J. Austral. Math. Soc. (Series A) 63 (1997), 16–31

SPECTRAL AND ASYMPTOTIC PROPERTIES OF DOMINATED OPERATORS

FRANK RÄBIGER and MANFRED P. H. WOLFF

(Received 1 November 1995; revised 26 July 1996)

Communicated by P. G. Dodds

Abstract

We investigate the relationship between the peripheral spectrum of a positive operator T on a Banach lattice E and the peripheral spectrum of the operators S dominated by T, that is, $|Sx| \leq T|x|$ for all $x \in E$. This can be applied to obtain inheritance results for asymptotic properties of dominated operators.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 47B65, 47A10, 47A35, 47A53.
Keywords and phrases: Banach lattice, dominated operators, peripheral spectrum, essential spectrum, almost periodicity, uniform ergodicity.

Introduction

The investigation of operators on Banach lattices leads to the natural question which properties of a positive operator T on a Banach lattice E are inherited by the operators S *dominated by* T, that is, $|Sx| \leq T|x|$ for all $x \in E$. For certain properties one has to impose the additional assumption, that the operator S is also positive.

There are numerous results on inheritance of operator properties such as compactness, weak compactness, or being a kernel or a Dunford-Pettis operator (see, for example, [1, 4, 7, 9, 13, 14, 20, 27]; see also [2, 19, 22, 28] for a comprehensive survey and further developments). Only recently the inheritance of spectral and asymptotic properties of an operator has been investigated (see, for example, [3, 5, 17, 18, 21, 23–25]).

In the present paper we are mainly interested in properties of the peripheral spectrum of a dominated operator. We always assume that the dominating operator T satisfies a certain growth condition (G). Then for positive operators S dominated by T one has

This paper is part of a research project supported by the Deutsche Forschungsgemeinschaft DFG

$\sigma(S) \cap r(T)\Gamma \subseteq \sigma(T) \cap r(T)\Gamma$, that is, either the spectral radii satisfy $r(S) < r(T)$ or the peripheral spectrum $\sigma(S) \cap r(S)\Gamma$ of S is contained in the peripheral spectrum $\sigma(T) \cap r(T)\Gamma$ of T (see Theorem 1.4). If T satisfies an ergodicity condition and/or the Banach lattice E has order continuous norm or is a KB-space, one obtains the corresponding inclusion $P\sigma(S) \cap r(T)\Gamma \subseteq P\sigma(T) \cap r(T)\Gamma$ for the point spectrum (see Theorem 2.2, Corollary 2.4 and Theorem 2.6). If $r(T)$ is a Riesz point of T and S a (not necessarily positive) operator dominated by T, then $r(S) < r(T)$ or the peripheral spectrum of S contains only Riesz points (see Theorem 3.1). This generalizes a result of Caselles [5, Theorem 4.1], where S is assumed to be positive. Finally we apply the above results and investigate inheritance of asymptotic properties such as uniform convergence of $S^{n+1} - S^n$ to 0 (see Theorem 4.1), almost periodicity and strong convergence of the powers S^n (see Theorem 4.2 and Corollary 4.3), and uniform ergodicity of S (see Theorem 4.5). In particular we generalize a result of Caselles [5, Corollary 4.6] and extend results of Räbiger [24, 25].

Our notation is standard and follows mainly the books of Meyer-Nieberg [19] and Schaefer [26]. Unexplained terminology can be found there. We briefly recall some frequently used notions. By $\Gamma := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ we denote the *unit circle*. Throughout the whole paper we consider spaces over $\mathbb{C}$. If E is a Banach space, then $\mathscr{L}(E)$ is the space of all bounded linear operators on E and E' the (*topological*) *dual* of E. For $T \in \mathscr{L}(E)$ let $T' \in \mathscr{L}(E')$ be the *adjoint* of T. Moreover, $\sigma(T)$ denotes the *spectrum*, $r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$ the *spectral radius*, $P\sigma(T) := \{\lambda \in \sigma(T) : \lambda$ is an eigenvalue of $T\}$ the *point spectrum*, and $\rho(T) := \mathbb{C} \setminus \sigma(T)$ the *resolvent set* of T. For $\lambda \in \rho(T)$ we set $R(\lambda, T) := (\lambda I - T)^{-1}$.

Now let E be a (complex) Banach lattice with modulus $|\,.\,|$. Then $E_+ := \{x \in E : x = |x|\}$ is the set of all *positive* elements in E. The dual space E' is again a Banach lattice and $x' \in E'$ is positive if and only if $\langle x', x\rangle \geq 0$ for all $x \in E_+$. For operators $S, T \in \mathscr{L}(E)$ we write $S \leq T$ if $(T - S)E_+ \subseteq E_+$, and T is called *positive* if $0 \leq T$.

1. The peripheral spectrum of dominated positive operators

In this section we show that for operators $0 \leq S \leq T$ on a Banach lattice E one always has $\sigma(S) \cap r(T)\Gamma \subseteq \sigma(T) \cap r(T)\Gamma$ provided that T satisfies a certain growth condition (G).

At first we recall some well-known facts and fix some notations. Let $T \in \mathscr{L}(E)$ be a bounded linear operator on a Banach space E. If G is a closed linear subspace of E such that $TG \subseteq G$ we denote by $T_| = T_{|G}$ the restriction of T to G and by $T_/ = T_{/G}$ the induced operator on the quotient space E/G given by $T_/(x + G) := Tx + G$, $x \in E$. The following lemma is well-known (see [26, V, Exercise 5]).

LEMMA 1.1. *Under the conditions above one has*

(i) $(\sigma(T_{|}) \cup \sigma(T_{/})) \cap r(T)\Gamma \subseteq \sigma(T) \cap r(T)\Gamma$.

(ii) $\max(r(T_{|}), r(T_{/})) = r(T)$.

(iii) $\lambda \in r(T)\Gamma$ *is a pole of the resolvent* $R(., T)$ *(of order* k*) if and only if* λ *is a pole of* $R(., T_{|})$ *and* $R(., T_{/})$ *(of order* $k_{|}$ *and* $k_{/}$*, respectively), and then* $\sup(k_{|}, k_{/}) \leq k \leq k_{|} + k_{/}$. *Moreover, if* P *is the residuum of* $R(., T)$ *at* λ*, then* $PG \subseteq G$ *and* $P_{|}$ *and* $P_{/}$ *is the residuum of* $R(., T_{|})$ *and* $R(., T_{/})$ *at* λ*, respectively.*

In the sequel we make use of the following construction. For details we refer to [26, V.1]. If E is a Banach space let $l_\infty(E)$ be the space of bounded E-valued sequences endowed with the sup-norm. For a free ultrafilter $\mathscr{U}$ on $\mathbb{N}$ we consider the closed subspace $c_{\mathscr{U}}(E) := \{(x_n) \in l_\infty(E) : \lim_{\mathscr{U}} \|x_n\| = 0\}$. The quotient space $E_{\mathscr{U}} := l_\infty(E)/c_{\mathscr{U}}(E)$ is called *ultrapower* or $\mathscr{U}$*-power* of E. For $(x_n) + c_{\mathscr{U}}(E) \in E_{\mathscr{U}}$ we also write $\widehat{(x_n)}$. The mapping $x \mapsto (x, x, \dots)\hat{}$ is an isometric embedding of E into $E_{\mathscr{U}}$ and thus E can be considered a closed subspace of $E_{\mathscr{U}}$. Every operator $T \in \mathscr{L}(E)$ induces an operator $T_{\mathscr{U}} \in \mathscr{L}(E_{\mathscr{U}})$ by means of $T_{\mathscr{U}}\widehat{(x_n)} := \widehat{(Tx_n)}$. Its restriction to E satisfies $T_{\mathscr{U}|E} = T$. Moreover, the following holds (see [26, V.1]).

LEMMA 1.2. (i) $\|T_{\mathscr{U}}\| = \|T\|$.

(ii) $\sigma(T_{\mathscr{U}}) = \sigma(T)$.

(iii) $\sigma(T_{\mathscr{U}}) \cap r(T)\Gamma \subseteq P\sigma(T_{\mathscr{U}})$.

(vi) $R(\lambda, T_{\mathscr{U}}) = R(\lambda, T)_{\mathscr{U}}$ *for all* $\lambda \in \rho(T)$.

(v) $\lambda \in \sigma(T)$ *is a pole of order* k *of* $R(., T)$ *if and only if the same holds for* $R(., T_{\mathscr{U}})$.

If E is a Banach lattice and $T \in \mathscr{L}(E)$ is a positive operator, then $E_{\mathscr{U}}$ is again a Banach lattice and $T_{\mathscr{U}} \in \mathscr{L}(E_{\mathscr{U}})$ is positive. For $\widehat{(x_n)} \in E_{\mathscr{U}}$ one has $|\widehat{(x_n)}| = \widehat{(|x_n|)}$.

Now let $0 \leq S \leq T$ be operators on a Banach lattice E. In the following lemma we present a condition under which an eigenvalue of S is also an eigenvalue of T.

LEMMA 1.3. *Let* E *be a Banach lattice and let* $S, T \in \mathscr{L}(E)$ *be such that* $0 \leq S \leq T$. *Suppose there is* $\alpha \in \Gamma$ *and* $x \in E$ *such that* $Sx = \alpha x$ *and* $T|x| = |x|$. *Then* $Tx = \alpha x$.

PROOF. The assumptions imply $|x| \leq S|x| \leq T|x| = |x|$. Thus $0 \leq |(T - S)x| \leq (T - S)|x| = |x| - S|x| \leq 0$, and hence $Tx = Sx = \alpha x$.

REMARK. The conditions of the lemma are satisfied if $0 \leq S \leq T$, $Sx = \alpha x$ for $\alpha \in \Gamma$ and $x \in E$, and there is a strictly positive linear form $x' \in E'_+$ such that $T'x' \leq x'$. (Recall that $x' \in E'_+$ is *strictly positive* if $\langle x', y\rangle > 0$ for all $y \in E_+ \setminus \{0\}$.) In fact, from $0 \leq T|x| - |x|$ and $0 \leq \langle T|x| - |x|, x'\rangle = \langle |x|, (T' - I)x'\rangle \leq 0$ we obtain $T|x| = |x|$ by the strict positivity of x'.

Now we come to the main result of this section. Recall that an operator T on a Banach space E satisfies the *growth condition* (G) if $\limsup_{\lambda\downarrow r(T)} \|(\lambda-r(T))R(\lambda,T)\| < \infty$. Lemma 1.2 implies that then $T_{\mathscr{U}} \in \mathscr{L}(E_{\mathscr{U}})$ has also property (G) for every ultrapower $E_{\mathscr{U}}$ of E. Clearly every operator with uniformly bounded powers and spectral radius 1 satisfies (G). Moreover a positive operator T on a Banach lattice E with $r(T) = 1$ has property (G) if and only if the *Cesaro means* $T_n := n^{-1}\sum_{k=0}^{n-1} T^k$, $n \in \mathbb{N}$, are uniformly bounded (see [10, 1.5, 1.7]).

THEOREM 1.4. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ such that $0 \leq S \leq T$ and T satisfies (G). Then $\sigma(S)\cap r(T)\Gamma \subseteq \sigma(T)\cap r(T)\Gamma$.*

PROOF. The assumptions imply $0 \leq r(S) \leq r(T)$. If $r(T) = 0$ there is nothing to prove. Otherwise we may assume $r(T) = 1$ and, by passing to an ultrapower, $\sigma(S)\cap\Gamma \subseteq P\sigma(S)$. Let $\alpha \in \sigma(S)\cap\Gamma$ and choose $0 \neq x \in E$ such that $Sx = \alpha x$. Then $|x| \leq S|x| \leq T|x|$. For $y \in E$ let $p(y) := \limsup_{\lambda\downarrow 1}(\lambda-1)\|R(\lambda,T)|y|\|$ (see also [19, proof of 4.1.11]). Since T satisfies (G) the mapping p is a continuous lattice seminorm. Then $J := \ker p$ is a closed ideal in E. From $p(Ty) \leq \|T\|p(y)$ we obtain $TJ \subseteq J$ and $SJ \subseteq J$. Let $S_/$ and $T_/$ be the operators on E/J induced by S and T, respectively. From $|x| \leq T|x|$ it follows that $p(x) \geq \|x\| > 0$, and hence $\tilde{x} := x + J \neq 0$. Clearly $S_/\tilde{x} = \alpha\tilde{x}$. Moreover, since T satisfies (G),

$$\begin{aligned} p(T|x|-|x|) &= \limsup_{\lambda\downarrow 1}(\lambda-1)\|R(\lambda,T)(T-\lambda+\lambda-1)|x|\| \\ &= \limsup_{\lambda\downarrow 1}(\lambda-1)^2\|R(\lambda,T)|x|\| = 0. \end{aligned}$$

Thus $T|x|-|x| \in J$; that is, $T_/|\tilde{x}| = |\tilde{x}|$. Now Lemma 1.3 implies $T_/\tilde{x} = \alpha\tilde{x}$. Hence $\alpha \in \sigma(T)$ by Lemma 1.1.

As in the proof of [26, V.4.9], one can extend Theorem 1.4 to operators T which are (G)-solvable. Recall that a positive operator T on a Banach lattice E is (G)-*solvable*, if there exist finitely many closed T-invariant ideals $\{0\} = I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = E$ such that the operator T_k induced on I_k/I_{k-1} satisfies (G) for all $2 \leq k \leq n$.

COROLLARY 1.5. *Let E be a Banach lattice and $S, T \in \mathscr{L}(E)$ operators such that $0 \leq S \leq T$. If T is (G)-solvable, then $\sigma(S)\cap r(T)\Gamma \subseteq \sigma(T)\cap r(T)\Gamma$.*

If T is a positive operator and $r(T)$ is a pole of the resolvent map $\lambda \mapsto R(\lambda, T)$, then T is (G)-solvable (see [26, p.326, Example 4]). In particular, this is the case if $r(T)$ is a Riesz point of T, that is, a pole of the resolvent map $R(\,.\,,T)$ with finite dimensional residuum.

COROLLARY 1.6. *Let E be a Banach lattice and $S, T \in \mathscr{L}(E)$ operators such that $0 \leq S \leq T$. If $r(T)$ is a Riesz point, then $\sigma(S) \cap r(T)\Gamma \subseteq \sigma(T) \cap r(T)\Gamma$.*

REMARK. If $r(S) = r(T)$, then by a result of Caselles [5, Theorem 4.1] $r(S)$ is a Riesz point of S, and hence $\sigma(S) \cap r(S)\Gamma$ consists entirely of Riesz points (see [26, V.5.5]). In Theorem 3.1 we will show that this conclusion actually holds for any operator S such that $|Sx| \leq T|x|$ for all $x \in E$.

2. The peripheral point spectrum of dominated positive operators

In this section we give analogues of Theorem 1.4 for the point spectrum. At first we recall some well-known facts from ergodic theory and the theory of Banach lattices. The following proposition is a special case of a general ergodic theorem due to Eberlein [9, Theorem 3.1].

PROPOSITION 2.1. *Let $T \in \mathscr{L}(E)$ be an operator on a Banach space E and suppose that $r(T) = 1$ and T satisfies (G). Then for $x \in E$ the following assertions are equivalent:*

(i) $\lim_{\lambda \downarrow 1}(\lambda - 1)R(\lambda, T)x$ *exists in E.*
(ii) $((\lambda - 1)R(\lambda, T)x)_{\lambda > 1}$ *has a weak cluster point (as $\lambda \to 1$).*

In this case $y := \lim_{\lambda \downarrow 1}(\lambda - 1)R(\lambda, T)x$ satisfies $Ty = y$.

An operator $T \in \mathscr{L}(E)$ is called *Abel ergodic* if

$$P_T x := \lim_{\lambda \downarrow r(T)} (\lambda - r(T))R(\lambda, T)x \text{ exists for all } x \in E.$$

From $(\lambda - r(T))R(\lambda, T) = (a\lambda - ar(T))R(a\lambda, aT)$, $\lambda \in \rho(T), a > 0$, it follows that T is Abel ergodic if and only if aT is Abel ergodic for all $a > 0$. If T is Abel ergodic and $r(T) > 0$, then $P_T \in \mathscr{L}(E)$ is a projection, $P_T E = \{x \in E : Tx = r(T)x\}$, and $\ker P_T = \overline{(r(T) - T)E}$ (see [16, 2.1.9]). By the uniform boundedness principle, every Abel ergodic operator satisfies (G).

For our next theorem we need a construction from the theory of Banach lattices (see [26, II.8, Example 1]). Let E be a Banach lattice and $y' \in E'_+$. The mapping $p : E \to \mathbb{R}_+;\ x \mapsto \langle y', |x| \rangle$ is a continuous lattice seminorm on E with kernel $\ker p = N(y') := \{x \in E : \langle y', |x| \rangle = 0\}$. Then p induces a lattice norm on $E/\ker p$. Let (E, y') be its (norm) completion, which is again a Banach lattice, and let $j_{y'} : E \to (E, y')$ be the lattice homomorphism induced by the quotient map $q : E \to E/\ker p$. It turns out that (E, y') is an AL-space, that is, on $(E, y')_+$ the norm is additive . If $T \in \mathscr{L}(E)$ is a positive operator such that $T'y' \leq y'$, then $TN(y') \subseteq N(y')$. Hence T induces an operator $T_/$ on $E/\ker p$ which is a positive contraction for the norm induced by p. Thus T_l has a unique contractive positive extension $\tilde{T} \in \mathscr{L}((E, y'))$. We call $\tilde{T}$ the operator on (E, y') induced by T.

Now we can state the following inheritance result for the point spectrum.

THEOREM 2.2. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ such that $0 \leq S \leq T$ and αT is Abel ergodic for all $\alpha \in \Gamma$. Then $P\sigma(S) \cap r(T)\Gamma \subseteq P\sigma(T) \cap r(T)\Gamma$.*

PROOF. Firstly let $r(T) = 0$. Clearly, T satisfies (G), and hence $\sup_{\lambda>0} \|\lambda R(\lambda, T)\| < \infty$. From $|\lambda R(\lambda, T)x| \leq |\lambda| R(|\lambda|, T)|x|$, $\lambda \in \mathbb{C} \setminus \{0\}$, $x \in E$, we obtain $\sup_{\lambda \in \mathbb{C}\setminus\{0\}} \|\lambda R(\lambda, T)\| < \infty$. Then $\lambda \mapsto \lambda R(\lambda, T) = I + TR(\lambda, T)$ has a removable singularity at 0. Thus $TR(\lambda, T)$ has a holomorphic extension to the whole complex plane. Since $\lim_{|\lambda|\to\infty} TR(\lambda, T) = 0$, Liouville's theorem implies $TR(\,.\,, T) = 0$, and hence $T = 0$.

Now let $r(T) > 0$. Without loss of generality we may assume $r(T) = 1$. Let $\alpha \in P\sigma(S) \cap \Gamma$ and choose $0 \neq x \in E$ such that $Sx = \alpha x$. Then $|x| \leq S|x| \leq T|x|$. Since T is Abel ergodic, $y := \lim_{\lambda\downarrow 1} (\lambda - 1)R(\lambda, T)|x|$ exists and $0 \leq |x| \leq y = Ty$. Let $x' \in E'_+$ be such that $\langle x', |x| \rangle > 0$. Again by Abel ergodicity, $y' := \sigma(E', E)$-$\lim_{\lambda\downarrow 1} (\lambda - 1)R(\lambda, T)'x'$ exists and $0 \leq S'y' \leq T'y' = y'$. Moreover, $\langle y', |x| \rangle = \lim_{\lambda\downarrow 1} \langle (\lambda - 1)R(\lambda, T)'x', |x| \rangle = \langle x', y \rangle \geq \langle x', |x| \rangle > 0$. Let $\tilde{S}$ and $\tilde{T}$ be the operators on the AL-space (E, y') induced by S and T, respectively. Then $0 \leq \tilde{S} \leq \tilde{T}$ and $\tilde{S}$ and $\tilde{T}$ are contractions. Let $\tilde{x} := j_{y'}x$. We have $\tilde{S}\tilde{x} = \alpha\tilde{x}$ and $\|\tilde{x}\| = \langle y', |x| \rangle > 0$, hence $\alpha \in P\sigma(\tilde{S})$. Moreover $|\tilde{x}| \leq \tilde{S}|\tilde{x}| \leq \tilde{T}|\tilde{x}|$. Since the norm of (E, y') is strictly monotone on $(E, y')_+$ and $\tilde{T}$ is contractive we obtain $\tilde{T}|\tilde{x}| = |\tilde{x}|$. Then Lemma 1.3 implies $\tilde{T}\tilde{x} = \alpha\tilde{x}$. Now $\alpha^{-1}T$ is Abel ergodic. Then $z := \lim_{\lambda\downarrow 1} (\lambda - 1)R(\lambda, \alpha^{-1}T)x$ exists in E and $Tz = \alpha z$. Thus

$$
\begin{aligned}
j_{y'}z &= \lim_{\lambda\downarrow 1} j_{y'}(\lambda - 1)R(\lambda, \alpha^{-1}T)x = \lim_{\lambda\downarrow 1} (\lambda - 1)R(\lambda, \alpha^{-1}\tilde{T})j_{y'}x \\
&= \lim_{\lambda\downarrow 1} (\lambda - 1)R(\lambda, \alpha^{-1}\tilde{T})\tilde{x} = \tilde{x} \neq 0.
\end{aligned}
$$

Hence $z \neq 0$ which shows $\alpha \in P\sigma(T) \cap \Gamma$.

REMARK. The proof shows that a positive operator T is the zero operator if $r(T) = 0$ and T satisfies (G).

If the powers of T converge strongly, then αT is Abel ergodic for all $\alpha \in \Gamma$ and $P\sigma(T) \cap \Gamma \subseteq \{1\}$. Thus we obtain the following result.

COROLLARY 2.3. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ be such that $0 \leq S \leq T$ and (T^n) is strongly convergent. Then $P\sigma(S) \cap \Gamma \subseteq P\sigma(T) \cap \Gamma \subseteq \{1\}$.*

If E is a Banach lattice with order continuous norm we can relax the conditions on the operator T. Recall that a Banach lattice E has *order continuous norm* if every decreasing net $(x_\alpha)_{\alpha\in A}$ in E_+, such that $\inf_\alpha x_\alpha = 0$ satisfies $\lim_\alpha \|x_\alpha\| = 0$. Examples of such spaces are c_0, L^p for $1 \le p < \infty$, and all reflexive Banach lattices. Order continuity of the norm is equivalent to the fact that for every relatively weakly compact set $C \subseteq E_+$, the *solid hull* $\mathrm{so}C := \{y \in E : |y| \le x \text{ for some } x \in C\}$ is relatively weakly compact as well (see [2, 13.8]), or that every closed ideal in E is a projection band (see [26, II.5.14]).

COROLLARY 2.4. *Let E be a Banach lattice with order continuous norm and let $S, T \in \mathscr{L}(E)$ such that $0 \le S \le T$ and T is Abel ergodic. Then $P\sigma(S) \cap r(T)\Gamma \subseteq P\sigma(T) \cap r(T)\Gamma$.*

PROOF. We have to consider only the case $r(T) > 0$ (see the remark after Theorem 2.2) and without loss of generality we may assume $r(T) = 1$. Now let $\alpha \in \Gamma$. If $\lambda > 1$ and $x \in E$ then

$$\begin{aligned}|(\lambda - 1)R(\lambda, \alpha^{-1}T)x| &\le (\lambda - 1)\sum\nolimits_{n\ge 0} \lambda^{-(n+1)}|\alpha^{-n}T^n||x| \\ &\le (\lambda - 1)R(\lambda, T)|x|.\end{aligned}$$

Since T is Abel ergodic, T and hence $\alpha^{-1}T$ satisfies (G). Moreover $C := \{(\lambda - 1)R(\lambda, T)|x| : 1 < \lambda \le 2\}$ is relatively compact and $D := \{(\lambda - 1)R(\lambda, \alpha^{-1}T)x : 1 < \lambda \le 2\}$ is contained in the solid hull of C. The order continuity of the norm then implies that D is relatively weakly compact. Thus $\alpha^{-1}T$ is Abel ergodic by Proposition 2.1. The assertion follows now from Theorem 2.2.

The next lemma is a pointwise version of Corollary 2.4.

LEMMA 2.5. *Let E be a Banach lattice with order continuous norm and let $S, T \in \mathscr{L}(E)$ be such that $0 \le S \le T$ and T satisfies (G). Let $\alpha \in r(T)\Gamma$ and $0 \ne x \in E$ be such that $Sx = \alpha x$. If $\lim_{\lambda\downarrow r(T)}(\lambda - r(T))R(\lambda, T)|x|$ exists in E, then $\alpha \in P\sigma(T)$.*

PROOF. If $r(T) = 0$, then by the remark after Theorem 2.2 we have $T = 0$, and hence $0 \in P\sigma(T)$. Now let $r(T) > 0$. Without loss of generality we may assume $r(T) = 1$. Then $|x| \le S|x| \le T|x|$. Since T satisfies (G) there exists $x' \in E'_+$ such that $T'x' = x'$ and $\langle x', |x|\rangle > 0$ (see [26, V.4.8]). Let $N(x') := \{y \in E : \langle x', |y|\rangle = 0\}$. Since E has order continuous norm the closed ideal $N(x')$ is a projection band (see [26, II.5.14]), and hence $E = N(x') \oplus N(x')^\perp$, where $N(x')^\perp := \{u \in E : \inf(|u|, |v|) = 0 \text{ for all } v \in N(x')\}$. Let Q be the band projection from E onto $N(x')^\perp$. Since $0 \le S'x' \le T'x' = x'$ the operators S and T leave $N(x')$ invariant, that is, $S(I-Q)E \subseteq (I-Q)E$ and $T(I-Q)E \subseteq (I-Q)E$. If we represent

S and T as operator matrices according to the decomposition $E = N(x') \oplus N(x')^{\perp}$, we obtain

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

where $S_3 = QS_{|QE}$ and $T_3 = QT_{|QE}$. In particular $0 \le S_3 \le T_3$. Let $\tilde{x}' := x'_{|QE}$. Then $\tilde{x}' \in (QE)'$ is strictly positive and for $y \in QE$ we have

$$\begin{aligned} \langle \tilde{x}', T_3 y \rangle &= \langle x', QTy \rangle = \langle x', QTy \rangle + \langle x', (I - Q)Ty \rangle \\ &= \langle x', Ty \rangle = \langle x', y \rangle = \langle \tilde{x}', y \rangle; \end{aligned}$$

that is, $T_3'\tilde{x}' = \tilde{x}'$. Let $\tilde{x} := Qx$. From $|x| = |Qx| + |(I - Q)x|$ it follows that

$$\langle \tilde{x}', |\tilde{x}| \rangle = \langle x', |Qx| + |(I - Q)x| \rangle = \langle x', |x| \rangle > 0,$$

and hence $\tilde{x} \neq 0$. The S-invariance of $N(x')$ yields $S_3 Qx = QSx$, and hence $S_3\tilde{x} = \alpha\tilde{x}$. The remark after Lemma 1.3 then implies $T_3\tilde{x} = \alpha\tilde{x}$.

Let now $\lambda > 1$ and $y \in E$. Then

$$|R(\lambda, \alpha^{-1}T)y| = \left| \sum_{n \ge 0} \frac{T^n y}{\alpha^n \lambda^{n+1}} \right| \le \sum_{n \ge 0} \frac{T^n |y|}{\lambda^{n+1}} = R(\lambda, T)|y|.$$

Thus

$$\sup_{\lambda > 1} \|(\lambda - 1)R(\lambda, \alpha^{-1}T)\| \le \sup_{\lambda > 1} \|(\lambda - 1)R(\lambda, T)\| < \infty \quad \text{and}$$
$$\{(\lambda - 1)R(\lambda, \alpha^{-1}T)x : 1 < \lambda \le 2\} \subseteq \operatorname{so}\{(\lambda - 1)R(\lambda, T)|x| : 1 < \lambda \le 2\}.$$

Since the limit $\lim_{\lambda \downarrow 1}(\lambda - 1)R(\lambda, T)|x|$ exists and E has order continuous norm, $\{(\lambda - 1)R(\lambda, \alpha^{-1}T)x : 1 < \lambda \le 2\}$ is relatively weakly compact. Proposition 2.1 then implies that $z := \lim_{\lambda \downarrow 1}(\lambda - 1)R(\lambda, \alpha^{-1}T)x$ exists in E, and $Tz = \alpha z$.

It remains to show $z \neq 0$. The T-invariance of $N(x')$ implies $T_3 Qy = QTy$ for every $y \in E$. Hence $T_3^n Qy = QT^n y$ for all $n \in \mathbb{N}$ and $y \in E$. Thus

$$\begin{aligned} Qz &= \lim_{\lambda \downarrow 1}(\lambda - 1) \sum_{n \ge 0} \frac{Q(\alpha^{-1}T)^n x}{\lambda^{n+1}} = \lim_{\lambda \downarrow 1}(\lambda - 1) \sum_{n \ge 0} \frac{\alpha^{-n} T_3^n \tilde{x}}{\lambda^{n+1}} \\ &= \lim_{\lambda \downarrow 1}(\lambda - 1) \sum_{n \ge 0} \frac{\tilde{x}}{\lambda^{n+1}} = \tilde{x} \neq 0, \end{aligned}$$

and hence $z \neq 0$.

If in Theorem 2.2 the Banach lattice E is a KB-space we can further relax the conditions on T. Recall that a Banach lattice E is a *KB-space* if E is a (projection) band in its bidual E''. In this case every increasing uniformly bounded sequence (x_n) in E_+ converges in norm (see [26, II.5.15]). Note that every KB-space has order continuous norm. Examples of KB-spaces are L^p for $1 \leq p < \infty$, and all reflexive Banach lattices.

THEOREM 2.6. *Let E be a KB-space and let $S, T \in \mathscr{L}(E)$ be such that $0 \leq S \leq T$ and T satisfies* (G). *Then $P\sigma(S) \cap r(T)\Gamma \subseteq P\sigma(T) \cap r(T)\Gamma$.*

PROOF. If $r(T) = 0$ the assertion follows from the remark after Theorem 2.2. As above, the case $r(T) > 0$ can be reduced to the case $r(T) = 1$. Let $Sx = \alpha x$ for $\alpha \in \Gamma$ and $0 \neq x \in E$. Then $|x| \leq S|x| \leq T|x|$. Hence the sequence $(T^n|x|)$ is increasing. On the other hand, if $\lambda > 1$, then $(\lambda - 1)R(\lambda, T)|x| \geq (\lambda - 1)\sum_{m \geq n} T^m|x|/\lambda^{m+1} \geq \lambda^{-n}T^n|x|$ for every $n \in \mathbb{N}$. Property (G) implies that $(T^n|x|)$ is uniformly bounded. Since E is a KB-space $y := \lim_n T^n|x|$ exists in E and $|x| \leq y = Ty$. Thus if $\lambda > 1$ then $0 \leq (\lambda - 1)R(\lambda, T)|x| \leq y$. Hence $\{(\lambda - 1)R(\lambda, T)|x| : \lambda > 1\}$ is contained in the order interval $[0, y]$ which is weakly compact (see [26, II.5.10]). By Proposition 2.1 the limit $\lim_{\lambda \downarrow 1}(\lambda - 1)R(\lambda, T)|x|$ exists. Now Lemma 2.5 implies $\alpha \in P\sigma(T)$ and the proof is finished.

The following example shows that in Theorem 2.2 and Corollary 2.4 the condition on T (Abel ergodicity) and in Theorem 2.6 the condition on E (KB-space) cannot be omitted.

EXAMPLE 2.7. Let $E = c_0$ be the space of all sequences converging to 0. Define operators S and T on E by $Sx := (\xi_1, 0, \xi_2, \xi_3, \dots)$ and $Tx := (\xi_1, \xi_1, \xi_2, \xi_3, \dots)$, $x = (\xi_n) \in E$. Then $0 \leq S \leq T$, $\|T\| = 1$ and $Se_1 = e_1$ where $e_1 = (1, 0, 0, \dots)$. In particular $1 \in P\sigma(S)$. On the other hand, let $x = (\xi_n) \in E$ be such that $Tx = x$. Then $\xi_1 = \xi_2 = \dots$. However, the only constant sequence belonging to E is the zero sequence, hence $1 \notin P\sigma(T)$. Since $(\lambda - 1)R(\lambda, T)e_1$ does not converge as $\lambda \downarrow 1$ the operator T is not Abel ergodic. Finally, if $\alpha \in \Gamma \setminus \{1\}$, then an easy computation shows that 1 is not an eigenvalue of $\alpha T'$. Thus, by [16, Theorem 2.1.4, Theorem 2.1.5], the operator αT is Abel ergodic for each $\alpha \in \Gamma \setminus \{1\}$.

From the results on the point spectrum we can deduce inheritance properties for the residual spectrum. In fact, if $T \in \mathscr{L}(E)$ is an operator on a Banach space E, consider $R\sigma(T) := \{\lambda \in \mathbb{C} : (\lambda - T)E \text{ is not dense in } E\}$, the *residual spectrum* of T. Then $R\sigma(T) = P\sigma(T')$ by the Hahn-Banach Theorem. Thus Theorem 2.6 leads to the following result.

THEOREM 2.8. *Let E be a Banach lattice such that E' has order continuous norm and let $S, T \in \mathscr{L}(E)$ be such that $0 \leq S \leq T$, and T satisfies (G). Then $R\sigma(S) \cap r(T)\Gamma \subseteq R\sigma(T) \cap r(T)\Gamma$.*

PROOF. The assumptions on S and T imply that $0 \leq S' \leq T'$ and T' satisfies (G). If E' has order continuous norm, then E' is already a KB-space (see [19, 2.4.14]). Thus Theorem 2.6 yields

$$R\sigma(S) \cap r(T)\Gamma = P\sigma(S') \cap r(T)\Gamma \subseteq P\sigma(T') \cap r(T)\Gamma = R\sigma(T) \cap r(T)\Gamma.$$

REMARK. One also obtains analogues of Theorem 2.2 and Corollaries 2.3 and 2.4 for the residual spectrum.

3. The essential spectrum of dominated operators

If $0 \leq S \leq T$ are operators on a Banach lattice E and $r(T)$ is a Riesz point of T, then by a result of Caselles [5, Theorem 4.1] either $r(S) < r(T)$ or $r(T)$ is a Riesz point of S. In this section we show that an analogous conclusion holds for every operator S which is *dominated by* T, that is, such that $|Sx| \leq T|x|$ for all $x \in E$.

Let us make this more precise. For an operator $T \in \mathscr{L}(E)$ on a Banach space E let $\Phi(T) := \{\lambda \in \mathbb{C} : \ker(\lambda - T)$ and $E/(\lambda - T)E$ are finite dimensional$\}$ be the *Fredholm domain*, $\sigma_{\text{ess}}(T) := \mathbb{C} \setminus \Phi(T)$ the *(Wolf) essential spectrum*, and $r_{\text{ess}}(T) := \sup\{|\lambda| : \lambda \in \sigma_{\text{ess}}(T)\}$ the *essential spectral radius* of T. If $\sigma_{\text{ess}}(T) = \emptyset$, we set $r_{\text{ess}}(T) = -\infty$. It is well-known that $\sigma_{\text{ess}}(T) \subseteq \sigma(T)$ is compact and $\sigma_{\text{ess}}(T) \neq \emptyset$ if E is infinite dimensional (see [11, XI, p.205]). Recall that $\lambda \in \sigma(T)$ is a *Riesz point* of T if λ is a pole of the resolvent map with residuum of finite rank. It turns out that $\{\lambda \in \sigma(T) : |\lambda| > r_{\text{ess}}(T)\}$ contains only Riesz points (see [11, XI.8.4]). Conversely, every Riesz point of T belongs to $\sigma(T) \setminus \sigma_{\text{ess}}(T)$ (see [11, XI.5.3]). If $r(T) = 0$ and 0 is a Riesz point, then T is nilpotent and hence E is finite dimensional (notice that the Neumann series is the Laurent expansion of the resolvent $R(\,.\,, T)$).

If T is a positive operator on a Banach lattice E and $r(T)$ is a Riesz point of T, then by a result of Niiro and Sawashima all elements of $\sigma(T) \cap r(T)\Gamma$ are poles of the resolvent. An inspection of the proof given by Lotz and Schaefer (see [26, V.5.5]) even shows that $\sigma(T) \cap r(T)\Gamma$ consists entirely of Riesz points (see also [18, Corollary 2.3]). Now the result of Caselles [5, Theorem 4.1] reads as follows (see also [18, Proposition 2.5]):

PROPOSITION. *If $0 \leq S \leq T$ are operators on a Banach lattice E such that $r(T)$ is a Riesz point of T, then $r_{\text{ess}}(S) < r(T)$.*

Our aim is to prove the following generalization of Caselles' result.

THEOREM 3.1. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ be operators such that S is dominated by T, and $r(T)$ is a Riesz point of T. Then $r_{\mathrm{ess}}(S) < r(T)$. In particular, $\sigma(S) \cap r(T)\Gamma$ contains only Riesz points.*

The proof of Theorem 3.1 is divided into several 'auxiliary results'. Our first lemma is due to Greiner [12, Proposition 1.32] (see also [6, Lemma 8.9]) and has its origin in a result of Schaefer [26, V.5.1, V.7.4]. If E is a Banach lattice and $z \in E_+$, then E_z denotes the ideal generated by z endowed with the norm $p_z(x) := \inf\{r > 0 : |x| \le rz\}$. The space E_z is a Banach lattice (see [26, II.7.2]). Moreover there is an isometric lattice isomorphism from E_z onto a space $C(K)$, K compact, which maps z to 1_K (see [26, II.7.2, II.7.4]).

LEMMA 3.2. *Let $S, T \in \mathscr{L}(E)$ be operators on a Banach lattice E such that S is dominated by T. Suppose there is $\alpha \in \Gamma$ and $0 \neq z \in E$ such that $Sz = \alpha z$ and $T|z| = |z|$. Then there is a surjective isometry $V \in \mathscr{L}(\overline{E_{|z|}})$ such that $Sx = \alpha V T V^{-1} x$ for all $x \in \overline{E_{|z|}}$.*

If T is an operator on a Banach space E, $G \subseteq E$ a closed T- invariant subspace and $\lambda \in \sigma(T) \cap r(T)\Gamma$ a Riesz point of T, then Lemma 1.1 implies that λ is a Riesz point or belongs to the resolvent set of the induced operators $T_|$ and $T_/$ on G and E/G, respectively. In case G is an ideal in a Banach lattice E, Caselles [5, Lemma 4.4] has shown that the converse is true. We formulate his result in a slightly different form.

LEMMA 3.3. *Let E be a Banach lattice, $T \in \mathscr{L}(E)$, $I \subseteq E$ a closed T-invariant ideal, and $T_|$ and $T_/$ the induced operators on I and E/I, respectively. Suppose that $\lambda \in \mathbb{C}$ is a Riesz point of $T_|$ and $T_/$, or a Riesz point of either $T_|$ or $T_/$ and belongs to the resolvent set of the other operator. Then λ is a Riesz point of T.*

Now we prove a special case of Theorem 3.1. Recall that a positive operator T on a Banach lattice E is *irreducible* if $\{0\}$ and E are the only closed T-invariant ideals in E. We call $u \in E_+$ a *topological order unit* if the ideal generated by u is dense in E. For $z' \in E'$ and $z \in E$ we denote by $z' \otimes z$ the operator given by $(z' \otimes z)x := \langle z', x\rangle z$, $x \in E$.

LEMMA 3.4. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ be operators such that S is dominated by T, $r(T)$ is a Riesz point of T, and T is irreducible. Then $r_{ess}(S) < r(T)$.*

PROOF. If E is finite dimensional there is nothing to prove. Otherwise we have $r(T) > 0$ and without loss of generality we may assume $r(T) = 1$. If $r(S) < 1$

the assertion holds. Now let $r(S) = 1$ and $\lambda \in \sigma(S) \cap \Gamma$. Consider an ultrapower $E_{\mathcal{U}}$ of E and the operator $S_{\mathcal{U}}$ induced by S. Then $\lambda \in P\sigma(S_{\mathcal{U}})$ by Lemma 1.2. Choose $0 \neq \hat{v} \in E_{\mathcal{U}}$ such that $S_{\mathcal{U}}\hat{v} = \alpha\hat{v}$. Clearly, $S_{\mathcal{U}}$ is dominated by $T_{\mathcal{U}}$. The irreducibility of T implies that 1 is a pole of order 1 of $R(., T)$ and the residuum P at 1 is given by $P = z' \otimes u$, where $z' \in E'_+$ is strictly positive and $u \in E_+$ is a topological order unit (see [26, V.5.1, V.5.2]). Lemma 1.2 implies that 1 is a pole of order 1 of $R(., T_{\mathcal{U}})$ with residuum $P_{\mathcal{U}} = \hat{z}' \otimes \hat{u}$, where $\hat{u} := (u, u, \dots)\hat{} \in E_{\mathcal{U}}$ and $\hat{z}' \in (E_{\mathcal{U}})'_+$ is given by $\langle \hat{z}', \hat{x} \rangle := \lim_{\mathcal{U}} \langle z', x_n \rangle$, $\hat{x} = \widehat{(x_n)} \in E_{\mathcal{U}}$. Thus $P_{\mathcal{U}}$ has rank 1, and hence 1 is a Riesz point of $T_{\mathcal{U}}$.

Since $T'_{\mathcal{U}}\hat{z}' = \hat{z}'$ the closed ideal $I := \{\hat{x} \in E_{\mathcal{U}} : \langle \hat{z}', |\hat{x}| \rangle = 0\} \subseteq E_{\mathcal{U}}$ is invariant for $T_{\mathcal{U}}$ and $S_{\mathcal{U}}$, respectively. Let $(T_{\mathcal{U}})_|$ and $(S_{\mathcal{U}})_|$ be the induced operators on I. From $P_{\mathcal{U}} I = \{0\}$ and Lemma 1.1 it follows that $1 \in \rho((T_{\mathcal{U}})_|)$. Since $(T_{\mathcal{U}})_|$ is positive this implies $r((T_{\mathcal{U}})_|) < 1$ (see [26, V.4.1]), and hence $r((S_{\mathcal{U}})_|) < 1$. In particular $\hat{v} \notin I$.

Let $W := (T_{\mathcal{U}})_/$ and $U := (S_{\mathcal{U}})_/$ be the induced operators on $F := E_{\mathcal{U}}/I$ and let z and y be the canonical images of $\hat{v}$ and $\hat{u}$ in F, respectively. Then U is dominated by W, 1 is a Riesz point of W with corresponding residuum $Q := (P_{\mathcal{U}})_/ = \hat{z}' \otimes y$, and $0 \neq \alpha z = Uz$. Since $|\hat{v}| = |S_{\mathcal{U}}\hat{v}| \leq T_{\mathcal{U}}|\hat{v}|$ and $T'_{\mathcal{U}}\hat{z}' = \hat{z}'$ we obtain $T_{\mathcal{U}}|\hat{v}| - |\hat{v}| \in I$, and hence $W|z| = |z| = \lambda y$ for some $\lambda > 0$.

Let J be the closed ideal in F generated by y. Then J is invariant for W, Q and U. Let $W_|$, $U_|$ and $W_/$, $Q_/$, $U_/$ be the induced operators on J and F/J, respectively. Then $Q_/ = 0$, and hence $1 \in \rho(W_/)$. Since $W_/$ is positive and $U_/$ is dominated by $W_/$ we obtain $r(U_/) \leq r(W_/) < 1$. On the other hand, 1 is a Riesz point of $W_|$ and $W_|$, $U_|$, α and z satisfy the assumptions of Lemma 3.2. Thus the operators $U_|$ and $\alpha W_|$ are similar, and hence α is a Riesz point of $U_|$. Now Lemma 3.3 and $r(U_/) < 1$ imply that α is a Riesz point of $U = (S_{\mathcal{U}})_/$. Since $r((S_{\mathcal{U}})_|) < 1$ by the same argument, we obtain that α is a Riesz point of $S_{\mathcal{U}}$. Thus α is a Riesz point of $S = S_{\mathcal{U}|E}$.

Now we prove Theorem 3.1. We follow the lines of a proof of Lotz and Schaefer (see [26, V.5.5] and [5, Theorem 4.1]).

PROOF OF THEOREM 3.1. If E is finite dimensional the assertion is obvious. Now let E be infinite dimensional. Since $r(T)$ is a Riesz point we have $r(T) > 0$. Then without loss of generality we may assume $r(T) = 1$. The proof is now divided into three steps.

(1) We first assume that $r(T) = 1$ is a pole of order one of $R(., T)$ and its residuum P is strictly positive, that is, $Px \in E_+ \setminus \{0\}$ for all $x \in E_+ \setminus \{0\}$. Then $PE = \mathrm{Fix}(T)$ (see [8, Theorem 2.17]) and from [26, III.11.5] it follows that PE is a finite dimensional sublattice of E. Thus PE is the linear span of normalized, mutually orthogonal vectors $e_1, \dots, e_n \in (PE)_+$. Let J_k, $1 \leq k \leq n$, be the closed ideal in E generated by e_k. Then $TJ_k \subseteq J_k$ and by [26, III.8.5] each $T_k := T_{|J_k}$ is irreducible.

Since S is dominated by T, each J_k is invariant for S. Hence we can apply Lemma 3.4 to T_k and $S_k := S_{|J_k}$ and obtain $r_{\mathrm{ess}}(S_k) < 1$ for $1 \leq k \leq n$. Now $J := \sum_{k=1}^{n} J_k$ is a closed ideal (see [26, III.1.2]) which is invariant for T and S. Since $PE \subseteq J$ and T is positive, the induced operator $T_/$ on E/J satisfies $r(T_/) < 1$. Therefore the same holds for the induced operator $S_/$ on E/J. On the other hand, $r_{\mathrm{ess}}(S_{|J}) < 1$ by the foregoing reasoning. Hence the assertion follows from Lemma 3.3.

(2) Next, let $r(T) = 1$ be a pole of order one of $R(., T)$ with (not necessarily strictly positive) residuum P. Since $TP = PT$, the ideal $J := \{x \in E : P|x| = 0\}$ is invariant under T and $r(T_{|J}) < 1$. Thus $SJ \subseteq J$ and $r(S_{|J}) < 1$. For the induced operators $T_/$ and $S_/$ on E/J we are in the situation of (1). Hence the assertion follows from Lemma 3.3.

(3) Finally, let $r(T) = 1$ be a pole of order $k > 1$. Then $Q := \lim_{\lambda\downarrow 1}(\lambda-1)^k R(\lambda, T)$ is a positive operator on E satisfying $Q^2 = 0$. Since $TQ = QT$ the ideal $J := \{x \in E : Q|x| = 0\}$ is T-invariant. For the induced operators $T_|$ and $T_/$ on J and E/J, respectively, we obtain that 1 is a pole of order $k-1$ of $R(., T_|)$ and a pole of order 1 of $R(., T_/)$. Moreover, $SJ \subseteq J$ and the induced operators $S_|$ and $S_/$ on J and E/J are dominated by $T_|$ and $T_/$, respectively. Thus the assertion follows by induction over k and applying Lemma 3.3.

4. Asymptotic properties of dominated operators

In this section we apply the previous results to investigate inheritance of asymptotic properties. Recall that by the theorem of Katznelson-Tzafriri [15, Theorem 1] an operator T on a Banach space E with uniformly bounded powers T^n satisfies $\lim_n \|T^n - T^{n+1}\| = 0$ if and only if $\sigma(T) \cap \Gamma \subseteq \{1\}$. Now the following result is an immediate consequence of Theorem 1.4.

THEOREM 4.1. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ be such that $0 \leq S \leq T$, $\sup_n \|T^n\| < \infty$ and $\lim_n \|T^n - T^{n+1}\| = 0$. Then $\lim_n \|S^n - S^{n+1}\| = 0$.*

In [24] and [25] it is shown that for operators $0 \leq S \leq T$ on a Banach lattice E with order continuous norm strong convergence of (T^n) to a projection P_T of finite rank implies strong convergence of (S^n). We will see that the rank condition on P_T can be replaced by a spectral condition on T.

At first we prove an inheritance result for a property which is slightly more general than strong convergence of the powers T^n, $n \in \mathbb{N}$. An operator T on a Banach space E is called *almost periodic*, if $\{T^n x : n \in \mathbb{N}\}$ is relatively compact for all $x \in E$. In this case the Jacobs-Glicksberg-deLeeuw splitting theorem (see [16, §2.4]) yields a decomposition $E = E_0 \oplus E_r$ of E where

$$E_0 = E_0(T) = \{x \in E : \lim_n T^n x = 0\} \quad \text{and}$$

$$E_r = E_r(T) = \overline{\text{lin}}\{x \in E : Tx = \alpha x \text{ for some } \alpha \in \Gamma\}.$$

Now we obtain the following inheritance result for almost periodicity (see [24, Proposition 3.10] and [25, Theorem 4.6]). Notice that we do not impose any restriction on the projection Q_T from E onto E_r.

THEOREM 4.2. *Let E be a Banach lattice with order continuous norm and let $S, T \in \mathscr{L}(E)$ be such that $0 \leq S \leq T$ and T is almost periodic. If $\sigma(T) \cap \Gamma \neq \Gamma$, then S is almost periodic.*

PROOF. By the uniform boundedness principle, $\sup_n \|S^n\| \leq \sup_n \|T^n\| < \infty$, and hence $r(T) \leq 1$. If $r(T) < 1$, then $r(S) \leq r(T) < 1$, which implies $\lim_n \|S^n\| = 0$. Thus we may assume $r(T) = 1$. By a result of Lotz (see [26, V.4.9]) $\sigma(T) \cap \Gamma$ is cyclic, that is, $\lambda \in \sigma(T) \cap \Gamma$ implies $\lambda^n \in \sigma(T)$ for all $n \in \mathbb{Z}$. Since $\sigma(T) \cap \Gamma$ is not the whole unit circle it must be a finite union of finite subgroups of Γ. Hence there exists $m \in \mathbb{N}$ such that $\sigma(T^m) \cap \Gamma = \{1\}$. By Theorem 1.4 we have $\sigma(S^m) \cap \Gamma \subseteq \sigma(T^m) \cap \Gamma = \{1\}$. On the other hand, $\{T^{mn}x : n \in \mathbb{N}\}$ is relatively compact for $x \in E$. Since E has order continuous norm $\{S^{mn}x : n \in \mathbb{N}\} \subseteq \text{so}\{T^{mn}|x| : n \in \mathbb{N}\}$ is relatively weakly compact for all $x \in E$. If $\lambda > 1$ then $(\lambda - 1)R(\lambda, S^m)x$ is in the closed convex hull of $\{S^{mn}x : n \in \mathbb{N}\}$ which is again weakly compact by Eberlein's theorem. Then Proposition 2.1 implies that S^m is Abel ergodic and $E = \text{Fix}(S^m) \oplus \overline{(I - S^m)E}$. Since $\sigma(S^m) \cap \Gamma \subseteq \{1\}$, the Katznelson-Tzafriri theorem (see [15, Theorem 1]) yields $\lim_n \|S^{mn} - S^{m(n+1)}\| = 0$. Thus $\lim_n S^{mn}x = 0$ for all $x \in \overline{(I - S^m)E}$, and hence $(S^{mn})_{n \in \mathbb{N}}$ is strongly convergent. Thus $\{S^n x : n \in \mathbb{N}\} \subseteq \bigcup_{k=1}^m S^k\{S^{m(n-1)}x : n \in \mathbb{N}\}$ is relatively compact for all $x \in E$.

If (T^n) is strongly convergent we obtain strong convergence of (S^n).

COROLLARY 4.3. *Let E be a Banach lattice with order continuous norm and let $S, T \in \mathscr{L}(E)$ be such that $0 \leq S \leq T$ and (T^n) is strongly convergent. If $\sigma(T) \cap \Gamma \neq \Gamma$, then (S^n) is strongly convergent.*

PROOF. By Theorem 4.1 the operator S is almost periodic. Then the Jacobs-Glicksberg-deLeeuw decomposition yields $E = E_0(S) \oplus E_r(S)$. By Corollary 2.3 we have $P\sigma(S) \cap \Gamma \subseteq P\sigma(T) \cap \Gamma \subseteq \{1\}$. Thus $E_r(S) = \text{Fix}(S)$. Hence (S^n) is strongly convergent.

If (T^n) is uniformly convergent, then $1 \in \rho(T)$ or 1 is isolated in $\sigma(T)$ (see [16, 2.2.7]). So we obtain the following result.

COROLLARY 4.4. *Let E be a Banach lattice with order continuous norm and let $S, T \in \mathscr{L}(E)$ be such that $0 \leq S \leq T$ and (T^n) is uniformly convergent. Then (S^n) is strongly convergent.*

REMARKS. (a) In Corollary 4.4 one cannot expect a better convergence of the sequence (S^n) (see [23, 2.6 Remark (a)]).

(b) We do not know if the conclusion of Theorem 4.2 and Corollary 4.3 still holds without the spectral condition on T. At least in that case one knows that $\{S^n x : n \in \mathbb{N}\}$ is relatively weakly compact for all $x \in E$, that is, S is *weakly almost periodic*.

We conclude with an application of Theorem 3.1. Recall that an operator T on a Banach space E is *uniformly ergodic* if the Cesaro means $T_n := \sum_{k=0}^{n-1} T^k/n$, $n \in \mathbb{N}$, are uniformly convergent. The limit $P_T := \lim_n T_n$ is called the *ergodic projection* corresponding to T. It is well known that T is uniformly ergodic if and only if $\lim_n \|T^n\|/n = 0$ and 1 is a pole of the resolvent $R(., T)$ (see [8, Theorem 3.16]). In this case P_T coincides with the spectral projection corresponding to the spectral set $\{1\}$ (see [8, Theorem 2.23]). Thus T is uniformly ergodic with ergodic projection of finite rank if and only if $\lim_n \|T^n\|/n = 0$ and 1 is a Riesz point of T.

Now we obtain the following generalization of a result of Caselles [5, Corollary 4.6].

THEOREM 4.5. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ be operators such that S is dominated by T. If T is uniformly ergodic with ergodic projection of finite rank, then S is uniformly ergodic with ergodic projection of finite rank.*

PROOF. Our assumptions imply $r(S) \leq r(T) \leq 1$. If $r(S) < 1$ there is nothing to prove. If $r(S) = r(T) = 1$, then 1 is a Riesz point of T. Theorem 3.1 yields $r_{\text{ess}}(S) < 1$. In particular 1 is a Riesz point of S. On the other hand, $\|S^n\| \leq \|T^n\|$, $n \in \mathbb{N}$, and hence $\lim_n \|S^n\|/n = 0$. Thus the assertion follows from the previous discussion.

FINAL REMARK. The authors obtained corresponding results for pseudo-resolvents. This is the subject of a forthcoming paper.

References

[1] C. D. Aliprantis and O. Burkinshaw, 'On weakly compact operators on Banach lattices', *Proc. Amer. Math. Soc.* **83** (1981), 573–578.

[2] ———, *Positive operators* (Academic Press, London, 1985).

[3] F. Andreu, V. Caselles, J. Martinez and J. M. Mazon, 'The essential spectrum of AM-compact operators', *Indag. Math.* (N.S.) **2** (1991), 149–158.

[4] A. V. Bukhvalov, 'Integral representations of linear operators', *J. Soviet. Math.* **8** (1978), 129–137.

[5] V. Caselles, 'On the peripheral spectrum of positive operators', *Israel J. Math.* **58** (1987), 144–160.

[6] Ph. Clément, H. J. A. M. Heijmans, S. Angenent, C. J. van Duijn and B. de Pagter, *One-parameter semigroups* (North-Holland, Amsterdam, 1987).

[7] P. G. Dodds and D. H. Fremlin, 'Compact operators in Banach lattices', *Israel J. Math.* **34** (1979), 287–320.

[8] N. Dunford, 'Spectral theory. I Convergence to projections', *Trans. Amer. Math. Soc* **54** (1943), 185–217.

[9] W. F. Eberlein, 'Abstract ergodic theorems and weak almost periodic functions', *Trans. Amer. Math. Soc.* **67** (1949), 217–240.

[10] R. Emilion, 'Mean bounded operators and mean ergodic theorems', *J. Funct. Anal.* **61** (1985), 1–14.

[11] I. Gohberg, S. Goldberg and M. A. Kaashoek, *Classes of linear operators*, 1 (Birkhäuser, Basel, 1990).

[12] G. Greiner, *Über das Spektrum stark stetiger Halbgruppen positiver Operatoren* (Dissertation, Tübingen, 1980).

[13] W. Haid, *Sätze vom Radon-Nikodym-Typ für Operatoren auf Banachverbänden* (Dissertation, Tübingen, 1982).

[14] N. J. Kalton and P. Saab, 'Ideal properties of regular operators between Banach lattices', *Illinois J. Math.* **29** (1985), 382–400.

[15] Y. Katznelson and L. Tzafriri, 'On power bounded operators', *J. Funct. Anal.* **68** (1986), 313–328.

[16] U. Krengel, *Ergodic theorems* (de Gruyter, Berlin, 1985).

[17] J. Martinez, 'The essential spectral radius of dominated positive operators', *Proc. Amer. Math. Soc.* **118** (1993), 419–426.

[18] J. Martinez and J. M. Mazon, 'Quasi-compactness of dominated positive operators and C_0-semigroups', *Math. Z.* **207** (1991), 109–120.

[19] P. Meyer-Nieberg, *Banach lattices* (Springer, Berlin, 1991).

[20] B. de Pagter, 'The components of a positive operator', *Indag. Math.* **86** (1983), 229–241.

[21] B. de Pagter and A. R. Schep, 'Measures of non-compactness of operators on Banach lattices', *J. Funct. Anal.* **78** (1988), 31–55.

[22] F. Räbiger, *Absolutstetigkeit und Ordnungsabsolutstetigkeit von Operatoren*, Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Math.-Naturwiss. Klasse, Jahrgang 1991, 1. Abhandlung, 1–132, (Springer, Berlin, 1991).

[23] ———, 'Stability and ergodicity of dominated semigroups, I. The uniform case', *Math. Z.* **214** (1993), 43–54.

[24] ———, 'Stability and ergodicity of dominated semigroups, II. The strong case', *Math. Ann.* **297** (1993), 103–116.

[25] ———, 'Attractors and asymptotic periodicity of positive operators on Banach lattices', *Forum Math.* **7** (1995), 665–683.

[26] H. H. Schaefer, *Banach lattices and positive operators* (Springer, Berlin, 1974).

[27] A. R. Schep, *Kernel operators* (Ph.D. Thesis, University of Leiden, Netherlands, 1977).

[28] A. C. Zaanen, *Riesz spaces II* (North-Holland, Amsterdam, 1983).

Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
D-72076 Tübingen
Germany
frra@michelangelo.mathematik.uni-tuebingen.de
manfred.wolff@uni-tuebingen.de