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EQUIVARIANT HOLOMORPHIC MAPS INTO THE SIEGEL DISC AND THE METAPLECTIC REPRESENTATION

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Abstract

We restrict the metaplectic representation to subgroups G of the symplectic group associated to equivariant holomorphic maps into the Siegel disc. We describe the invariant subspaces of the decomposition, and reduce the problem to the decomposition of a space of 'harmonic' polynomials under the action of the maximal compact subgroup of G.

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0. Introduction

The decomposition of the metaplectic representation to subgroups of the symplectic group Sp_{2N} (more exactly to their pull-back in the metaplectic group which is a 2-fold covering of Sp_{2N}) has been determined explicitly by Kashiwara and Vergne (see [8]) for two examples:

(i) $G = \mathrm{Sp}_{2n}$ and $G' = O(k)$ as subgroups of Sp_{2nk}

(ii) $G = U(p,q)$ and $G = U(k)$ as subgroups of $U(pk, qk) \subset \mathrm{Sp}_{2(p+q)k}$.

G and G' form a dual reductive pair, as defined by Howe ([5]); moreover G' is compact, and G is of hermitian type, in the sence that the associated symmetric space of G is hermitian. In fact, as G' is compact, it commutes with the 1-dimensional center of a maximal compact subgroup of $\mathrm{Sp}_{2N}(\approx \mathbb{U}(N))$. This 1-dimensional center belongs to G, because G is the commutant of G', and yields the desired complex structure on the corresponding symmetric space G/K. These facts were tacitly used in [8], but the role of the hermitian space G/K in its bounded realization was even more transparent in a recent work [2] by Davidson, where the second example is again studied.

These remarks suggest it might be worthwhile to recast and extend at least part of these results in the context of equivariant holomorphic maps into the Siegel disc. The latter were studied and classified by Satake ([10, 11]), and offer intriguing examples which might deserve further study. We hope to give more details for specific examples in a further paper.

1. The Siegel disc

Let $\mathrm{Sym}_N(\mathbb{C})$ denote the space of $N \times N$ symmetric matrices with complex entries, and let

$$\Delta\ (= \Delta_N) = \{Z \in \mathrm{Sym}_N(\mathbb{C}) \mid \mathbb{I}_N - Z^*Z \gg 0\}. \tag{1.1}$$

Δ is a bounded symmetric domain, holomorphically equivalent to the Siegel half-plane and will be referred to as the Siegel disc.

To describe the group of holomorphic diffeomorphisms of Δ, let us first consider $\mathbb{C}^N$ equipped with the standard inner product $(\zeta\,|\,\eta) = \sum_{i=1}^N \zeta_i \bar{\eta}_i$, and consider the form $\mathrm{Im}(\zeta\,|\,\eta)$: this is a non-degenerate skew-symmetric form on the (real) vector space $\mathbb{C}^N$. An ($\mathbb{R}$-linear) transform g of $\mathbb{C}^N$ preserving this symplectic structure can be realized in a unique way as $g.\ \zeta = A\zeta + B\bar{\zeta}$ where A and B are $N \times N$ complex matrices which satisfy

$$\begin{cases} AA^* - BB^* = \mathbb{I}_N \\ A^tB - B^tA = 0, \end{cases} \tag{1.2}$$

or equivalently

$$\begin{cases} A^*A - {}^tB\bar{B} = \mathbb{I}_N \\ B^*A - {}^tA\bar{B} = 0. \end{cases} \tag{1.2'}$$

To such a transform we associate the $2N \times 2N$ complex matrix $g = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}$. Let also

$$\mathrm{Sp}_c = \left\{ \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \quad A, B \text{ satisfy (1.2) or (1.2')} \right\}. \tag{1.3}$$

Now Sp_c is isomorphic to the real symplectic group $\mathrm{Sp}_{2N}(\mathbb{R})$; on the other hand Sp_c is a real form of the complex symplectic group $\mathrm{Sp}_{2N}(\mathbb{C})$, where

$$\mathrm{Sp}_{2N}(\mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \begin{matrix} A^tD - B^tC = \mathbb{I}_N \\ A^tB = B^tA, \quad C^tD = D^tC \end{matrix} \right\}. \tag{1.4}$$

The conjugation σ of $\mathrm{Sp}_{2N}(\mathbb{C})$ with respect to Sp_c is $\sigma\colon \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{pmatrix}$.

Now, if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ belongs to $\mathrm{Sp}_{2N}(\mathbb{C})$, and if $Z \in \mathrm{Sym}_N(\mathbb{C})$, the holomorphic action $g(Z)$ is defined if $CZ + D$ is non-singular, and then

(1.5) $$g(Z) = (AZ + B)(CZ + D)^{-1}.$$

When $g = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}$ belongs to Sp_c, and $Z \in \Delta$, the condition is always satisfied, and (1.5) specializes to

(1.5′) $$g(Z) = (AZ + B)(\bar{B}Z + \bar{A})^{-1}.$$

Fix O as the origin in Δ. The stabilizer of the point O in Sp_c is the subgroup $\left\{ \begin{pmatrix} A & O \\ O & \bar{A} \end{pmatrix},\quad AA^* = \mathbb{I}_N \right\}$, which is isomorphic to $\mathbb{U}(N)$.

The Lie algebra of the stabilizer has a one-dimensional center generated by the element

(1.6) $$\frac{1}{2} J_0 = \begin{pmatrix} i/2\,\mathbb{I}_N & O \\ O & -i/2\,\mathbb{I}_N \end{pmatrix}.$$

With respect to the adjoint action of this element, the Lie algebra $\mathfrak{sp}_{2N}$ of $\mathrm{Sp}_{2N}(\mathbb{C})$ has the decomposition

(1.7)
$$\begin{aligned}\mathfrak{sp}_{2N} = {} & \left\{ \begin{pmatrix} O & Z \\ O & O \end{pmatrix},\ Z \in \mathrm{Sym}_N(\mathbb{C}) \right\} \oplus \left\{ \begin{pmatrix} X & O \\ O & -{}^tX \end{pmatrix},\ X \in M_N(\mathbb{C}) \right\} \\ & \oplus \left\{ \begin{pmatrix} O & O \\ W & O \end{pmatrix},\ W \in \mathrm{Sym}_N(\mathbb{C}) \right\}.\end{aligned}$$

There corresponds, at least on a dense open set, a decomposition for the elements of $\mathrm{Sp}_{2N}(\mathbb{C})$:

(1.8)
$$\begin{aligned} g &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{I}_N & BD^{-1} \\ O & \mathbb{I}_N \end{pmatrix} \begin{pmatrix} {}^tD^{-1} & O \\ O & \mathbb{I}_N \end{pmatrix} \begin{pmatrix} \mathbb{I}_N & O \\ D^{-1}C & \mathbb{I}_N \end{pmatrix}, \end{aligned}$$

where $q \in \mathrm{Sp}_{2N}(\mathbb{C})$ is such that D is non-singular.

On the open set $\{\det D \neq O\}$ define, for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

$$X^+(g) = BD^{-1}, \qquad X^-(g) = D^{-1}C, \qquad K(g) = D. \tag{1.9}$$

This clearly defines three holomorphic maps, with values the two first in $\mathrm{Sym}_N(\mathbb{C})$, the last one in $GL(n, \mathbb{C})$.

If, moreover $g = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in \mathrm{Sp}_c$, then the condition is always satisfied, and (1.9) specializes to:

$$X^+(g) = B\bar{A}^{-1}, \qquad X^-(g) = \bar{A}^{-1}\bar{B}, \qquad K(g) = \bar{A}. \tag{1.10}$$

By abuse of notation, we denote, for $Z \in \mathrm{Sym}_N(\mathbb{C})$, $\exp Z = \begin{pmatrix} \mathbb{I}_N & Z \\ O & \mathbb{I}_N \end{pmatrix}$. Now it is easily seen that if $Z \in \mathrm{Sym}_N(\mathbb{C})$ and $g \in \mathrm{Sp}_{2N}(\mathbb{C})$ is such that $g(Z)$ is defined, then

$$X^+(g \exp Z) = g(Z) = (AZ + B)(CZ + D)^{-1}. \tag{1.11}$$

Finally, let us introduce the canonical factor of automorphy

$$J(g, Z) = K(g \exp Z) = CZ + D. \tag{1.12}$$

This factor is easily seen to verify the following relation, whenever it makes sense:

$$J(gg', Z) = J(g, g'(Z))J(g', Z). \tag{1.13}$$

Let also $j(g, Z) = \det J(g, Z)$, and for $Z, W \in \mathrm{Sym}_N(\mathbb{C})$, define whenever it makes sense

$$K(Z, W) = J(\sigma(\exp W)^{-1}, Z)^{-1} = (1 - \bar{W}Z)^{-1} \tag{1.14}$$

and $k(Z, W) = \det K(Z, W)$.

Notice that K is well defined if Z and W belong to Δ.

2. The Fock model for the metaplectic representation

The Fock space $\mathscr{F}$ is the space of holomorphic functions $F: \mathbb{C}^N \to \mathbb{C}$, such that

$$\|F\|^2_{\mathscr{F}} = \int_{\mathbb{C}^N} |F(\zeta)|^2 d\mu(\zeta) < +\infty, \tag{2.1}$$

where $d\mu$ is the measure on $\mathbb{C}^N$ defined by

$$d\mu(\zeta) = e^{-\pi(\zeta|\zeta)} d\lambda(\zeta), \tag{2.2}$$

where $d\lambda$ is the ordinary $2n$-dimensional Lebesgue measure on $\mathbb{C}^N$.

The Fock space is a Hilbert space when equipped with $\|F\|_{\mathscr{F}}$ as a norm; it admits a reproducing kernel. Explicitly:

$$F(\zeta) = \int_{\mathbb{C}^N} F(\xi) e^{-\pi(\zeta|\xi)} d\mu(\xi), \tag{2.3}$$

for $F \in \mathscr{F}$ and $\zeta \in \mathbb{C}^N$.

The metaplectic representation is realized on the Fock space in a very convenient way ([1, 7, 4]). We only sketch the construction. First, consider the Heisenberg group $\mathbb{H}_{2N+1} = \mathbb{C}^N \times \mathbb{R}$, with the group law

$$(\zeta, s)(\xi, t) = (\zeta + \xi, s + t + \operatorname{Im}(\zeta|\xi)).$$

By the Stone–von Neumann theorem, an irreducible unitary representation U of $\mathbb{H}_{2N+1}$ which is given by a non-trivial character times the identity on the center $\{(0, s), s \in \mathbb{R}\}$ of $\mathbb{H}_{2N+1}$ is unique up to a unitary isomorphism.

The group Sp_c acts as a group of automorphisms of $\mathbb{H}_{2N+1}$: if $g \in \mathrm{Sp}_c$, define $g(\zeta, s) = (g.\zeta, s)$. So, for $g \in \mathrm{Sp}_c$, we may define a new representation U_g by $U_g(\zeta, s) = U(g.\zeta, s)$. As g fixes the center of $\mathbb{H}_{2N+1}$, it is clear that U_g is unitarily equivalent to U. It follows that there exists a unique (up to a complex factor of modulus 1) unitary operator T_g, such that $U_g \circ T_g = T_g \circ U$.

Now it is easily verified that if g and g' are two elements of Sp_c, then $T_g \circ T_{g'}$ is a unitary intertwining operator between $U_{gg'}$ and U, so that $T_{gg'} = c(g, g') T_g \circ T_{g'}$, where $c(g, g')$ is a complex number of modulus 1.

This produces a projective representation of Sp_c. It can be shown ([12, 9]) that it can be lifted to a unitary representation of a 2-fold covering of Sp_c, called the metaplectic group. It implies that one can choose T_g for $g \in \mathrm{Sp}_c$, such that $c(g, g') = \pm 1$.

The Fock space can be used to give a concrete realization of the unitary representation of the Heisenberg group (this amounts to using a complex polarization). Choose the character on the center to be $(0, s) \to e^{i\pi s}$, and let, for $F \in \mathscr{F}$ and $(\zeta, s) \in \mathbb{H}_{2N+1}$,

$$U(\zeta, s) F(\xi) = e^{i\pi s} e^{-\pi/2(\zeta|\zeta)} e^{\pi(\xi|\bar{\zeta})} F(\xi - \bar{\zeta}).$$

It is easily verified that this defines an irreducible unitary representation on $\mathscr{F}$, which satisfies $U(0, s) = e^{i\pi s} \mathbb{I}_{\mathscr{F}}$. In this context we have (see [1, 7, 4]):

PROPOSITION 1. *For* $g = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in \mathrm{Sp}_c$, *define the operator* T_g (*modulo* ± 1) *on* $\mathscr{F}$ *by*

$$(T_g F)(\xi) = \int_{\mathbb{C}^N} K_g(\xi, \eta) F(\eta)\, d\mu(\eta), \tag{2.4}$$

where

$$(2.5)\quad K_g(\xi,\eta) = \pm(\det \bar{A})^{-1/2} \exp \frac{\pi}{2}\left\{{}^t\xi B\bar{A}^{-1}\xi + 2(\bar{A}^{-1}\xi|\eta) - {}^t\bar{\eta}\bar{A}^{-1}\bar{B}\bar{\eta}\right\}.$$

Then for all $g, h \in \mathrm{Sp}_c$ and $(\xi, s) \in \mathbb{H}_{2N+1}$

(i) T_g *is unitary*
(ii) $T_g \circ U(\zeta, s) = U(g.\zeta, s) \circ T_g$
(iii) $T_g \circ T_h = \pm T_{gh}$.

REMARK 1. There are in fact two metaplectic representations, depending on the choice of the character on the center of $\mathbb{H}_{2N+1}$. The two representations are the contragradient of one another. Notice that, as our choices of the group law and the character differ from those in [1] and [4], we obtain a different result. The reason for this choice is to have holomorphic formulas (as opposed to anti-holomorphic) later on.

Now, using the Harish Chandra decomposition of Sp_c (formulas 1.10), we can re-write (2.5) as

$$(2.6)\, K_g(\xi,\eta) = \pm \det K(g)^{-1/2} \exp \frac{\pi}{2}\left\{{}^t\xi X^+(g)\xi + 2(K(g)^{-1}\xi|\eta) - {}^t\bar{\eta}X^-(g)\bar{\eta}\right\}.$$

REMARK 2. Formula (2.6) shows that the metaplectic representation has a holomorphic contituation to a large open set in $\mathrm{Sp}_{2N}(\mathbb{C})$. In [6] Howe found an interesting holomorphic semi-group which is part of our extension and which acts by contractions. See also related results in [3].

3. Equivariant holomorphic map into the Siegel disc

Let $\mathscr{D}$ be a hermitian symmetric space of the non-compact type, and assume G is a reductive Lie group acting holomorphically and transitively on $\mathscr{D}$. For technical reasons, we assume that G is the connected component of the identity in a Zariski-connected reductive algebraic $\mathbb{R}$-group. We fix an origin o in $\mathscr{D}$, let K be the stabilizer of o in G, and θ the Cartan involution of G such that the fixed points of θ is K.

A pair $(\rho, \rho_{\mathscr{D}})$ of an $\mathbb{R}$-homomorphism $\rho: G \to \mathrm{Sp}_c$ and a holomorphic map $\rho_{\mathscr{D}}: \mathscr{D} \to \Delta$ is an *equivariant holomorphic map* if the following conditions are satisfied:

$$(3.1)\qquad \rho_{\mathscr{D}}(g(z)) = \rho(g)(\rho_{\mathscr{D}}(z)), \quad \text{for } g \in G,\ z \in \mathscr{D}$$

and

$$\rho_{\mathscr{D}}(o) = O,$$
$$(3.2) \qquad \rho(\theta(g)) =_{\rho} (g)^*)^{-1}, \quad \text{for } g \in G.$$

These axioms imply that $\rho_{\mathscr{D}}$ is a totally geodesic map for the Bergman metrics. As G is of hermitian type, there exists an element H_0 in $\mathfrak{k} = \mathrm{Lie}\,(K)$, such that$H_0$ belongs to the center of $\mathfrak{k}$ and ad H_0 defines the complex structure on $\mathfrak{p} = \{X \in \mathfrak{g} = \mathrm{Lie}\,(G) \mid \theta X = -X\}$ which is identified with the tangent plane at o. The fact that $\rho_{\mathscr{D}}$ is holomorphic, together with (3.1) imply that

$$(3.3) \qquad \left[\rho(H_0) - \tfrac{1}{2}J_0,\ \rho(X)\right] = 0, \qquad \forall X \in \mathfrak{g}.$$

Although some of our results are true in this general situation, we require a stronger condition, called the (H_2) condition in [11], to which we refer for more details on equivariant holomorphic maps:

$$(3.4) \qquad \rho(H_0) = \tfrac{1}{2}J_0 \qquad \text{`}(H_2)\text{ condition'}.$$

Consider now the complexification of the whole situation. Still denote by ρ the extension of ρ to $G_{\mathbb{C}}$, the complex algebraic group corresponding to G (or more precisely to the algebraic group whose component of the identity is G) into $\mathrm{Sp}_{2N}(\mathbb{C})$, and let σ be the conjugation of $G_{\mathbb{C}}$ with respect to G. We may decompose $\mathfrak{g}_{\mathbb{C}}$ with respect to the action of ad H_0, and get

$$(3.5) \qquad \mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-,$$

where $\mathfrak{p}^+, \mathfrak{k}_{\mathbb{C}}, \mathfrak{p}^-$ are the eigenspaces of ad H_0 for the eigenvalues respectively $i, 0, -i$.

There is a global analogue, the Harish Chandra decomposition on a dense open subset of $G_{\mathbb{C}}$:

$$(3.6) \qquad g = \exp x^+(g)k(g)\exp x^-(g),$$

where $x^+(g) \in \mathfrak{p}^+, k(g) \in K_{\mathbb{C}}, x^-(g) \in \mathfrak{p}^-$.

The assumptions (and notably (3.3)) imply that the decompositions (3.5) and (3.6) are ρ-compatible, in the following sense:

$$(3.7) \qquad \begin{aligned} X^+(\rho(g)) &= \rho(x^+(g)),\\ K(\rho(g)) &= \rho(k(g)),\\ X^-(\rho(g)) &= \rho(x^-(g)), \quad g \in G. \end{aligned}$$

Finally, for $g \in G$, and $z \in \mathscr{D}$, introduce the automorphy factors (we again abuse notation):

$$J(g, z) = J(\rho(g), \rho_{\mathscr{D}}(z)), \tag{3.8}$$

$$j(g, z) = \det J(\rho(g), \rho_{\mathscr{D}}(z)), \tag{3.9}$$

where we use notation from Section 1.

Again, from the cocycle relation (1.13) and the assumptions on ρ, it is immediate that J and j satisfy the following property:

$$\begin{aligned} J(gg', z) &= J(g, g'(z))J(g', z), \\ j(gg', z) &= j(g, g'(z))j(g', z). \end{aligned} \tag{3.10}$$

We also set

$$K(z, w) = K(\rho_{\mathscr{D}}(z), \rho_{\mathscr{D}}(w)), \quad \text{and} \quad k(z, w) = k(\rho_{\mathscr{D}}(z), \rho_{\mathscr{D}}(w)). \tag{3.11}$$

We further assume that $\mathscr{D}$ is realized in the Harish Chandra imbedding, so we view $\mathscr{D}$ as an open set in $\mathfrak{p}^+$, and $\rho_{\mathscr{D}}$ is just the restriction to $\mathscr{D}$ of the linear map, still denoted by ρ, from $\mathfrak{p}^+$ into $\mathrm{Sym}_N(\mathbb{C})$ which is obtained from $\rho: \mathfrak{g} \to \mathfrak{sp}_c$ by complexification and restriction.

A last comment on the (H_2) condition (3.4): it implies that the commutant of $\rho(G)$ in Sp_c is compact; in fact the center of the maximal compact subgroup of Sp_c, which is $\left\{\begin{pmatrix} e^{i\theta}\mathbb{I}_N & O \\ O & e^{-i\theta}\mathbb{I}_N \end{pmatrix} = \exp\theta J_0, \theta \in \mathbb{R}/2\pi\mathbb{Z}\right\}$ belongs to $\rho(G)$, and its commutant in Sp_c is (isomorphic to) $\mathbb{U}(N)$. In other terms the map $\rho_{\mathscr{D}}$ is rigid (see [11]); conversely, it is easily seen that if $\rho_{\mathscr{D}}$ is rigid, that is if the commutant of $\rho(G)$ is compact, then one can enlarge G to a group $\tilde{G}$ such that the (H_2) condition is satisfied, and without changing the commutant in Sp_c.

4. Special gaussians and harmonic polynomials

One of the advantages of using the Fock model is that $\mathscr{F}$ has two families of elements which can be used for computations: the (holomorphic) gaussians and the (holomorphic) polynomials.

For $Z \in \mathrm{Sym}_N(\mathbb{C})$ define the associated gaussian Γ_Z by

$$\Gamma_Z(\xi) = e^{\pi/2 {}^t\xi Z \xi} \quad \text{for} \quad \xi \in \mathbb{C}^N. \tag{4.1}$$

Now Γ_Z belongs to $\mathscr{F}$ if and only if $Z \in \Delta$ (see [4]), and moreover

$$\|\Gamma_Z\|^2_{\mathscr{F}} = \det(1 - Z^*Z)^{-1/2}. \tag{4.2}$$

Denote by $\mathscr{P}$ the set of holomorphic polynomials on $\mathbb{C}^N$. If $p \in \mathscr{P}$, denote by $\partial(p)$ the coefficients differential operator on $\mathbb{C}^N$ such that

$$\partial(p)e^{\pi{}^t\xi\eta} = p(\eta)e^{\pi{}^t\xi\eta},$$

where $\partial(p)$ acts on the ξ variable. In other words we set $\partial(\xi_j) = (1/\pi)\partial/\partial\xi_j$ $(1 \leq j \leq N)$ and extend as an algebra homomorphism. Then we have the following expression for the inner product:

$$(4.3) \qquad (p|q)_{\mathscr{F}} = \partial(p)(\bar{q})(O).$$

Notation of Section 3 being in force, recall that ρ denotes (among other things) a $\mathbb{C}$-linear map from $\mathfrak{p}^+$ into $\mathrm{Sym}_N(\mathbb{C})$.

If $Z \in \mathrm{Sym}_N(\mathbb{C})$, denote by q_Z the associated quadratic polynomial $q_Z(\xi) = {}^t\xi Z\xi$.

For $z \in \mathfrak{p}^+$, set $q_z = q_{\rho(z)}$, so that

$$(4.4) \qquad q_z(\xi) = {}^t\xi\rho(z)\xi,$$

and let $\mathscr{I}$ be the subalgebra of $\mathscr{P}$ generated by the q_z, as z varies through $\mathfrak{p}^+$. Elements of $\mathscr{I}$ will be referred to as *special* polynomials.

Similarly, for $z \in \mathfrak{p}_+$, define the *special gaussian* Γ_z by $\Gamma_z = \Gamma_{\rho(z)}$. Clearly if $z \in \mathscr{D}$, $\Gamma_z \in \mathscr{F}$. Finally let $\mathscr{H}$ be the subspace of $\mathscr{P}$ defined by

$$(4.5) \qquad \mathscr{H} = \{p \in \mathscr{P}\partial(q_z)\bar{p} = O,\ \forall z \in \mathfrak{p}^+\}$$

Elements of $\mathscr{H}$ will be referred to as *harmonic* polynomials.

PROPOSITION 2. (a) $\mathscr{P} = \mathscr{I}\mathscr{H}$.

(b) *The space spanned by the products of a harmonic polynomial and a special gaussian is dense in* $\mathscr{F}$.

By using the standard graduation of $\mathscr{P}$ by the degree, it is clear that (a) is a statement for each $\mathscr{P}_m$ (= the space of homogeneous polynomials of degree m), so we may ignore any difficulty from using infinite dimensional vector spaces. Now if $h \in \mathscr{H}$, formula (4.3) shows that h is orthogonal to $\mathscr{I}^+$, the subspace of polynomials in $\mathscr{I}$ without constant terms, and even to the ideal $\mathscr{I}^+\mathscr{P}$ generated by $\mathscr{I}^+$ in $\mathscr{P}$. But as an element orthogonal to $\mathscr{I}^+\mathscr{P}$ is clearly in $\mathscr{H}$, we get $\mathscr{P} = \mathscr{H} \oplus \mathscr{I}^+\mathscr{P}$; but of course we may iterate this result to get that any polynomial in $\mathscr{P}$ can be written as a sum of products of a harmonic polynomial by an element in $\mathscr{I}$. This is what (a) means.

To prove (b), observe first that

$$\frac{d}{dt}(\Gamma_{tz})|_{t=0} = \tfrac{1}{2}\pi\, q_z, \qquad z \in \mathfrak{p}^+.$$

From this it follows that the closed space generated by the special gaussians contains $\mathscr{I}$, and hence the closed space generated by products of a harmonic polynomial and a special gaussian contains $\mathscr{I}\mathscr{H}$, which equals $\mathscr{P}$ by a part (a). As $\mathscr{P}$ is dense in $\mathscr{F}$, we get (b).

There is a natural action of $GL(N, \mathbb{C})$ on $\mathscr{P}$: if $g \in GL(N, \mathbb{C})$ and $p \in \mathscr{P}$, define $L(g)p$ by

$$(L(g)p)(\xi) = p(g^{-1}.\xi), \qquad \xi \in \mathbb{C}^N.$$

Then $L(g)q_Z = q_{g.Z}$, where

$$g.Z = (g^{-1})^t Z g^{-1}.$$

Using the identifications made in Section 1, this corresponds to the adjoint action of $GL(n, \mathbb{C})$ (complexification of $\mathbb{U}(N)$) on $\mathrm{Sym}_N(\mathbb{C})$.

Now if $g \in K_{\mathbb{C}}$ and $z \in \mathfrak{p}^+$, this implies that $L(\rho(g))q_z = q_{g.z}$, where $g.z = \mathrm{Ad}\, g(z)$. This clearly implies that $\mathscr{H}$ is stable under $L(\rho(K_{\mathbb{C}}))$, and gives rise to a representation of $K_{\mathbb{C}}$ in $\mathscr{H}$.

5. The main formula

In this section we describe the action of the metaplectic representation on the product of a special gaussian by a harmonic polynomial.

THEOREM 1. *Let $z \in \mathscr{D}$, $h \in \mathscr{H}$ and $g \in G$. Then*

(5.1) $$T_{\rho(g)}(\Gamma_z h) = \pm j(g, z)^{-1/2}\Gamma_{g(z)} L(J(g, z))h.$$

We first need a simple lemma which generalizes the orthogonality relation between $\mathscr{H}$ and $\mathscr{I}_+\mathscr{P}$.

LEMMA 1. *Let $h \in \mathscr{H}$, $f \in \mathscr{F}$ and $p \in \mathscr{I}_+$. Then*

(5.2) $$\int_{\mathbb{C}^N} h(\eta)\overline{f(\eta)}\;\overline{p(\eta)}\, d\mu(\eta) = 0.$$

This is a consequence of the density of $\mathscr{P}$ in $\mathscr{F}$.

Let us first prove (5.1) when $z = o$. Notice that $\Gamma_0 \equiv 1$, $J(g, o) = K(\rho(g))$, $g(o) = x_+(g)$, $j(g, o) = \det K(\rho(g))$. Using this and formulas (2.4) and (2.6), the left-hand side of (5.1) can be written as

$$T_{\rho(g)}h(\xi) = \pm j(g, o)^{-1/2}\Gamma_{g(o)}(\xi) \int_{\mathbb{C}^N} h(\eta)\Gamma_{-x^-(g)}(\eta) e^{\pi(J(g,o)^{-1}\xi|\eta)}\, d\mu(\eta).$$

As $\Gamma_{-x^-(g)}$ can be expanded as 1 plus a converging series of elements in $\mathscr{I}_+$, we see from (5.2) that the value of the integral is just

$$\int_{\mathbb{C}^N} h(\eta) e^{\pi(J(g,o)^{-1}\xi|\eta)}\, d\mu(\eta).$$

Using now the reproducing property (2.3), this integral is equal to $h(J(g,o)^{-1}\xi) = L(J(g,o))h(\xi)$, hence formula (5.1) is true in this special case. For the general case, choose $g_0 \in G$, such that $z = g_0(o)$. Then $h.\Gamma_z = j(g_0, o)^{+1/2} T_{\rho(g_0)}(L(J(g_0,o)^{-1})h)$, by using the part of the formula we already proved. As T is a representation, we obtain

$$\begin{aligned} T_{\rho(g)}(h\Gamma_Z) &= \pm j(g_0,o)^{1/2} T_{\rho(gg_0)}(L(J(g_0,o)^{-1})h) \\ &= \pm j(g_0,o)^{1/2} j(gg_0,o)^{-1/2} \Gamma_{gg_0(o)}(L(J(gg_0,o)J(g_0,o)^{-1})h), \end{aligned}$$

where we use again the part of the formula we already proved. It just remains to use the cocycle relations (3.10) to conclude.

In deriving consequences of the main formula, we need another result which is proven along similar lines.

PROPOSITION 3. *Let* $z, w \in \mathscr{D}$, $h, \ell \in \mathscr{H}$. *Then*

$$(\Gamma_z h, \Gamma_w \ell)_{\mathscr{F}} = k(z,w))^{1/2} (K(z,w)^{-1} h|\ell). \tag{5.3}$$

Here we use notation from Section 1; the determination of the square root is chosen so that $k(z,z)^{1/2}$ is positive.

Choose g and γ in G such that $z = g(o)$, $w = \gamma(o)$. Then

$$\begin{aligned} h\Gamma_z &= j(g,o)^{1/2} T_{\rho(g)}(L(J(g,o)^{-1})h \quad \text{and} \\ \ell\Gamma_w &= j(\gamma,o)^{1/2} T_{\rho(\gamma)}(L(J(\gamma,o)^{-1})\ell. \end{aligned}$$

So

$$\begin{aligned} (\Gamma_z h, \Gamma_w \ell)_{\mathscr{F}} &= j(g,o)^{1/2} \overline{j(\gamma,o)}^{1/2} (T_{\rho(g)}(L(J(g,o)^{-1})h), T_{\rho(\gamma)}(L(J(\gamma,o)^{-1})\ell))_{\mathscr{F}} \\ &= j(g,o)^{1/2} \overline{j(\gamma,o)}^{1/2} (T_{\rho(\gamma^{-1}g)}(L(J(g,o)^{-1})h), (L(J(\gamma,o)^{-1})\ell))_{\mathscr{F}} \\ &= j(g,o)^{1/2} \overline{j(\gamma,o)}^{1/2} j(\gamma^{-1}g,o)^{-1/2} \\ &\quad \cdot (\Gamma_{\gamma^{-1}g(o)}(L(J(\gamma^{-1}g,o)J(g,o)^{-1})h), L(J(\gamma,o)^{-1})\ell)_{\mathscr{F}} \\ &= j(g,o)^{1/2} \overline{j(\gamma,o)}^{1/2} j(\gamma^{-1}g,o)^{-1/2} \\ &\quad \cdot (L(J(\gamma^{-1}g,o)J(g,o)^{-1})h), L(J(\gamma,o)^{-1})\ell)_{\mathscr{F}} \end{aligned}$$

where we use (5.2) again.

Now $L(g)^* = L(\sigma(g)^{-1})$, and so by transposition we get

$$(\Gamma_z h, \Gamma_w \ell)_{\mathscr{F}} = j(g,o)^{1/2} \overline{j(\gamma,o)}^{1/2} J(\gamma^{-1}g,o)^{-1/2} \\ \cdot (L(\sigma(J(\gamma,o)))J(\gamma^{-1}g,o)J(g,o)^{-1}h,\ell)_{\mathscr{F}}.$$

Consider now the function $\sigma(J(\gamma,o))J(\sigma(\gamma)^{-1}g,o)J(g,o)^{-1}$. It is holomorphic in g, antiholomorphic in γ in a (connected) open set in $G_{\mathbb{C}} \times G_{\mathbb{C}}$, and coincides on $G \times G$ with $\sigma(J(\gamma,o))J(\gamma^{-1}g,o)J(g,o)^{-1}$. It is unchanged if we multiply g (and γ) by any element in $K_{\mathbb{C}}$, as $K_{\mathbb{C}}$ normalizes $\mathfrak{p}^+$ and $\mathfrak{p}^-$. These remarks allow us to substitute $g = \exp z$ and $\gamma = \exp w$ in the computation; but this gives immediately $K(z,w)^{-1}$ (see condition (3.11)). The same argument can be used to compute the scalar factor $j(g,o)^{1/2} j(\overline{\gamma,o})^{1/2} J(\gamma^{-1}g,o)^{-1/2}$. Of course the formula is proven up to a sign. But for fixed h and ℓ, the inner product is clearly a function in $\mathscr{D} \times \mathscr{D}$ which is holomorphic in z and antiholomorphic in w; this forces the choice of the sign, because this must be positive for $z = w$.

6. Consequences of the main formula

The main formula suggests that the decomposition of $T|_G$ is connected with decomposition of $L|_{\rho(K_{\mathbb{C}})}$ on the space $\mathscr{H}$.

For $\mathscr{L}$ an invariant subspace of $\mathscr{H}$ under $L(\rho(K_{\mathbb{C}}))$, form

(6.1) $\quad \mathscr{F}_{\mathscr{L}} =$ the closed subspace of $\mathscr{F}$ generated by all products $h\Gamma_z$,

where $h \in \mathscr{L}$ and $z \in \mathscr{D}$.

Clearly, by an argument similar to the one used in Section 4, $\mathscr{F}_{\mathscr{L}}$ is also the closure of the subspace generated by all products hq, where $h \in \mathscr{L}$ and $q \in \mathscr{I}$.

Thanks to the main formula (5.1), $\mathscr{F}_{\mathscr{L}}$is stable under $T_{|_G}$.

THEOREM 2. *Assume $\mathscr{L}$ is irreducible under $L(\rho(K_{\mathbb{C}}))$. Then $\mathscr{F}_{\mathscr{L}}$ is irreducible under $T_{|_G}$.*

To prove this result, we need to consider the metaplectic representation as a unitary representation $\tilde{T}$ of $\widetilde{\mathrm{Sp}}_c$ a two-fold covering of Sp. We denote by $\tilde{G}$ the corresponding covering of G, and $\tilde{\rho}$ the lifting of ρ to $\tilde{G}$. Let U be the compact subgroup $\{\exp tH_0\}_{t\in\mathbb{R}}$, and $\tilde{U}$ the pullback of U in $\tilde{G}$. Now, formula (2.6) implies in this case

$$\tilde{T}(\tilde{\rho}(\tilde{u})) = \det\tilde{\rho}(\tilde{u})^{-1/2} L(\tilde{\rho}(\tilde{u})), \tag{6.2}$$

where $\det\tilde{\rho}(\tilde{u})^{-1/2}$ is a (univalued) character on $\tilde{U}$. As $\mathscr{L}$ is irreducible under $L(\rho(K_{\mathbb{C}}))$, and H_o is in the center of $\mathfrak{k}$, $L(\tilde{\rho}(\tilde{u}))|_{\mathscr{L}}$ is, by Schur lemma, a multiple of $\mathbb{I}_{\mathscr{L}}$.

These facts together imply that there exists a character χ of $\tilde{U}$, such that

$$\tilde{T}(\tilde{\rho}(\tilde{u}))|_{\mathscr{L}} = \chi(\tilde{u})\,\mathrm{I\!I}_{\mathscr{L}}. \tag{6.3}$$

Now define the following operator P on $\mathscr{F}_{\mathscr{L}}$:

$$Pf = \int_{\tilde{U}} \chi(\tilde{u})^{-1}\tilde{T}(\tilde{u})f\,d\tilde{u}. \tag{6.4}$$

This clearly defines a bounded operator on $\mathscr{F}_{\mathscr{L}}$. Moreover $P^2 = P$, and because of the unitarity of $\tilde{T}$, it is easily seen that P is self-adjoint. Now, if $h \in \mathscr{L}$, then $Pf = f$ as a consequence of (6.3). Moreover, if $g \in \mathscr{I}_+$, and $h \in \mathscr{L}$,

$$\begin{aligned} P(qh) &= \int_{\tilde{U}} \chi(\tilde{u})^{-1} \det \tilde{\rho}(u)^{-1/2} L(\tilde{\rho}(\tilde{u}))(qh)du \\ &= \int_{\tilde{U}} \chi(\tilde{u})^{-1} \det \tilde{\rho}(\tilde{u})^{-1/2} L(\tilde{\rho}(\tilde{u}))(q). \det \tilde{\rho}(\tilde{u})^{1/2}.\chi(\tilde{u}).h d\tilde{u} \\ &= \left(\int_{\tilde{U}} L(\tilde{\rho}(\tilde{u}))(q)d\tilde{u}\right).h. \end{aligned}$$

But $\int_{\tilde{U}} L(\tilde{\rho}(\tilde{u}))(q)d\tilde{u} = 0$, as q is the sum of holomorphic homogeneous polynomials of strictly positive degree.

From these results, we conclude that P is the orthogonal projection of $\mathscr{F}_{\mathscr{L}}$ onto $\mathscr{L}$. Now the rest of the proof is standard: if $\mathscr{V}$ is a closed invariant subspace in $\mathscr{F}_{\mathscr{L}}$. then P leaves $\mathscr{V}$ invariant, and same is true for $\mathscr{V}^{\perp}$, its orthogonal complement in $\mathscr{F}_{\mathscr{L}}$. So we may assume that $\mathscr{V}$ does contain an element f such that $Pf \neq 0$ (otherwise use $\mathscr{V}^{\perp}$ instead). So $\mathscr{V} \cap \mathscr{L}$ is not reduced to $\{0\}$, so $\mathscr{V} \supset \mathscr{L}$ by the irreducibility of $\mathscr{L}$ under $L(\rho(K_{\mathbb{C}}))$. But (5.1) implies that $\Gamma_z h \in \mathscr{V}$ for $z \in \mathscr{D}$ and $h \in \mathscr{L}$. So $\mathscr{V} \supset \mathscr{F}_{\mathscr{L}}$, and this obviously shows the irreducibility of $\mathscr{F}_{\mathscr{L}}$.

REMARK 3. As K is compact, an irreducible subspace $\mathscr{L}$ is always finite dimensional. Moreover, $K_{\mathbb{C}}$ has a center, which (through ρ) acts on $\mathbb{C}^N$ by complex dilations (this is essentially the 'H_2 assumption' (3.4)). From this we conclude that $\mathscr{L}$ consists in homogeneous polynomials and two irreducible subspaces corresponding to equivalent representations of $K_{\mathbb{C}}$ correspond to the same degree of homogeneity. So $\mathscr{H}$ decomposes under $\rho(K_{\mathbb{C}})$ with finite multiplicity.

REMARK 4. Let $\mathscr{P}_{\mathscr{L}}$ be the vector subspace spanned by all products hq, where h runs through $\mathscr{L}$ and q runs through $\mathscr{I}$. Then $\mathscr{P}_{\mathscr{L}}$ is dense in $\mathscr{F}_{\mathscr{L}}$, and is graded space under the natural graduation on $\mathscr{P}$, as a consequence of the last remark. But $\mathscr{P} \cap \mathscr{F}_{\mathscr{L}}$ is also a graded space (because of (6.2)), and each homogeneous component of these two spaces are clearly equal. So $\mathscr{L}$ is the space of elements in $\mathscr{P} \cap \mathscr{F}_{\mathscr{L}}$ of minimal degree.

THEOREM 3. *Let $\mathscr{L}$ and $\mathscr{L}'$ be two invariant irreducible subspaces. Then $\mathscr{F}_{\mathscr{L}}$ and $\mathscr{F}_{\mathscr{L}'}$ are equivalent as representations of G if and only if $\mathscr{L}$ and $\mathscr{L}'$ are equivalent as K-representations.*

Let first assume $\mathscr{F}_{\mathscr{L}}$ and $\mathscr{F}_{\mathscr{L}'}$ are equivalent, and let A be a (unitary) intertwining operator form $\mathscr{F}_{\mathscr{L}}$ into $\mathscr{F}_{\mathscr{L}'}$.

A commutes with $L(\rho(u))$ for $u \in U$, which implies that A maps $\mathscr{P} \cap \mathscr{F}_{\mathscr{L}}$ into $\mathscr{P} \cap \mathscr{F}_{\mathscr{L}'}$, preserves the degree, so maps $\mathscr{L}$ into $\mathscr{L}'$ by Remark 4. But for $k \in K$, $T_{\rho(g)}$ coincides with $L(\rho(k))$ (except for the caracter $\det \rho(k)^{-1/2}$) and hence $\mathscr{L}$ and $\mathscr{L}'$ are equivalent as K-representations.

Conversely, let $\mathscr{L}$ and $\mathscr{L}'$ be two equivalent subspaces of $\mathscr{H}$. Let B be a (unitary) intertwining operator. Let $f = \sum_{i=1}^{n} h_i \Gamma_{z_i}$, with $h_i \in \mathscr{L}$ and $z_i \in \mathscr{D}$. Define $Af = \sum_{i=1}^{n} Bh_i \Gamma_{z_i}$. To see that A is well defined on $\mathscr{P} \cap \mathscr{F}_{\mathscr{L}}$, observe that $\|Af\|_{\mathscr{F}} = \|f\|_{\mathscr{F}}$, as a consequence of formula (5.3). Now extend A to $\mathscr{F}_{\mathscr{L}}$ by continuity. The intertwining property is clearly a consequence of the main formula. This proves Theorem 3.

THEOREM 4. *The restriction of the metaplectic representation T to G decomposes discretely and with finite multiplicities. Each component is of the form $\mathscr{F}_{\mathscr{L}}$, for some invariant irreducible subspace $\mathscr{L}$, and the multiplicity of the corresponding representation of G in $\mathscr{F}$ equals the multiplicity of the corresponding reprsentation of K in $\mathscr{H}$.*

In fact, let $\mathscr{H} = \bigoplus_{i \in I} \mathscr{H}_i$ be a decomposition of $\mathscr{H}$ into irreducible subspaces. Let $\mathscr{F}_i = \mathscr{F}_{\mathscr{H}_i}$. Then, as the $\mathscr{H}_i$ are mutually orthogonal, the same is true for the $\mathscr{F}_i$ (see formula (5.3)). As $\mathscr{P} = \mathscr{I}\mathscr{H}$, we see by density that $\mathscr{F} = \bigoplus_{i \in I} \mathscr{F}_i$ (direct orthogonal sum).

The rest of the proof is an easy consequence of Theorems 2 and 3.

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