{n-1}$, so that $f(g) = f_n$. Therefore, if (8) does not hold for some unbounded sequence of $r \notin F$, then we easily deduce that

$$\overline{N_n}\left((e+1)r, \frac{1}{f_n - z}\right) \geq (1-o(1))\frac{1}{1403}\log M(r, f_n).$$

This is (1). Now, we can assume that for all $r \notin F$, we have (8) and further deduce the claim. By the same argument as in the proof of Theorem 2, we can find a simple curve $\Gamma \subset \{\tilde{t} \leq |w| < \tilde{t}^{1500}\}$ going around the origin once and on which (15) holds. Then the argument of Lemma 5 implies the existence of a simple curve $\Gamma_0 \subset \{\tilde{t}^{1/1406} \leq |w| < \tilde{t}^{1500}\}$ which surrounds the origin once and on which $|f(g(z))| = M(R, f(g))$, where $R = \tilde{t}^{\beta} \notin F$, $1/1406 \leq \beta < 1/1403$. Define $G_0 = \operatorname{int}\Gamma_0$, $G_1 = g(G_0)$, $G_2 = f(G_1)$ and $\Gamma_j = \partial G_j$ $(0 \leq j \leq 2)$. Obviously, by the Maximum Principle, all the G_j are simply connected and all the Γ_j are simple curves and surround the origin once. Assume that $g(z)$ (respectively, $f(z)$) describes p_1 (respectively, p_2) times the curve Γ_1 (respectively, Γ_2) as z describes the Γ_0 (respectively, Γ_1) once. Then $f(g)$ has $P_2 = p_1 p_2$ zeros in G_0. By Rouché's theorem, $f(g(z)) - z$ also has P_2 zeros in G_0. From the main ideas in Bergweiler [5], by a little modification of his proof, we can prove that

$$\overline{P_2} = (1-o(1))P_2, \tag{18}$$

where $\overline{P_2}$ denotes the number of distinct zeros of $f(g(z)) - z$ in G_0. For completeness, we shall give the proof of (18). First of all, we want to prove that

$$\overline{f_j(G_0)} \subset G_1,\ 0 \leq j < n-1. \tag{19}$$

It is obvious that G_1 is a domain and contains the origin. By the maximum principle, $\overline{f_j(G_0)} \subset \{|w| \le M(\tilde{t}^{1500}, f_j)\}$, $0 \le j < n-1$. Since on Γ_0, $|f(g(z))| = M(R, f(g)) \ge M(\frac{1}{2}M(R,g), f)$, we have that $|g(z)| \ge M(\tilde{t}^{1500}, f_j)$, $0 \le j < n-1$, on Γ_0, noting that f is transcendental. Thus (19) follows from $\Gamma_1 \subset g(\Gamma_0) \subset \{M(\tilde{t}^{1500}, f_j) < |w|\}$, $0 \le j < n-1$.

Let $z_0 \in G_0$ be a zero of $f(g(z)) - z$ with multiplicity $m+1$. Then we have that $f_n'(z_0) = 1$ and z_0 is a periodic point with period $k \le n$. Let p be the smallest positive integer such that $(f_k'(z_0))^p = 1$, and further $kp \le n$, and let $m_1 + 1$ be the multiplicity of the zero z_0 of $f_{kp}(z) - z$. Then it is easy to see that $s = n/(kp)$ is a positive integer, and $f_n = (f_{kp})_s$, and therefore $m = m_1$. By a result of Fatou (see [5, Lemma 6]), it follows that there exist m/p cycles of Leau domains, each of which contains at least one singularity of the inverse function of f, and which are in G_1, by [5, Lemma 8] and (19). Therefore G_1 contains at least m/p critical points of f, for from the claim it is easy to see that f has no asymptotic values. However, f has at most $p_2 - 1$ critical points in G_1. Hence we have

$$P_2 - \overline{P_2} \le \sum km = \sum kp\frac{m}{p} \le n(p_2 - 1),$$

where $\sum$ is taken over all the zeros of $f(g(z)) - z$ in G_0. This implies (18), since $p_1 \to \infty$, as $r \to \infty$.

By $(P_2)_j$ we denote the number of zeros of $f_j(z) - z$ in G_0. Obviously, it follows from $|f_n(z)| = M(R, f_n)$ on Γ_0 that $|f_j(z)| > |z|$ $(0 \le j \le n-1)$ on Γ_0, and hence $(P_2)_j$ is equal to the number of zeros of $f_j(z)$ in G_0, that is, the winding number of $f_j(\Gamma_0)$ around the origin. Obviously, $(P_2)_j \le (P_2)_{n-1} = p_2$. Since

$$(1 - o(1))P_2 \le \overline{P_2} \le \overline{(P_2)_n} + \sum_{j=1}^{n-1} (P_2)_j \le \overline{(P_2)_n} + (n-1)p_2,$$

we have

$$\overline{(P_2)_n} \ge (1 - o(1))P_2,$$

where $\overline{(P_2)_n}$ denotes the number of distinct periodic points of period n of $f(z)$ in G_0, and further we have for arbitrary $a \in G_0$

$$\begin{aligned}\overline{n_n}\left(R^v, \frac{1}{f_n - z}\right) &\ge \overline{n_n}\left(\Gamma_0, \frac{1}{f_n - z}\right) \ge (1 - o(1))n\left(\Gamma_0, \frac{1}{f_n - z}\right)\\ &= (1 - o(1))n\left(\Gamma_0, \frac{1}{f_n - a}\right) \ge (1 - o(1))n\left(R, \frac{1}{f_n - a}\right),\end{aligned}$$

where $v = 1500^2$, and $\overline{n_n}(\Gamma_0, 1/(f_n - z))$ is the number of distinct periodic points of period n of f in int Γ_0.

Thus Theorem 1 follows.

References

[1] I. N. Baker, 'Zusammensetzungen ganzer Funktionen', *Math. Z.* **69** (1958), 121–163.
[2] ———, 'Fixpoints and iterates of entire functions', *Math. Z.* **71** (1959), 146–153.
[3] ———, 'Some entire functions with fixpoints of every order', *J. Austral. Math. Soc. (Series A)* **1** (1959/1960), 203–209.
[4] W. Bergweiler, 'On proof of a conjecture of Gross concerning fix-points', *Math. Z.* **204** (1990), 381–390.
[5] ———, 'Periodic points of entire functions: Proof of a conjecture of Baker', *Complex Variables Theory Appl.* **17** (1991), 57–72.
[6] J. Clunie, 'The composition of entire and meromorphic functions', in: *Math. essays dedicated to A. J. Macintyre* (ed. H. Shankar) (Athens, Ohio, 1970) pp. 75–92.
[7] W. K. Hayman, *Meromorphic functions* (Clarendon Press, Oxford, 1964).
[8] ———, *Research problems in function theory* (Athlone Press, London, 1967).
[9] G. Valiron, 'Sur la dérivée des fonctions algébroides', *Bull. Soc. Math. France* **59** (1929), 17–39.
[10] L. Yang, *New researches on value distribution theory*, (in Chinese) (Science Press, 1982).
[11] J. H. Zheng and C. C. Yang, 'Further results on fix-points and zeros of entire functions', *Trans. Amer. Math. Soc.* (1) **347** (1995), 37–50.

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